



Communication-Avoiding Algorithms for Linear Algebra, ML and Beyond

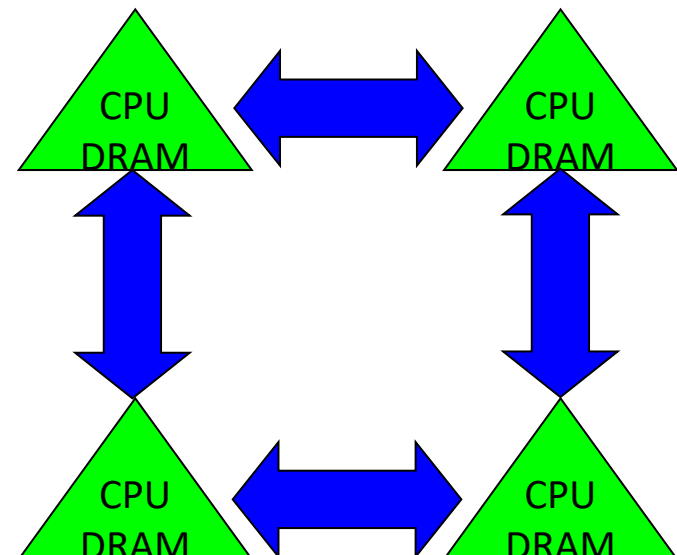
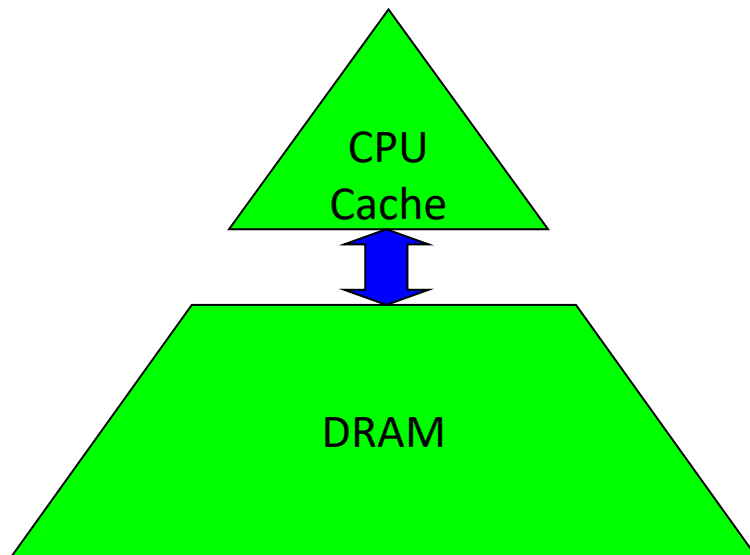
Jim Demmel, EECS & Math Depts., UC Berkeley

And many, many others ...

Why avoid communication? (1/3)

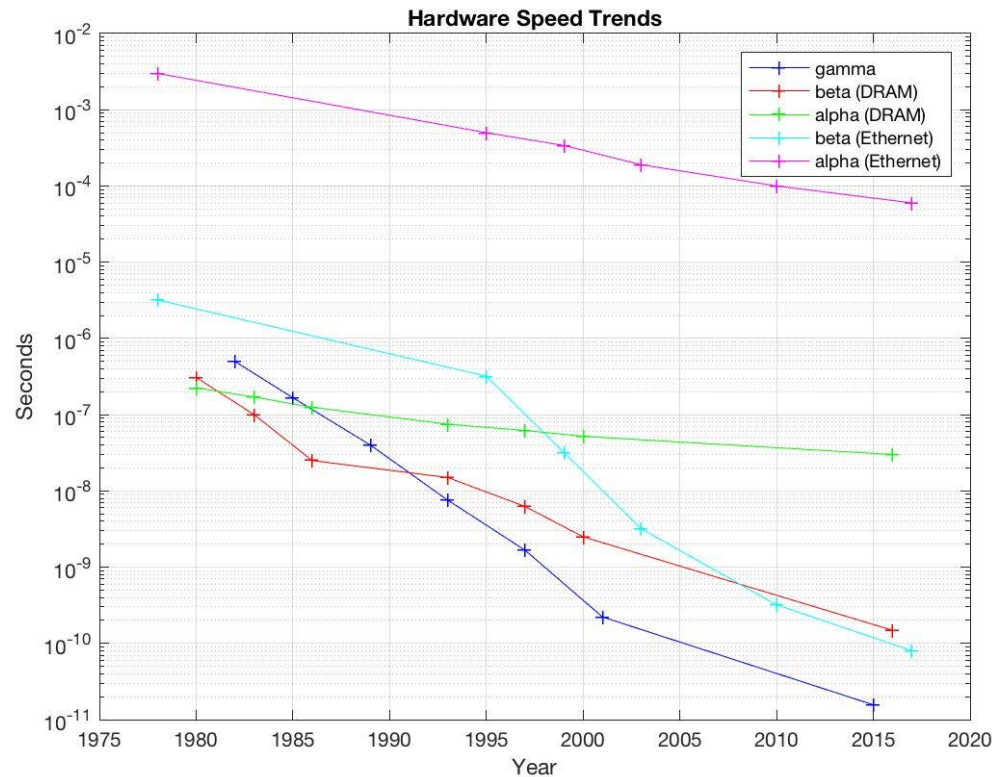
Algorithms have two costs (measured in time or energy):

1. Arithmetic (FLOPS)
2. Communication: moving data between
 - levels of a memory hierarchy (sequential case)
 - processors over a network (parallel case).



Why avoid communication? (2/3)

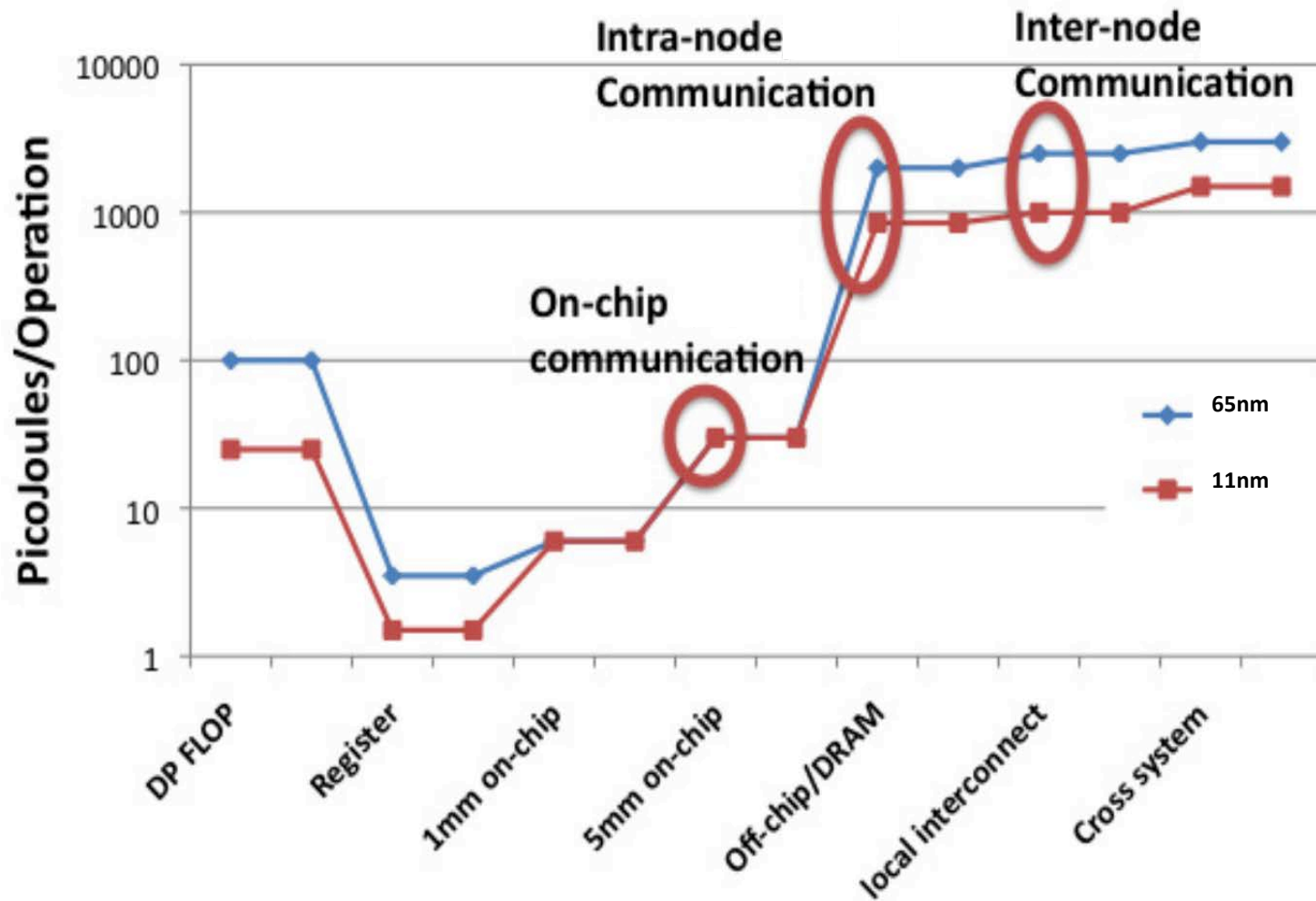
- Running time of an algorithm is sum of 3 terms:
 - # flops * time_per_flop
 - # words moved / bandwidth
 - # messages * latency } communication
- Time_per_flop (γ) \ll 1/ bandwidth (β) \ll latency (α)



