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GENERALIZED CROSS-VALIDATION FOR TOTAL VARIATION IMAGE RESTORATION *

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In Memory of Gene Golub

Abstract. In this paper, we consider and study total variation (TV) image restoration. In literature, there are several regularization parameter selection methods for Tikhonov regularization problems (e.g., the discrepancy principle and the generalized cross-validation method). However, these selection methods have not been developed for TV regularization problems. The main aim of this paper is to develop a fast TV image restoration method with automatic selection of regularization parameters to restore blurred and noisy images. The method exploits the generalized cross-validation (GCV) technique to determine inexpensively how much regularization used in each restoration step. By updating these regularization parameters in the iterative procedure, the restored image can be obtained by choosing the one having the minimum GCV value. Our experimental results show that the quality of restored images by the proposed method with no prior knowledge of the original image is quite well. We will also demonstrate the proposed method is also very efficient.

Keywords: image restoration, generalized cross-validation, regularization parameters, total variation

1. Introduction. Digital image restoration and reconstruction play an important part in various areas of applied sciences such as medical and astronomical imaging, film restoration, image and video coding. In this paper, we focus on a common degradation model: an ideal image $\boldsymbol{f} \in \mathbb{R}^{n^2}$ is observed in the presence of a spatial-invariant blur matrix $\boldsymbol{H} \in \mathbb{R}^{n^2 \times n^2}$ and an additive zero-mean Gaussian white noise $\boldsymbol{n} \in \mathbb{R}^{n^2}$ of standard deviation σ. Thus the observed image $\boldsymbol{g} \in \mathbb{R}^{n^2}$ is obtained by:

$$\boldsymbol{g} = \boldsymbol{H}\boldsymbol{f} + \boldsymbol{n}. \tag{1.1}$$

Our aim is to determine $\boldsymbol{f}$ by knowing the blurring matrix $\boldsymbol{H}$ and the observed image $\boldsymbol{g}$, but without the knowledge of $\boldsymbol{f}$ and σ.

1.1. Total Variation Image Restoration. It is well-known that restoring an image $\boldsymbol{f}$ is a very ill-conditioned problem. A regularization method should be used in the image restoration process. The total variation (TV) regularization, proposed by Rudin, Osher and Fatemi [16], has become very popular for this purpose. The main advantage of the TV formulation is the ability to preserve edges in the image due to the piecewise smooth regularization property of the TV norm. A discrete version of the unconstrained TV deblurring problem [16] is given by

$$\min_{\boldsymbol{f}} \ \lambda \|\boldsymbol{H}\boldsymbol{f} - \boldsymbol{g}\|_2^2 + ||\boldsymbol{f}||_{TV}, \tag{1.2}$$

where $\|\cdot\|_2$ denotes the Euclidean norm, λ is a positive regularization parameter which measures the trade off between a good fit and a regularized solution, and $\|\cdot\|_{TV}$ is

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the discrete TV regularization term. The discrete gradient operator $\nabla : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ is defined by

$$(\nabla \boldsymbol{f})_{j,k} = ((\nabla \boldsymbol{f})^x_{j,k}, (\nabla \boldsymbol{f})^y_{j,k})$$

with

$$(\nabla \boldsymbol{f})^x_{j,k} = \begin{cases} \boldsymbol{f}_{j+1,k} - \boldsymbol{f}_{j,k} & \text{if } j < n, \\ 0 & \text{if } j = n, \end{cases} \qquad (\nabla \boldsymbol{f})^y_{j,k} = \begin{cases} \boldsymbol{f}_{j,k+1} - \boldsymbol{f}_{j,k} & \text{if } k < n, \\ 0 & \text{if } k = n \end{cases}$$

for $j, k = 1, \ldots, n$. Here $\boldsymbol{f}_{j,k}$ refers to the $(jn + k)$th entry of the vector $\boldsymbol{f}$ (it is the (j, k)th pixel location of the image). The discrete total variation of $\boldsymbol{f}$ is defined by

$$\|\boldsymbol{f}\|_{TV} := \sum_{1 \le j,k \le n} |(\nabla \boldsymbol{f})_{j,k}|_2 = \sum_{1 \le j,k \le n} \sqrt{|(\nabla \boldsymbol{f})^x_{j,k}|^2 + |(\nabla \boldsymbol{f})^y_{j,k}|^2}.$$

Here $|\cdot|_2$ is the Euclidean norm in $\mathbb{R}^2$.

A number of numerical methods have been proposed for solving (1.2), see for instance [3] for references therein. Recently, Huang et al. [11] proposed and developed a fast total variation minimization method for image restoration. The proposed unconstrained TV deblurring problem is given by

$$\min_{\boldsymbol{f},\boldsymbol{u}} \mathcal{J}(\boldsymbol{f}, \boldsymbol{u}) \equiv \min_{\boldsymbol{f},\boldsymbol{u}} \alpha_1 \|\boldsymbol{H}\boldsymbol{f} - \boldsymbol{g}\|_2^2 + \alpha_2 ||\boldsymbol{f} - \boldsymbol{u}||_2^2 + ||\boldsymbol{u}||_{TV}, \tag{1.3}$$

where α_1 and α_2 are positive regularization parameters. The main difference between (1.2) and (1.3) is that a fitting term $||\boldsymbol{f} - \boldsymbol{u}||_2^2$ is added in the new minimization method. We can interpret the total variation minimization scheme to denoise the deblurred image $\boldsymbol{f}$. Here α_2 measures the trade off between a deblurred image $\boldsymbol{f}$ and a denoised image $\boldsymbol{u}$, and α_1 measures the amount of fitting to the observed image $\boldsymbol{g}$ by $\boldsymbol{f}$. An alternating minimization algorithm is employed to solve the proposed total variation minimization problem. Experimental results in [11] have shown that the quality of restored images by the proposed method are competitive with those restored by the existing total variational restoration methods.

Recently, Wang et al. [17] also proposed and developed the alternating minimization algorithm for deblurring and denoising jointly by solving a TV regularization problem. Their algorithm is derived from the well-known variable-splitting and penalty techniques in optimization. The idea is that at each pixel an auxiliary variable is introduced to transfer its gradient out of non-differentiable term. Extensive numerical results show that their algorithm performs favorably in comparison to several state-of-the-art algorithms.

1.2. Regularization Parameters. It is well known that blurring matrices are in general ill-conditioned and deblurring algorithms will be extremely sensitive to noise, see for instance Gonzalez and Woods [6]. The ill-conditioning of the blurring matrices stems from the wide range of magnitude of their eigenvalues, see Engl et al. [4]. Therefore, excess amplification of the noise at small eigenvalues can occur. The method of regularization is used to achieve stability for image restoration problems. In (1.3), stability is attained by using regularization which restricts the set of admissible solutions.

There are several possible strategies for choice of regularization parameters, e.g., the discrepancy principle [4, 7] and the generalized cross-validation method [5]. The

discrepancy principle is an a-posterior strategy for choosing a regularization parameter as a function of an error level (the input error level must be known). The generalized cross-validation method can be applied to determine regularization parameters when the input error level is not known, but we assume that the error is a Gaussian white noise. Another practical method for choosing a regularization parameter when data are noisy is the L-curve criterion [8]. The method is based on the plot of the norm of the regularized solution versus the norm of the corresponding data fitting residual. It is used to choose a regularization parameter related to the characteristic L-shaped "corner" of the plot. However, these regularization parameter selection methods have not been developed for total variation image restoration problems. For instance, the TV formulation in (1.2) or (1.3) is nonlinear, therefore the generalized cross-validation evaluation formula cannot be derived explicitly.

1.3. Outline. The main aim of this paper is to develop a fast TV image restoration method with automatic selection of regularization parameters in (1.3) to restore blurred and noisy images. The method exploits the generalized cross-validation technique to determine inexpensively how much regularization (α_1) used in each restoration step. At each iteration, we also make the regularization parameter α_2 become larger and larger. We note that when α_2 is sufficiently large, $\boldsymbol{f}$ and $\boldsymbol{u}$ are close to each other and therefore the total variation model in (1.3) is about the same as that in (1.2). By updating these regularization parameters in the iterative procedure, the restored image can be obtained by choosing the one having the minimum generalized cross-validation value. Our experimental results show that the quality of restored images by the proposed method with no prior knowledge of the original image is quite well. We will also demonstrate the proposed method is also very efficient.

The outline of this paper is as follows. In Section 2, the fast total variation based image restoration method is presented. In Section 3, we review the generalized cross validation technique and present the proposed algorithm. In Section 4, numerical examples are given to demonstrate the effectiveness of the proposed method. Finally, concluding remarks are given in Section 5.