Data from
Patterson &
Hennessey, 2019

Why avoid communication? (3/3)

Same story for saving energy





Goals

- Redesign algorithms to *avoid* communication
 - Between all memory hierarchy levels
 - $L1 \leftrightarrow L2 \leftrightarrow \text{DRAM} \leftrightarrow \text{network, etc}$
- Attain lower bounds if possible
 - Classical algorithms often far from lower bounds
 - Large speedups and energy savings possible
- Automate implementation of communication-avoiding (CA) algorithms



Sample Speedups

- Doing same operations, just in a different order
 - Up to **12x** faster for 2.5D dense matmul on 64K core IBM BG/P
 - Up to **100x** faster for 1.5D sparse-dense matmul on 1536 core Cray XC30
 - Up to **6.2x** faster for 2.5D All-Pairs-Shortest-Path on 24K core Cray XE6
 - Up to **11.8x** faster for direct N-body on 32K core IBM BG/P
- Mathematically identical answer, but different algorithm
 - Up to **13x** faster for Tall Skinny QR on Tesla C2050 Fermi NVIDIA GPU
 - Up to **6.7x** faster for symeig(band A) on 10 core Intel Westmere
 - Up to **4.2x** faster for BiCGStab (MiniGMG bottom solver) on 24K core Cray XE6
 - Up to **5.1x** faster for coordinate descent LASSO on 3K core Cray XC30
- Different algorithm, different approximate answer
 - Up to **16x** faster for SVM on a 1536 core Cray XC30
 - Up to **135x** faster for ImageNet training on 2K Intel KNL nodes



Sample Speedups

- Doing same operations, just in a different order

**Ideas adopted by Nervana, “deep learning” startup,
acquired by Intel in August 2016**

Kwasniewski, Hoefler, et al (Best Student Paper, SC’19)

- Mathematically identical answer, but different algorithm

SIAG on Supercomputing Best Paper Prize, 2016

(D., Grigori, Hoemmen, Langou)

Released in LAPACK 3.7, 2016

LAPACK 3.10: Householder Reconstruction, 2021

- Different algorithm, different approximate answer

IPDPS 2015 Best Paper Prize (You, D. Czechowski, Song, Vuduc)

ICPP 2018 Best Paper Prize (You, Zhang, Hsieh, D., Keutzer)

2019: Idea (LARS) adopted by industry standard benchmark MLPerf

Outline

- Linear Algebra
 - Communication Lower Bounds for classical direct linear algebra
 - CA 2.5D Matmul
 - TSQR - Tall-Skinny QR
 - Iterative Methods for linear algebra
- Machine Learning
 - Training Neural Nets – “ImageNet training in minutes”
 - Convolutional Neural Nets
- And Beyond
 - Extending communication lower bounds and optimal algorithms to general loop nests

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Summary of CA Linear Algebra

- “Direct” Linear Algebra
 - Lower bounds on communication for linear algebra problems like $Ax=b$, least squares, $Ax = \lambda x$, SVD, etc
 - Mostly not attained by algorithms in standard libraries
 - LAPACK, ScaLAPACK, ...
 - New algorithms needed to attain these lower bounds
 - New numerical properties, ways to encode answers, data structures, not just loop transformations
 - Autotuning to find optimal implementation (eg GPTune)
 - Sparse matrices: depends on sparsity structure
- Ditto for “Iterative” Linear Algebra

Lower bound for all “n³-like” linear algebra

- Let M = “fast” memory size (per processor)

$$\#words_moved \text{ (per processor)} = \Omega(\#flops \text{ (per processor)} / M^{1/2})$$

- Parallel case: assume either load or memory balanced
- Holds for
 - Matmul



Lower bound for all “n³-like” linear algebra

- Let M = “fast” memory size (per processor)

$$\#words_moved \text{ (per processor)} = \Omega(\#flops \text{ (per processor)} / M^{1/2})$$

$$\#messages_sent \geq \#words_moved / largest_message_size$$

- Parallel case: assume either load or memory balanced
- Holds for
 - Matmul, BLAS, LU, QR, eig, SVD, tensor contractions, ...
 - Some whole programs (sequences of these operations, no matter how individual ops are interleaved, eg A^k)
 - Dense and sparse matrices (where #flops $\ll n^3$)
 - Sequential and parallel algorithms
 - Some graph-theoretic algorithms (eg Floyd-Warshall)

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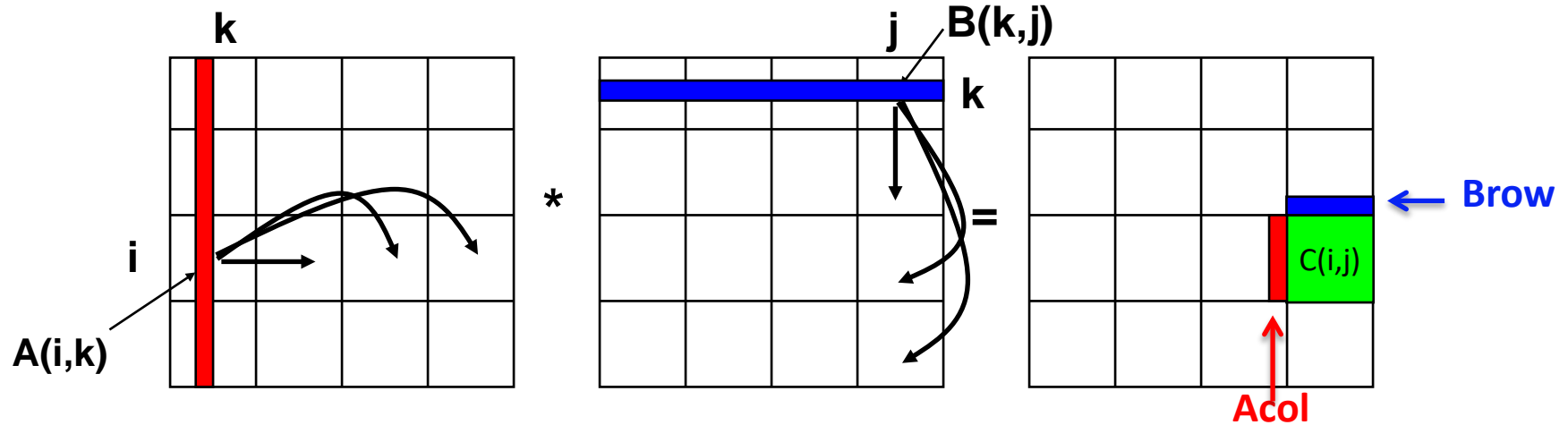
SIAM SIAG/Linear Algebra Prize, 2012

(Ballard, D., Holtz, Schwartz)

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SUMMA– $n \times n$ matmul on $P^{1/2} \times P^{1/2}$ grid (nearly) optimal using minimum memory $M=O(n^2/P)$



For $k=0$ to $n/b-1$... $b = \text{block size} = \# \text{cols in } A(i,k) = \# \text{rows in } B(k,j)$

for all $i = 1$ to $P^{1/2}$

owner of $A(i,k)$ broadcasts it to whole processor row (using binary tree)

for all $j = 1$ to $P^{1/2}$

owner of $B(k,j)$ broadcasts it to whole processor column (using bin. tree)

Receive $A(i,k)$ into $Acol$

Receive $B(k,j)$ into $Brow$

$C_{\text{myproc}} = C_{\text{myproc}} + Acol * Brow$

Summary of dense parallel algorithms attaining communication lower bounds

- Assume $n \times n$ matrices on P processors
- Minimum Memory per processor = $M = O(n^2 / P)$
- Recall lower bounds:
#words_moved = $\Omega((n^3 / P) / M^{1/2}) = \Omega(n^2 / P^{1/2})$
#messages = $\Omega((n^3 / P) / M^{3/2}) = \Omega(P^{1/2})$
- SUMMA attains this lower bound
- Does ScaLAPACK attain these bounds?
 - For #words_moved: mostly, except nonsym. Eigenproblem
 - For #messages: asymptotically worse, except Cholesky
- New algorithms attain all bounds, up to polylog(P) factors
 - Cholesky, LU, QR, Sym. and Nonsym eigenproblems, SVD

Can we do Better?