2. The Fast Total Variation Image Restoration Method. An alternating minimization algorithm [11] is used to solve (1.3). Starting from an initial guess $\boldsymbol{u}^{(0)}$, this method computes a sequence of iterates

$$\boldsymbol{f}^{(1)}, \boldsymbol{u}^{(1)}, \boldsymbol{f}^{(2)}, \boldsymbol{u}^{(2)}, \ldots, \boldsymbol{f}^{(i)}, \boldsymbol{u}^{(i)}, \ldots.$$

such that

$$\begin{cases} \boldsymbol{f}^{(i)} & = & \operatorname{argmin}_{\boldsymbol{f}} \ \alpha_1 \|\boldsymbol{H}\boldsymbol{f} - \boldsymbol{g}\|_2^2 + \alpha_2 \|\boldsymbol{f} - \boldsymbol{u}^{(i-1)}\|_2^2 \\ \boldsymbol{u}^{(i)} & = & \operatorname{argmin}_{\boldsymbol{u}} \ \alpha_2 \|\boldsymbol{f}^{(i)} - \boldsymbol{u}\|_2^2 + ||\boldsymbol{u}||_{TV} \end{cases}$$

for $i = 1, 2, \cdots$.

The first step of the method is to perform the deblurring. The minimizer of the optimization problem:

$$\min_{\boldsymbol{f}} \ \|\boldsymbol{H}\boldsymbol{f} - \boldsymbol{g}\|_2^2 + \frac{\alpha_2}{\alpha_1} \|\boldsymbol{f} - \boldsymbol{u}^{(i-1)}\|_2^2 \tag{2.1}$$

is equivalent to solving a linear system:

$$(\boldsymbol{H}^t\boldsymbol{H} + \alpha \boldsymbol{I})\boldsymbol{f} = \boldsymbol{H}^t\boldsymbol{g} + \alpha \boldsymbol{u}^{(i-1)}, \quad \text{where} \quad \alpha = \frac{\alpha_2}{\alpha_1} \tag{2.2}$$

Such system can be solved very efficiently by using fast transform-based methods and preconditioning techniques for Toeplitz-like matrices, see [12].

The second step of the method is to apply an exact TV denoising scheme to the image generated by the previous deblurring step. The minimizer of the optimization problem

$$\alpha_2 \|\boldsymbol{f}^{(i)} - \boldsymbol{u}\|_2^2 + ||\boldsymbol{u}||_{TV}$$

can be solved by many TV denoising methods like Chambolle's projection algorithm [1], semismooth Newton's method [13], multilevel optimization method [3] and graph-based optimization method [2]. In this paper, we employ the Chambolle projection algorithm in the denoising step.

3. The Proposed Method. In this section, we use the generalized cross-validation technique to determine inexpensively how much (α_1) regularization used in each restoration step. At each iteration, we also make the regularization parameter α_2 become larger and larger. By updating these regularization parameters in the iterative procedure, the restored image can be obtained by choosing the one having the minimum generalized cross-validation value. We first review the principle of the generalized cross validation method for selection of a regularization parameter.

3.1. The Generalized Cross-Validation Method. Generalized cross-validation (GCV) [5] is a technique that estimates a regularization parameter directly without requiring an estimate of the noise variance. It is based on the concept of prediction errors. The basic idea is to take kth observation out of all observed data, then to use the remaining observations to predict the kth observation. If the regularization parameter is a good choice, the kth component of the fitted data should be a good predictor for kth observation on average.

For the minimization problem in (2.1), for each $k = 1, \cdots, n^2$, let $\boldsymbol{v}_\alpha^{(k)}$ be the vector that minimizes the error measure:

$$\sum_{j=1, j\neq k}^{n^2} \left([\boldsymbol{g}]_j - [\boldsymbol{H}(\boldsymbol{v} + \boldsymbol{u}^{(i-1)})]_j \right)^2 \text{ or } \sum_{j=1, j\neq k}^{n^2} \left([\boldsymbol{g} - \boldsymbol{H}\boldsymbol{u}^{(i-1)}]_j - [\boldsymbol{H}\boldsymbol{v}]_j \right)^2, \tag{3.1}$$

where $\boldsymbol{v} = \boldsymbol{f} - \boldsymbol{u}^{(i-1)}$, $[\boldsymbol{x}]_j$ is the jth element of $\boldsymbol{x}$ and $[\boldsymbol{H}\boldsymbol{x}]_j$ is the jth element of $\boldsymbol{H}\boldsymbol{x}$.