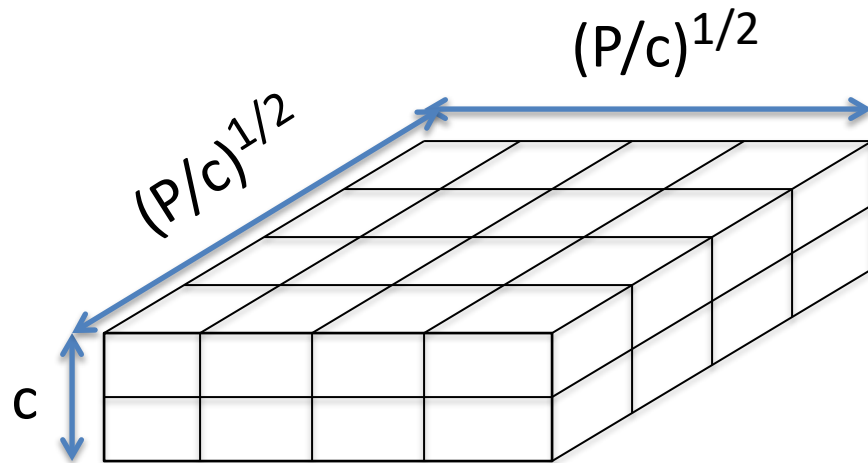


Can we do better?

- Aren't we already optimal?
- Why assume $M = O(n^2/p)$, i.e. minimal?
 - Lower bound still true if more memory
 - Can we attain it?
- Special case: “3D Matmul”
 - Uses $M = O(n^2/p^{2/3})$
 - Dekel, Nassimi, Sahni [81], Bernstein [89], Agarwal, Chandra, Snir [90], Johnson [93], Agarwal, Balle, Gustavson, Joshi, Palkar [95]
- Not always $p^{1/3}$ times as much memory available...

2.5D Matrix Multiplication

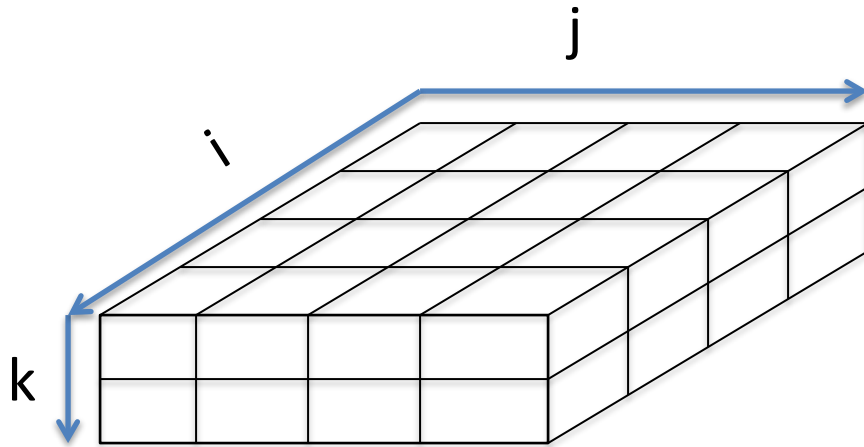
- Assume can fit cn^2/P data per processor, $c > 1$
- Processors form $(P/c)^{1/2} \times (P/c)^{1/2} \times c$ grid



Example: $P = 32$, $c = 2$

2.5D Matrix Multiplication

- Assume can fit cn^2/P data per processor, $c > 1$
- Processors form $(P/c)^{1/2} \times (P/c)^{1/2} \times c$ grid



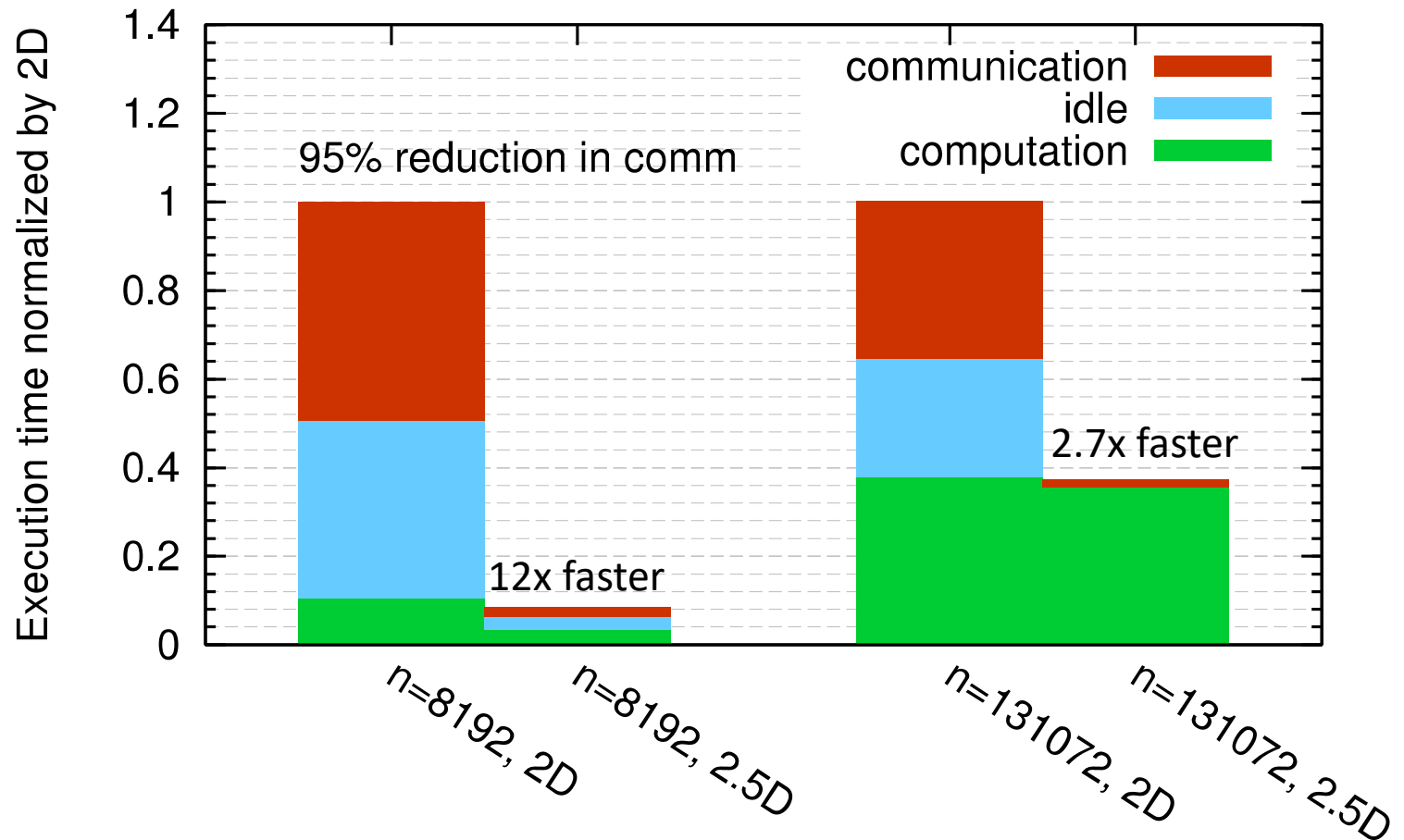
Initially $P(i,j,0)$ owns $A(i,j)$ and $B(i,j)$
each of size $n(c/P)^{1/2} \times n(c/P)^{1/2}$

- (1) $P(i,j,0)$ broadcasts $A(i,j)$ and $B(i,j)$ to $P(i,j,k)$
- (2) Processors at level k perform $1/c$ -th of SUMMA, i.e. $1/c$ -th of $\sum_m A(i,m) * B(m,j)$
- (3) Sum-reduce partial sums $\sum_m A(i,m) * B(m,j)$ along k -axis so $P(i,j,0)$ owns $C(i,j)$

2.5D Matmul on BG/P, 16K nodes / 64K cores

c = 16 copies

Matrix multiplication on 16,384 nodes of BG/P



Distinguished Paper Award, EuroPar'11 (Solomonik, D.)