If α_1 is such that $\boldsymbol{v}_\alpha^{(k)}$ is a good estimate of $\boldsymbol{v}$, then $[\boldsymbol{H}\boldsymbol{v}_\alpha^{(k)}]_k$ should be a good approximation of $[\boldsymbol{g} - \boldsymbol{H}\boldsymbol{u}^{(i-1)}]_k$ on average. For a given α_1, the average squared error between the predicted value $[\boldsymbol{H}\boldsymbol{v}_\alpha^{(k)}]_k$ and the observed value $[\boldsymbol{g} - \boldsymbol{H}\boldsymbol{u}^{(i-1)}]_k$ is given by

$$\frac{1}{n^2} \sum_{k=1}^{n^2} \left([\boldsymbol{g} - \boldsymbol{H}\boldsymbol{u}^{(i-1)}]_k - [\boldsymbol{H}\boldsymbol{v}_\alpha^{(k)}]_k \right)^2.$$

The generalized cross-validation is a weighted version of the above error:

$$\tau(\alpha_1) \equiv \frac{1}{n^2} \sum_{k=1}^{n^2} \left([\boldsymbol{g} - \boldsymbol{H}\boldsymbol{u}^{(i-1)}]_k - [\boldsymbol{H}\boldsymbol{v}_\alpha^{(k)}]_k \right)^2 \left[\frac{1 - m_{k,k}(\alpha_1)}{1 - \frac{1}{n^2} \sum_{j=1}^{n^2} m_{j,j}(\alpha)} \right],$$

where $m_{j,j}(\alpha)$ is the (j,j)th entry of the so-called influence matrix

$$\boldsymbol{M}(\alpha) = \boldsymbol{H}(\boldsymbol{H}^T\boldsymbol{H} + \alpha\boldsymbol{I})^{-1}\boldsymbol{H}^T.$$

In [5], Golub, Heath and Wahba have shown that $\tau(\alpha)$ can be written as

$$\tau(\alpha) = n^2 \frac{\|(\boldsymbol{I} - \boldsymbol{M}(\alpha))(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{u}^{(i-1)})\|_2^2}{\mathrm{trace}(\boldsymbol{I} - \boldsymbol{M}(\alpha))^2}.$$

The optimal regularization parameter is chosen to be the α that minimizes $\tau(\alpha)$. Since $\tau(\alpha)$ is a nonlinear function, the minimizer usually cannot be determined analytically. However, if $\boldsymbol{H}$ is a blurring matrix generated by a symmetric point spread function, $\boldsymbol{H}$ can be diagonalized by a fast transform matrix [12], then we can rewrite $\tau(\alpha)$ as

$$\tau(\alpha) = \frac{n^2 \sum_{j=1}^{n^2} \left\{ \mathrm{diag}\left(\frac{\alpha}{\lambda_j^2 + \alpha}\right) \boldsymbol{\Phi}\left[\boldsymbol{g} - \boldsymbol{H}\boldsymbol{u}^{(i-1)}\right] \right\}_j^2}{\left(\sum_{j=1}^{n^2} \frac{\alpha}{\lambda_j^2 + \alpha}\right)^2} \tag{3.2}$$

where λ_j is the jth eigenvalue of $\boldsymbol{H}$ and $\boldsymbol{\Phi}$ is the corresponding discrete transform matrix (e.g., discrete cosine transform matrix or discrete Fourier matrix). We recall that λ_j can be obtained by taking the fast transform of the first column of $\boldsymbol{H}$ [12].

When $\boldsymbol{H}$ cannot be diagonalized by the fast transform matrix, we can use the Hutchinson estimator to compute an approximation of $\tau(\alpha)$. It has been shown in [10] that the term $\mathrm{trace}(\boldsymbol{I} - \boldsymbol{M}(\alpha))$ can be rewritten as follows: $\alpha\mathrm{trace}((\boldsymbol{H}^T\boldsymbol{H} + \alpha\boldsymbol{I})^{-1})$.