Kwasniewski, Hoefler, et al (Best Student Paper, SC'19)

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TSQR: QR of a Tall, Skinny matrix

$$W = \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix}$$

$$\begin{pmatrix} R_{00} \\ R_{10} \\ R_{20} \\ R_{30} \end{pmatrix} = \begin{pmatrix} Q_{01} & R_{01} \\ Q_{11} & R_{11} \end{pmatrix}$$

$$\begin{pmatrix} R_{01} \\ R_{11} \end{pmatrix} = \begin{pmatrix} Q_{02} & R_{02} \end{pmatrix}$$

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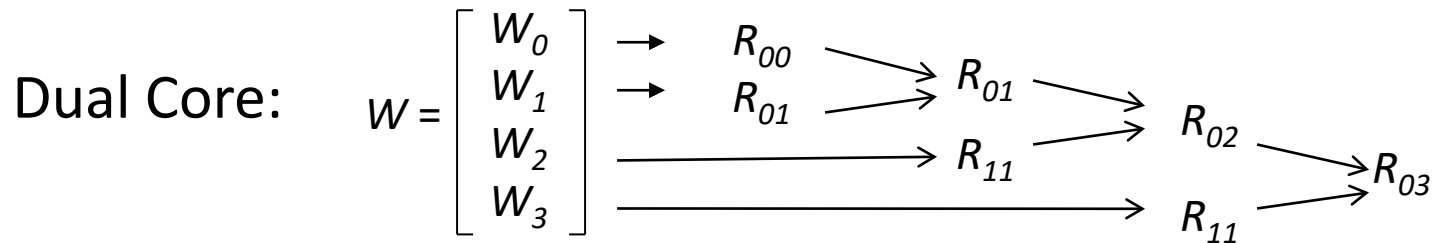
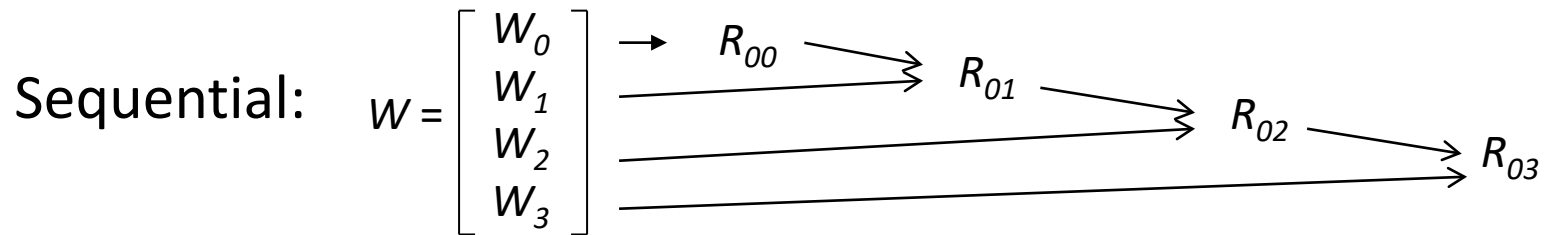
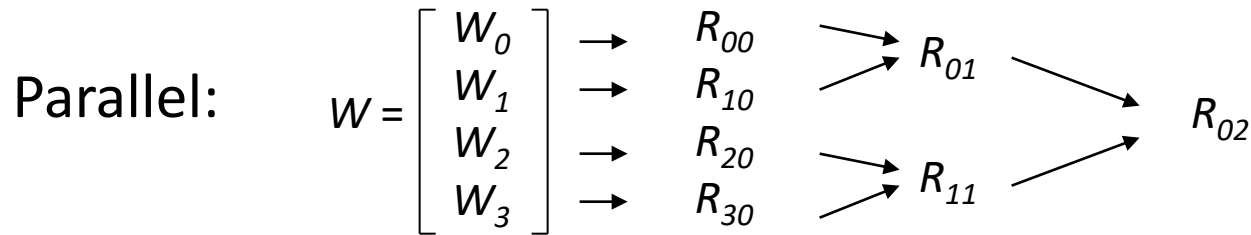
$$W = \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} Q_{00} & R_{00} \\ Q_{10} & R_{10} \\ Q_{20} & R_{20} \\ Q_{30} & R_{30} \end{pmatrix} = \begin{pmatrix} Q_{00} \\ Q_{10} \\ Q_{20} \\ Q_{30} \end{pmatrix} \cdot \begin{pmatrix} R_{00} \\ R_{10} \\ R_{20} \\ R_{30} \end{pmatrix}$$

$$\begin{pmatrix} R_{00} \\ R_{10} \\ R_{20} \\ R_{30} \end{pmatrix} = \begin{pmatrix} Q_{01} & R_{01} \\ Q_{11} & R_{11} \end{pmatrix} = \begin{pmatrix} Q_{01} \\ Q_{11} \end{pmatrix} \cdot \begin{pmatrix} R_{01} \\ R_{11} \end{pmatrix}$$

$$\begin{pmatrix} R_{01} \\ R_{11} \end{pmatrix} = \begin{pmatrix} Q_{02} & R_{02} \end{pmatrix}$$

Output = $\{ Q_{00}, Q_{10}, Q_{20}, Q_{30}, Q_{01}, Q_{11}, Q_{02}, R_{02} \}$

TSQR: An Architecture-Dependent Algorithm



Multicore / Multisocket / Multirack / Multisite / Out-of-core: ?

Can choose reduction tree dynamically



TSQR Performance Results

- Parallel
 - Intel Clovertown
 - Up to **8x** speedup (8 core, dual socket, 10M x 10)
 - Pentium III cluster, Dolphin Interconnect, MPICH
 - Up to **6.7x** speedup (16 procs, 100K x 200)
 - BlueGene/L
 - Up to **4x** speedup (32 procs, 1M x 50)
 - Tesla C 2050 / Fermi
 - Up to **13x** (110,592 x 100)
 - Grid – **4x** on 4 cities vs 1 city (Dongarra, Langou et al)
 - Cloud – **1.6x slower than just accessing data twice** (Gleich and Benson)
- Sequential
 - “**Infinite speedup**” for out-of-core on PowerPC laptop
 - As little as 2x slowdown vs (predicted) infinite DRAM
 - LAPACK with virtual memory never finished
- SVD costs about the same
- Joint work with Grigori, Hoemmen, Langou, Anderson, Ballard, Keutzer, others



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Avoiding Communication in Iterative Linear Algebra

- k-steps of iterative solver for sparse $Ax=b$ or $Ax=\lambda x$
 - Does k SpMV's with A and starting vector
 - Many such “Krylov Subspace Methods”
 - Conjugate Gradients (CG), GMRES, Lanczos, Arnoldi, ...
- Goal: minimize communication
 - Assume matrix “well-partitioned”
 - Serial implementation
 - Conventional: $O(k)$ moves of data from slow to fast memory
 - **New: $O(1)$ moves of data – optimal**
 - Parallel implementation on p processors
 - Conventional: $O(k \log p)$ messages (k SpMV calls, dot prods)
 - **New: $O(\log p)$ messages - optimal**
- Lots of speed up possible (modeled and measured)
 - Price: some redundant computation
 - Challenges: Poor partitioning, Preconditioning, Num. Stability

Minimizing Communication of GMRES to solve $Ax=b$

- GMRES: find x in $\text{span}\{b, Ab, \dots, A^k b\}$ minimizing $\|Ax - b\|_2$

Standard GMRES

for $i=1$ to k

$w = A \cdot v(i-1) \dots SpMV$

$MGS(w, v(0), \dots, v(i-1))$

update $v(i), H$

endfor

solve LSQ problem with H

Communication-avoiding GMRES

$W = [v, Av, A^2v, \dots, A^k v]$

$[Q, R] = \text{TSQR}(W)$

\dots “Tall Skinny QR”

build H from R

solve LSQ problem with H

Sequential case: #words moved decreases by a factor of k

Parallel case: #messages decreases by a factor of k

- **Oops – W from power method, precision lost!**
- **Fix: replace W by $[v, p_1(A)v, p_2(A)v, \dots, p_k(A)v]$**
- Up to **2.3x** speedup for GMRES on 8 core Intel Clovertown (Hoemmen)
- Up to **4.2x** speedup for BiCGStab on 24K core Cray XE6 (Carson)

Compute $r_0 = b - Ax_0$. Choose r_0^* arbitrary.

Set $p_0 = r_0$, $q_{-1} = 0_{N \times 1}$.

For $k = 0, 1, \dots$ until convergence, Do

$$P = [p_{sk}, Ap_{sk}, \dots, A^s p_{sk}]$$

$$Q = [q_{sk-1}, Aq_{sk-1}, \dots, A^s q_{sk-1}]$$

$$R = [r_{sk}, Ar_{sk}, \dots, A^s r_{sk}]$$

//Compute the $1 \times (3s + 3)$ Gram vector.

$$g = (r_0^*)^T [P, Q, R]$$

//Compute the $(3s + 3) \times (3s + 3)$ Gram matrix

$$G = \begin{bmatrix} P^T \\ Q^T \\ R^T \end{bmatrix} [P \quad Q \quad R]$$

For $\ell = 0$ to s ,

$$b_{sk}^\ell = [B_1(:, \ell)^T, 0_{s+1}^T, 0_{s+1}^T]^T$$

$$c_{sk-1}^\ell = [0_{s+1}^T, B_2(:, \ell)^T, 0_{s+1}^T]^T$$

$$d_{sk}^\ell = [0_{s+1}^T, 0_{s+1}^T, B_3(:, \ell)^T]^T$$

1. Compute $r_0 := b - Ax_0$; r_0^* arbitrary;
2. $p_0 := r_0$.
3. For $j = 0, 1, \dots$ until convergence Do:
4. $\alpha_j := (r_j, r_0^*) / (Ap_j, r_0^*)$
5. $s_j := r_j - \alpha_j Ap_j$
6. $\omega_j := (As_j, s_j) / (As_j, As_j)$
7. $x_{j+1} := x_j + \alpha_j p_j + \omega_j s_j$
8. $r_{j+1} := s_j - \omega_j As_j$
9. $\beta_j := \frac{(r_{j+1}, r_0^*)}{(r_j, r_0^*)} \times \frac{\alpha_j}{\omega_j}$
10. $p_{j+1} := r_{j+1} + \beta_j (p_j - \omega_j Ap_j)$
11. EndDo