THEOREM 3.1. *Let $\boldsymbol{H}$ be a matrix of full rank. Let X be the discrete random variable which takes the values +1 and -1 each with probability 1/2, and let $\boldsymbol{x}$ with independent entries from X. Then $\alpha\boldsymbol{x}^T(\boldsymbol{H}^T\boldsymbol{H} + \alpha\boldsymbol{I})^{-1}\boldsymbol{x}$ is an unbiased estimator of $\alpha\mathrm{trace}((\boldsymbol{H}^T\boldsymbol{H} + \alpha\boldsymbol{I})^{-1})$.*

We consider $\boldsymbol{x}$ denotes a random vector for the Hutchinson estimator. Therefore, we can define

$$\tilde{\tau}(\alpha) \equiv n^2 \frac{\|(\boldsymbol{I} - \boldsymbol{M}(\alpha))(\boldsymbol{g} - \boldsymbol{H}\boldsymbol{u}^{(i-1)})\|_2^2}{\alpha^2(\boldsymbol{x}^T(\boldsymbol{H}^T\boldsymbol{H} + \alpha\boldsymbol{I})^{-1}\boldsymbol{x})^2}. \tag{3.3}$$

We employ $\tilde{\tau}(\alpha)$ to approximate $\tau(\alpha)$ and estimate a suitable value of the regularization parameter used in the image restoration. We note that the evaluation of $\tau(\alpha)$ involves the solution of linear systems with the matrix $\boldsymbol{H}^T\boldsymbol{H} + \alpha\boldsymbol{I}$, which can be solved by the conjugate gradient method. Convergence can be improved using preconditioning techniques. Transform-based preconditioning techniques have been proved to be very successful [12].

We remark that numerical approximation techniques have been proposed and developed to further reduce the computational complexity of determination of the optimal regularization parameter for the minimization of $\tau(\alpha)$ or $\tilde{\tau}(\alpha)$, see [9]. In [10], the approach employs Gauss quadrature to compute lower and upper bounds on the generalized cross-validation function. This method requires matrix-vector multiplications and the factorization of large matrices can be avoided. Some recent methods are proposed and studied in [14, 15].

3.2. The Algorithm. The main idea of the algorithm is to employ the generalized cross-validation technique to determine the regularization parameter α_1 in the deblurring step. At each iteration, the parameter α_2 becomes larger and larger until we find the one giving the minimum generalized cross-validation value in the iterative method. We note that when α_2 is sufficiently large, $\boldsymbol{f}$ and $\boldsymbol{u}$ are close to each other and therefore the total variation model in (1.3) is about the same as that in (1.2), see [11, 17]. In [17], Wang et al. studied a fast total variation deconvolution algorithm using a continuation scheme of a regularization parameter. Here we adapt their idea and employ the continuation scheme in the proposed algorithm. The description of the proposed algorithm is given as follows:

GCV-based TV Image Restoration Algorithm (GTV):