CA-BiCGStab

For $j = 0$ to $\lfloor \frac{s}{2} \rfloor - 1$, Do

$$\alpha_{sk+j} = \frac{\langle g, d_{sk+j}^0 \rangle}{\langle g, b_{sk+j}^1 \rangle}$$

$$q_{sk+j} = r_{sk+j} - \alpha_{sk+j} [P, Q, R] b_{sk+j}^1$$

For $\ell = 0$ to $s - 2j + 1$, Do

$$c_{sk+j}^\ell = d_{sk+j}^\ell - \alpha_{sk+j} b_{sk+j-1}^{\ell+1}$$

//such that $[P, Q, R] c_{sk+j}^\ell = A^\ell q_{sk+j}$

$$\omega_{sk+j} = \frac{\langle c_{sk+j+1}^1, Gc_{sk+j+1}^0 \rangle}{\langle c_{sk+j+1}^1, Gc_{sk+j+1}^1 \rangle}$$

$$x_{sk+j+1} = x_{sk+j} + \alpha_{sk+j} p_{sk+j} + \omega_{sk+j} q_{sk+j}$$

$$r_{sk+j+1} = q_{sk+j} - \omega_{sk+j} [P, Q, R] c_{sk+j+1}^1$$

For $\ell = 0$ to $s - 2j$, Do

$$d_{sk+j+1}^\ell = c_{sk+j+1}^\ell - \omega_{sk+j} c_{sk+j+1}^{\ell+1}$$

//such that $[P, Q, R] d_{sk+j+1}^\ell = A^\ell r_{sk+j+1}$

$$\beta_{sk+j} = \frac{\langle g, d_{sk+j+1}^0 \rangle}{\langle g, d_{sk+j}^0 \rangle} \times \frac{\alpha}{\omega}$$

$$p_{sk+j+1} = r_{sk+j+1} + \beta_{sk+j} p_{sk+j} - \beta_{sk+j} \omega_{sk+j} [P, Q, R] b_{sk+j}^1$$

For $\ell = 0$ to $s - 2j$, Do

$$b_{sk+j+1}^\ell = d_{sk+j+1}^\ell + \beta_{sk+j} b_{sk+j}^\ell - \beta_{sk+j} \omega_{sk+j} b_{sk+j}^{\ell+1}$$

//such that $[P, Q, R] b_{sk+j+1}^\ell = A^\ell p_{sk+j+1}$.

EndDo

EndDo

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Training Neural Nets by

Mini-Batch Stochastic Gradient Descent (SGD)

(You, Zhang, Hsieh, D., Keutzer, IPDPS 18)

- Iterate:
 - Pick a mini-batch of B data points
 - Update weights $W = W - \eta \cdot \nabla L(W)$
 - η = learning rate
 - $\nabla L(W)$ = gradient
- Data parallel version on P processors
 - Data partitioned, each processor gets B/P points
 - W_i replicated
 - Each processor computes $\nabla L(W)_i$ wrt its data
 - All-reduce: each processor computes
$$W_i = W_i - (\eta/P) \cdot \sum_{i=1}^P \nabla L(W)_i$$



$$\text{SGD: } W_i = W_i - (\eta/P) \cdot \sum_{i=1}^P \nabla L(W)_i$$

- Increase P to go faster: What are the bottlenecks?
- B/P decreases \Rightarrow less work per processor
 - Small matrix operations \Rightarrow locally communication bound
- Cost of each reduction $\sum_i \nabla L(W)_i$ grows
- Solution: increase B along with P
 - Maintain B/P \Rightarrow maintain processor efficiency
 - Try to converge in same #epochs (passes over data)
 - Same overall work, fewer reductions
- Oops: Convergence can be much worse
 - Convergence rate, test accuracy



Improving SGD convergence as B grows

- Facebook's strategy: adjust learning rate η
 - Increase B to kB \Rightarrow increase η to $k\eta$
 - Warmup rule: Start with smaller η , then increase
- Only worked up to B=1K for AlexNet (tried lots of tuning)
- Fix: Add Layer-wise Adaptive Rate Scaling (LARS)
 - $\|W\|/\|\nabla L(W)\|$ can vary by 233x between AlexNet layers
 - Let η be proportional to $\|W\|/\|\nabla L(W)\|$
 - (You, Gitman, Ginsburg, 2017)
 - Also need momentum, weight decay

ImageNet Training in Minutes

Speedup for AlexNet (for batchsize = 32K, changed LRN to BN)

Batch Size	Epochs	Top-1 Accuracy	Platform	Time
256	100	58.7%	8-core + K20 GPU	144 hrs
512	100	58.8%	DGX-1 station	6h 10m
4096	100	58.4%	DGX-1 station	2h 19m
32k	100	58.6%	512 KNLs	24m
32k	100	58.6%	1024 CPUs	11m

Speedup for ResNet50

Batch Size	Epochs	Top-1 Accuracy	Platform	Time
32	90	75.3%	CPU + M40 GPU	336h
256	90	75.3%	16 KNLs	45h
32K	90	75.4%	512 KNLs	60m
32K	90	75.4%	1600 CPUs	32m
32K	90	75.4%	2048 KNLs	20m

135x

ImageNet Training in Minutes

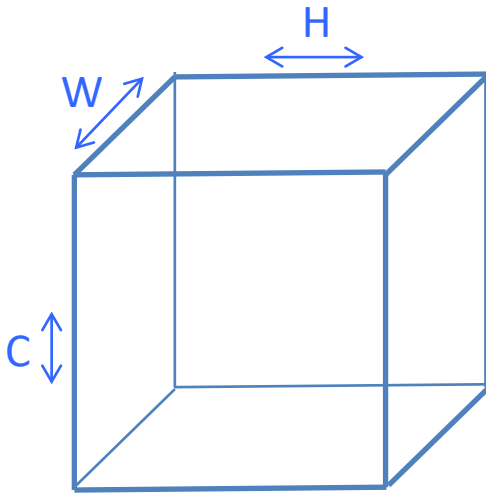
- Best Paper Prize at ICPP 2018
- Open Source in Caffe, NVIDIA Caffe, Facebook Caffe 2 (PyTorch)
- Media coverage by CACM, EureKalert, Intel, NSF, Science Daily, Science NewsLine, etc.
- Subsequent work at Tencent reached 4 minutes
- LARS adopted by industry standard benchmark MLPerf in 2019



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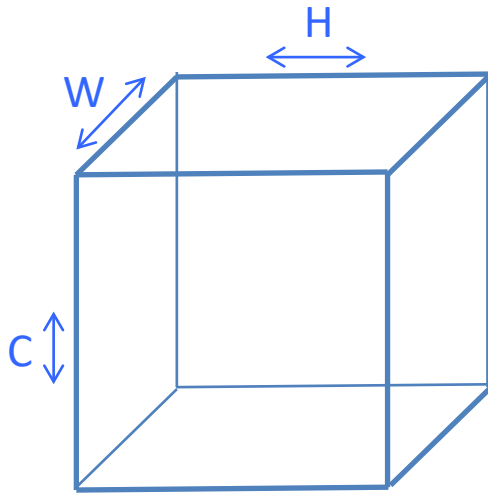
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What CNNs compute

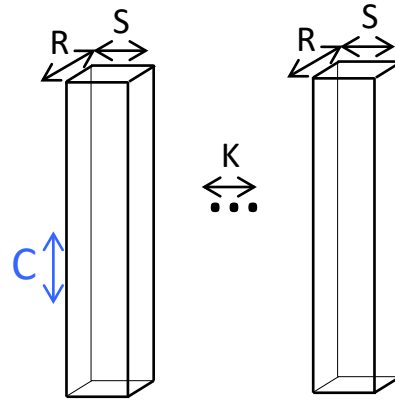


Image

What CNNs compute

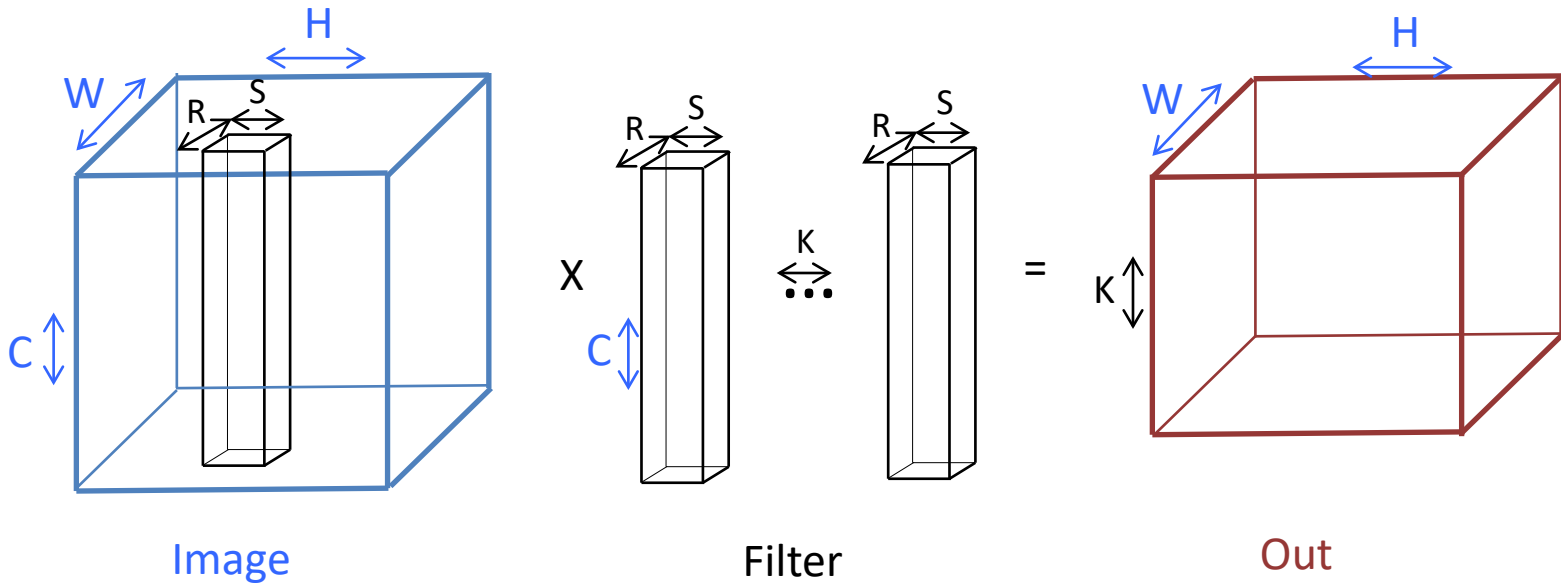


Image

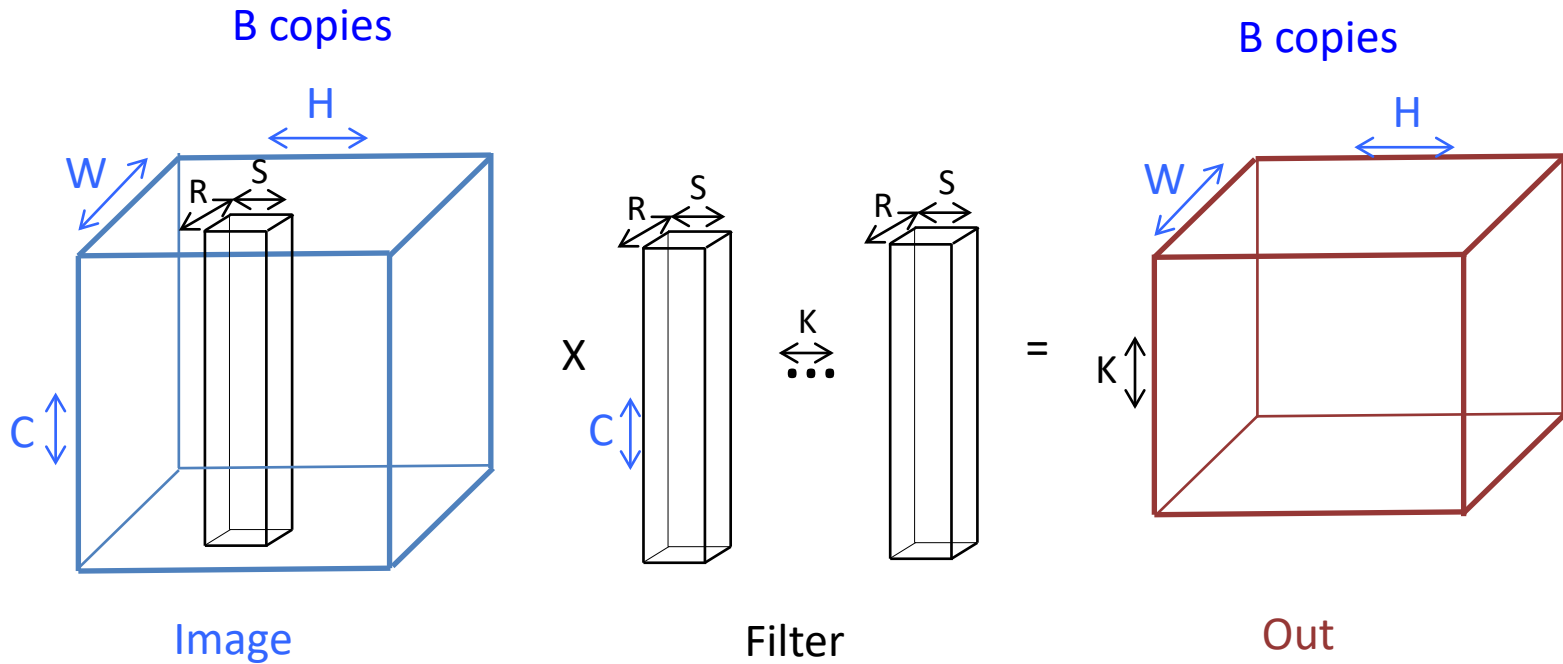


Filter

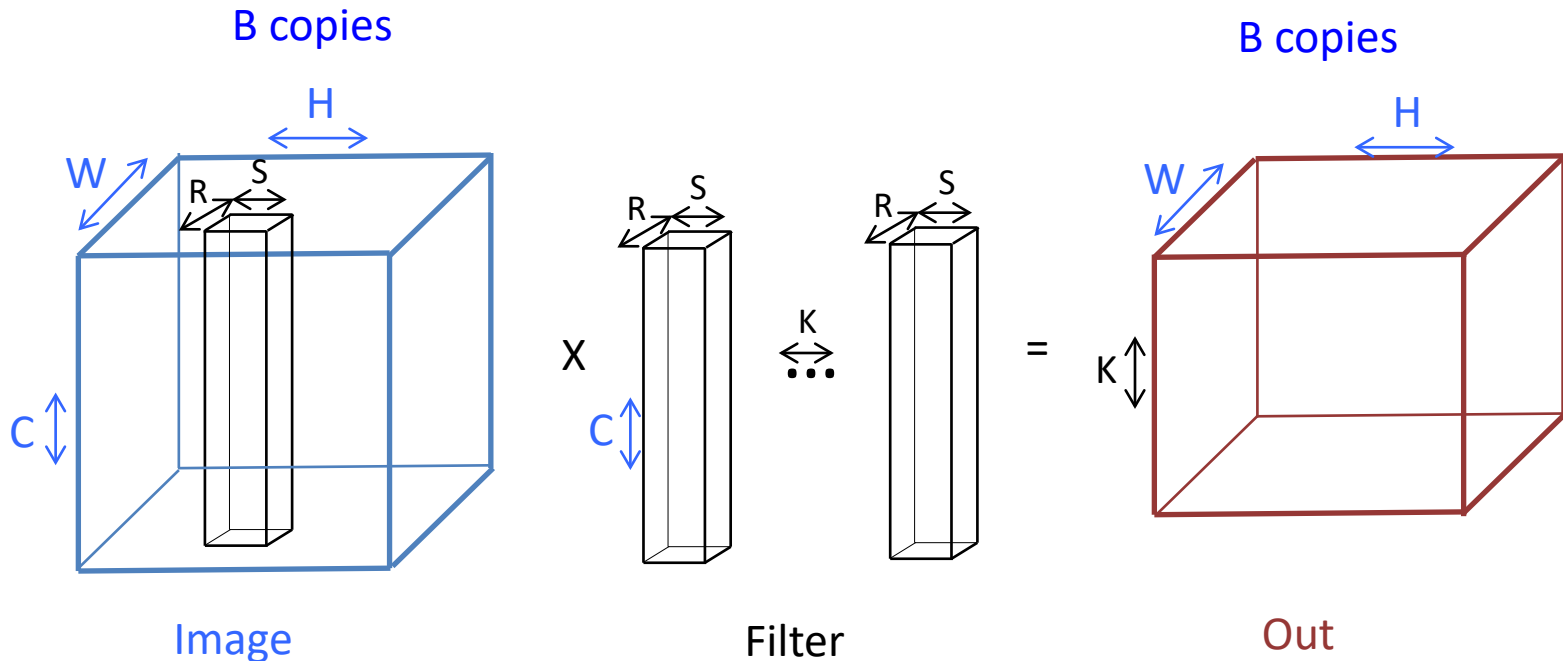
What CNNs compute



What CNNs compute



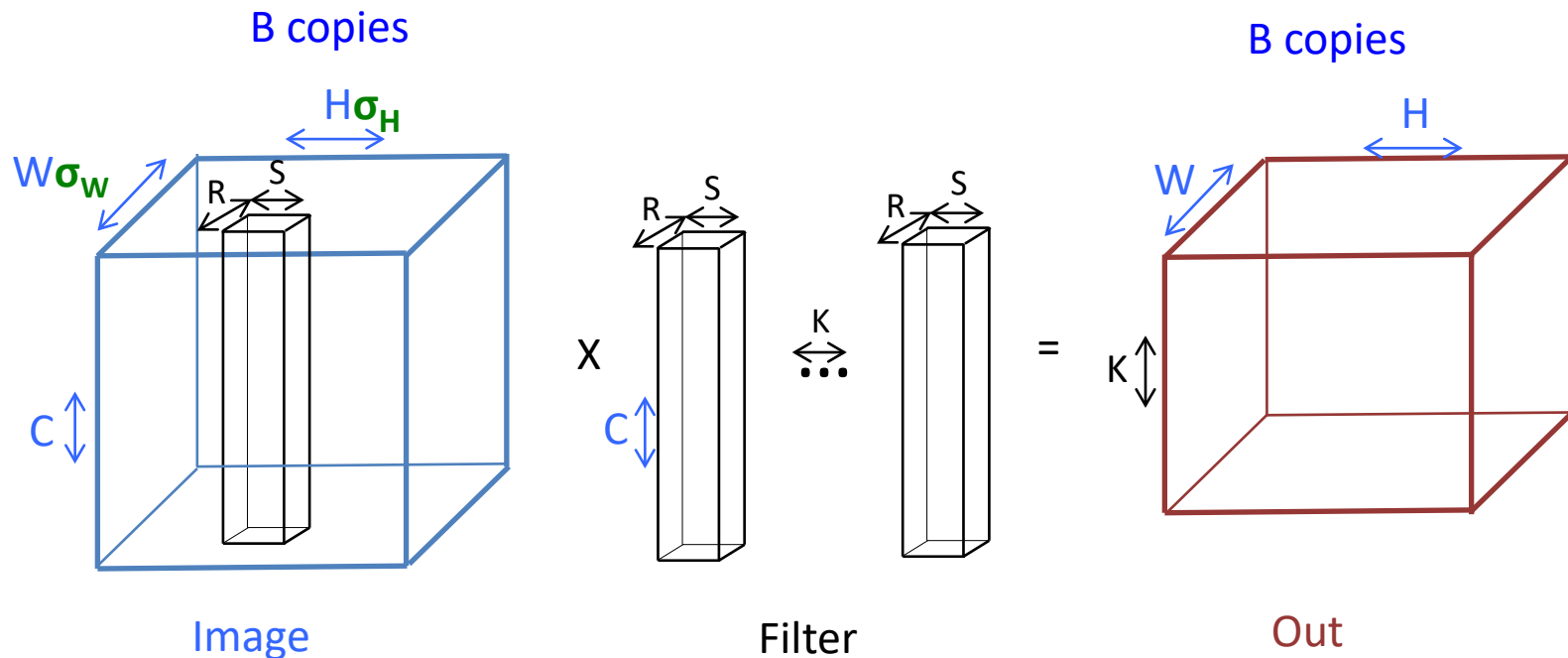
What CNNs compute



for $k=1:K$, for $h=1:H$, for $w=1:W$, for $r=1:R$,
for $s=1:S$, for $c=1:C$, for $b=1:B$

$$\text{Out}(k, h, w, b) += \text{Image}(r+w, s+h, c, b) * \text{Filter}(k, r, s, c)$$

What CNNs compute



for $k=1:K$, for $h=1:H$, for $w=1:W$, for $r=1:R$,
 for $s=1:S$, for $c=1:C$, for $b=1:B$

$$\text{Out}(k, h, w, b) += \text{Image}(r + \sigma_w w, s + \sigma_h h, c, b) * \text{Filter}(k, r, s, c)$$

Communication Lower Bound for CNNs

- Let $N = \#iterations = KHWRSCB$, $M = \text{cache size}$
- $\#words\ moved = \Omega(\max(\dots 5\ \text{terms})$
 - $BKHW$, ... size of Out
 - $\sigma_H \sigma_W BCWH$, ... size of Image
 - $CKRS$, ... size of Filter
 - N/M , ... lower bound for large loop bounds
 - $N/(M^{1/2} (RS/(\sigma_H \sigma_W))^{1/2})$... lower bound for small filters)
- Any one of 5 terms may be largest
- Bottommost bound beats matmul by factor $(RS/(\sigma_H \sigma_W))^{1/2}$
 - Applies in common case when data does not fit in cache, but one $R \times S$ filter does
 - Tile needed to attain N/M too big to fit in loop bounds
- Thm: Always attainable! (computer generated proof)
 - Beats im2col in data movement for various practical sizes
- Improved constants to appear in PASC'22
- Chen/Han/Wang (arxiv:1911.05662v3): HW accelerator

Outline

- Linear Algebra
 - Communication Lower Bounds for classical direct linear algebra
 - CA 2.5D Matmul
 - TSQR - Tall-Skinny QR
 - Iterative Methods for linear algebra
- Machine Learning
 - Training Neural Nets – “ImageNet training in minutes”
 - Convolutional Neural Nets
- **And Beyond**
 - **Extending communication lower bounds and optimal algorithms to general loop nests**

Communication lower bounds and optimal algorithms for general loop nests

- for $i = 1:n$, for $j=1:n$, for $k = 1:n$
 $C(i,j) = C(i,j) + A(i,k)*B(k,j)$
- #Words moved between main memory and cache of size $M = \Omega(n^3 / M^{1/2})$, attainable
- For $(i_1, i_2, \dots, i_k) \in S \subseteq \mathbb{Z}^k$, do something with
 - $A(i_1), B(i_2, i_3+i_4), C(i_1-i_2, i_2+3*i_3-5*i_4+2, \dots), \dots$
- Thm: #Words moved = $\Omega(|S| / M^{e_{HBL}})$
(Christ, D., Knight, Scanlon, Yelick)
 - HBL = Hölder / Brascamp / Lieb
 - Uses results by Christ, Tao, others
- Thm: There exists an optimal tiling that attains this lower bound (D., Rusciano)

What's left?

- Dealing with small loop bounds
 - Ex: Matvec special case of Matmul, not optimizable
 - Special cases: CNNs
 - Thm: If all subscripts like $(i), (i,j)$, etc, and $S =$ parallelepiped, \exists tighter, attainable lower bound $(D., Din_h)$
- Dealing with dependencies
 - Special cases: Linear algebra outside matmul, Floyd-Warshall, ...
- More realistic performance models than α, β, γ
 - Variable precision
 - Heterogeneous processors, accelerators, network topologies, differing costs of read and writes, ...
- Need to automate! (i.e. compilers)



Collaborators and Supporters

- **James Demmel, Kathy Yelick**, Vivek Bharadwaj, Grace Dinh, Tianyu Liang
- Peter Ahrens, Michael Anderson, Grey Ballard, Austin Benson, Erin Carson, Maryam Dehnavi, Aditya Devakonda, Michael Driscoll, David Eliahu, Andrew Gearhart, Evangelos Georganas, Mark Hoemmen, Shoaib Kamil, , Nicholas Knight, Penporn Koanantakool, Ben Lipshitz, Marghoob Mohiyuddin, Hong Diep Nguyen, Jason Riedy, Alex Rusciano, Oded Schwartz, Edgar Solomonik, Omer Spillinger, Yang You
- Abhinav Bhatele, Aydin Buluc, Michael Christ, Ioana Dumitriu, Kimon Fountoulakis, Armando Fox, David Gleich, Ming Gu, Jeff Hammond, Mike Heroux, Olga Holtz, Kurt Keutzer, Julien Langou, Xiaoye Li, Michael Mahoney, Devin Matthews, Tom Scanlon, Michelle Strout, Sam Williams, Hua Xiang, Zhao Zhang, Cho-Jui Hsieh,
- Jack Dongarra, Mark Gates, Jakub Kurzak, Dulceneia Becker, Ichitaro Yamazaki, ...
- Sivan Toledo, Alex Druinsky, Inon Peled, Greg Henry, Peter Tang,
- Laura Grigori, Sebastien Cayrols, Simplicie Donfack, Mathias Jacquelin, Amal Khabou, Sophie Moufawad, Mikolaj Szydlarski
- Members of SLICE, ADEPT, ASPIRE, BEBOP, ParLab, CACHE, EASI, FASTMath, MAGMA, PLASMA
- Thanks to DOE, NSF, UC Discovery, INRIA, Intel, Microsoft, Mathworks, National Instruments, NEC, Nokia, NVIDIA, Samsung, Oracle
- bebop.cs.berkeley.edu

For more details

- Bebop.cs.berkeley.edu
 - 155 page linear algebra survey in Acta Numerica (2014)
 - Book in progress (with Ballard, Carson, Grigori)
- CS267 – Berkeley’s Parallel Computing Course
 - Live broadcast in Spring 2021, now in 2022
 - www.cs.berkeley.edu/~demmel
 - All slides, video available
 - Prerecorded version broadcast since Spring 2013
 - www.xsede.org
 - Free supercomputer accounts to do homework
 - Free autograding of homework

Summary

Time to redesign all
linear algebra, machine learning, n-body, ...
algorithms and software, and compilers

Don't Communic...

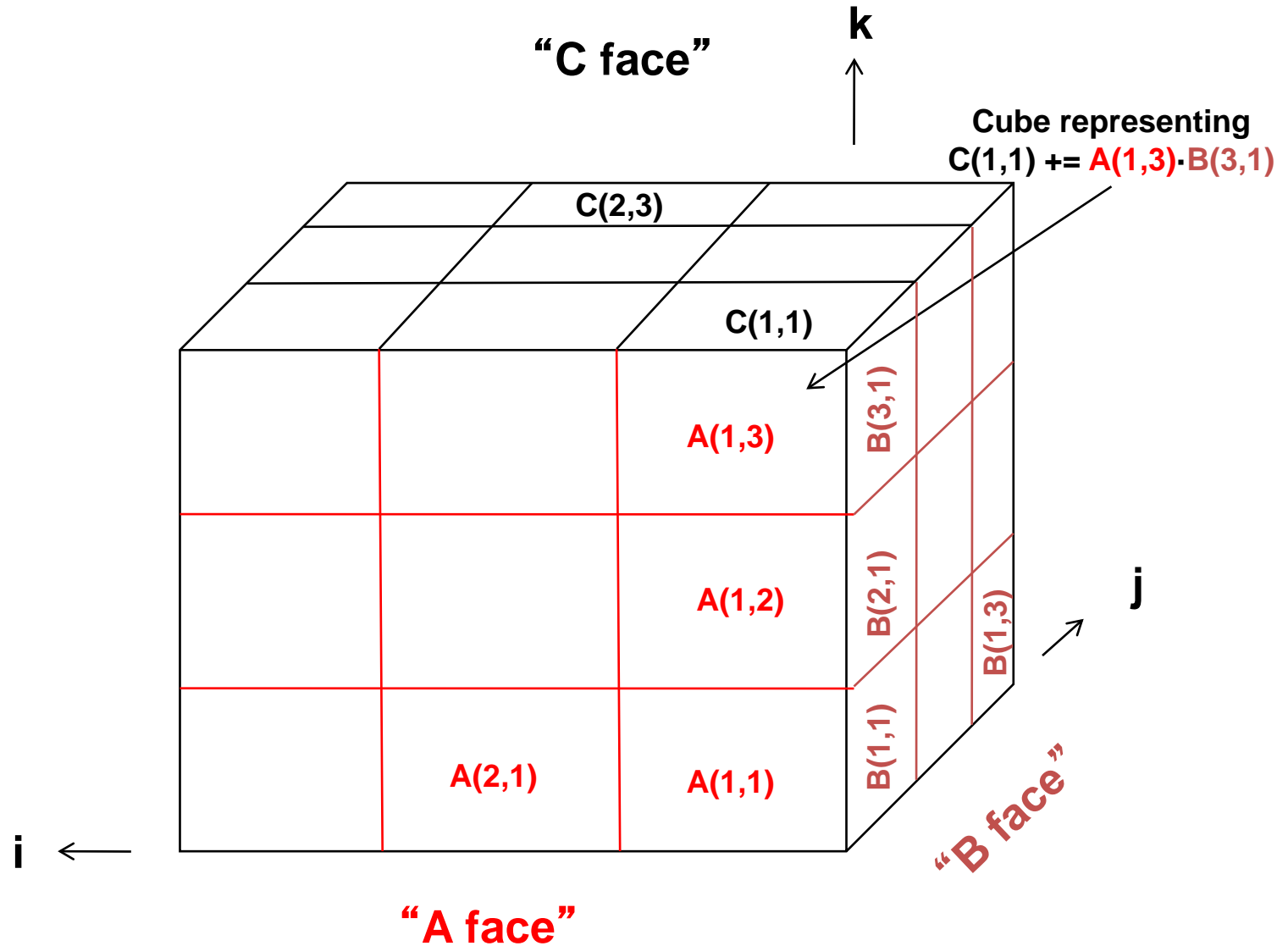
Backup slides



Proof of Communication Lower Bound on $C = A \cdot B$ (1/4)

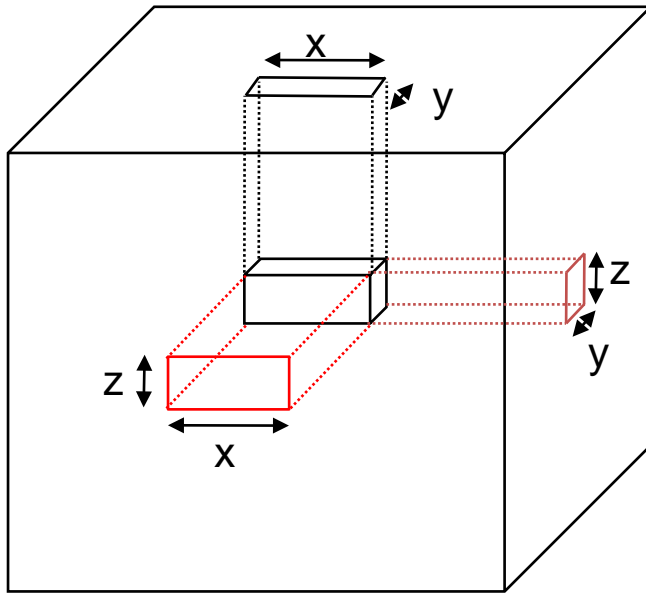
- Basic Counting Argument:
 - Only M entries of A , B and C are available in cache
 - Find an upper bound F on the number of different iterations $C(i,j) = C(i,j) + A(i,k) \cdot B(k,j)$ we can perform
 - Need to refill cache n^3/F times to complete algorithm
 - Need to read/write at least $M n^3/ F$ words to/from cache
- Represent iterations and data geometrically

Proof of Communication Lower Bound on $C = A \cdot B$ (2/4)

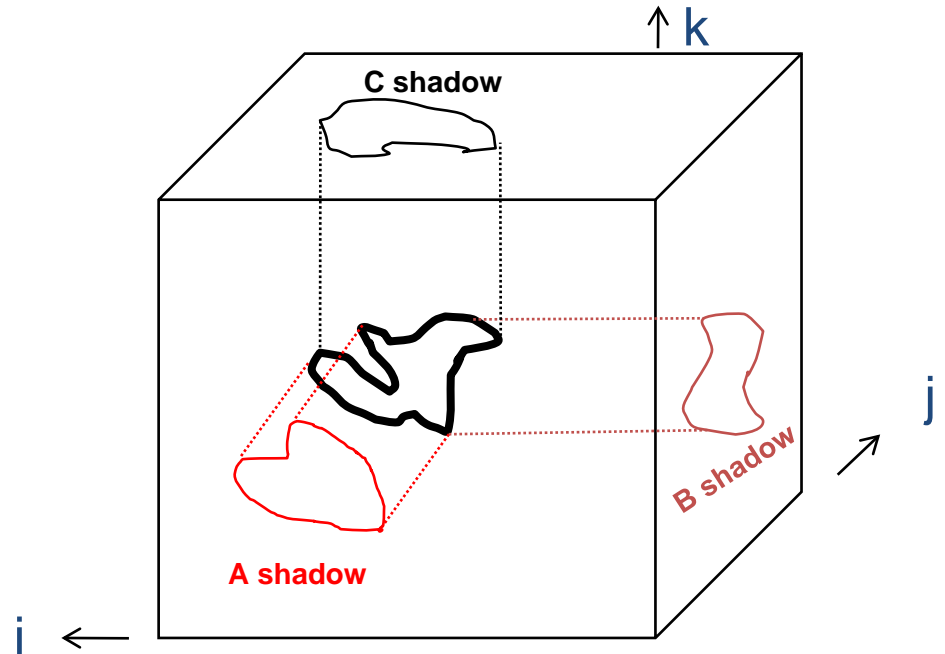


- If we have at most M “A squares”, “B squares”, and “C squares” on faces, how many cubes can we have?

Proof of Communication Lower Bound on $C = A \cdot B$ (3/4)



cubes in black box with
 side lengths x , y and z
 = Volume of black box
 = $x \cdot y \cdot z$
 = $(xz \cdot zy \cdot yx)^{1/2}$
 = $(\#A \square s \cdot \#B \square s \cdot \#C \square s)^{1/2}$



(i, k) is in **A shadow** if (i, j, k) in 3D set
 (j, k) is in **B shadow** if (i, j, k) in 3D set
 (i, j) is in **C shadow** if (i, j, k) in 3D set

Thm (Loomis & Whitney, 1949)

cubes in 3D set = Volume of 3D set
 $\leq (\text{area}(\mathbf{A shadow}) \cdot \text{area}(\mathbf{B shadow}) \cdot$
 $\text{area}(\mathbf{C shadow}))^{1/2}$



Proof of Communication Lower Bound on $C = A \cdot B$ (4/4)

- # loop iterations doable with M words of data = #cubes
 $\leq (\text{area}(A \text{ shadow}) \cdot \text{area}(B \text{ shadow}) \cdot \text{area}(C \text{ shadow}))^{1/2}$
 $\leq (M \cdot M \cdot M)^{1/2} = M^{3/2} = F$
- Need to read/write at least $M n^3 / F = \Omega(n^3 / M^{1/2}) = \Omega(\text{\#loop iterations} / M^{1/2})$ words to/from cache