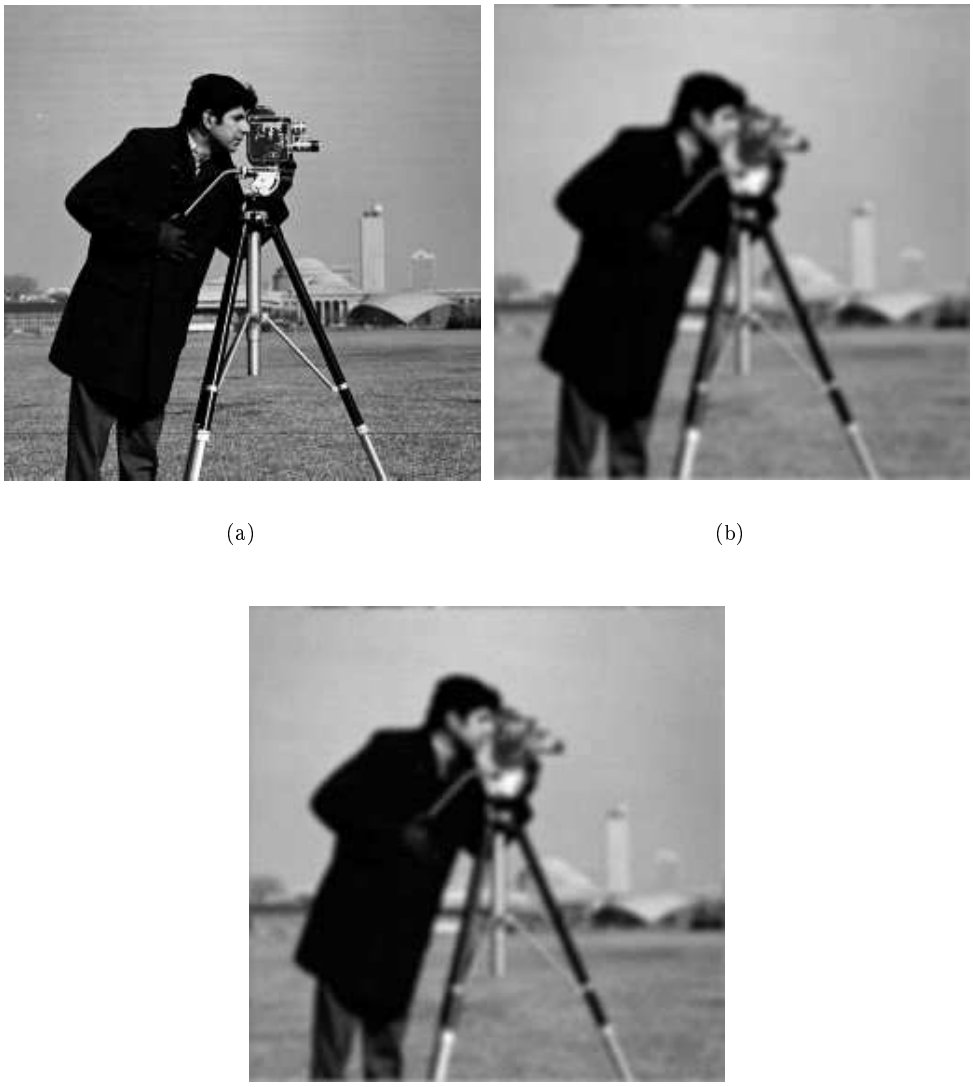
1. Set $k_{\max}$ and θ (the growth factor of α_2)
2. Set $k = 0$ and initialize $\boldsymbol{f}_k$ and $\alpha_2^{(k)}$
3. Solve $\boldsymbol{u}_k = \operatorname{argmin}_{\boldsymbol{u}}\{\alpha_2^{(k)}\|\boldsymbol{f}_k - \boldsymbol{u}\|_2^2 + \|\boldsymbol{u}\|_{TV}\}$
   While $k <= k_{\max}$ do
4. Set $k = k + 1$ and estimate $\alpha^{(k)} = \operatorname{argmin}_\alpha \tau(\alpha)$ in (3.2) or $\alpha^{(k)} = \operatorname{argmin}_\alpha \tilde{\tau}(\alpha)$ in (3.3) [this is referred to GCV-Hutchinson-based TV Image Restoration Algorithm (GHTV)]
5. Solve $\boldsymbol{f}_k = \operatorname{argmin}_f\{\|\boldsymbol{H}\boldsymbol{f} - \boldsymbol{g}\|_2^2 + \alpha^{(k)}\|\boldsymbol{f} - \boldsymbol{u}^{(k-1)}\|_2^2\}$
6. Set $\alpha_2^{(k)} = \alpha_2^{(k-1)} \cdot \theta$
7. Solve $\boldsymbol{u}_k = \operatorname{argmin}_{\boldsymbol{u}}\{\alpha_2^{(k)}\|\boldsymbol{f}_k - \boldsymbol{u}\|_2^2 + \|\boldsymbol{u}\|_{TV}\}$
   End while
8. Output the restored image $\boldsymbol{u}$ with the minimum generalized cross-validation value in Step 4

The solution in Step 5 can be determined by solving a linear system in the deblurring task. The solutions in Steps 3 and 7 can be determined by using the Chambolle projection algorithm in the denoising task. The evaluation of $\tau(\alpha)$ can be computed straightforwardly. The evaluation of $\tilde{\tau}(\alpha)$ can also be computed by using the Hutchinson estimator and then solving the corresponding linear system. We expect the overall algorithm should be very efficient. In the next section, we will demonstrate the performance of the GTV and GHTV algorithms.

4. Experimental Results. In this section, we illustrate the performance of the GTV and GHTV algorithms for image restoration problems. Signal to Noise ratio (SNR), BSNR and relative error are used to measure the quality of the restored images. They are defined as follows:

$$\mathrm{SNR} = 20\log_{10}\left(\frac{\|\boldsymbol{u}\|_2}{\|\boldsymbol{u}_c - \boldsymbol{u}\|_2}\right), \quad \mathrm{BSNR} = 20\log_{10}\left(\frac{\|\boldsymbol{g}\|_2}{\|\boldsymbol{n}\|_2}\right), \quad \mathrm{ReErr} = \frac{||\boldsymbol{u}_c - \boldsymbol{u}||_2^2}{||\boldsymbol{u}||_2^2}$$

where $\boldsymbol{u}$, $\boldsymbol{g}$, $\boldsymbol{u}_c$ and $\boldsymbol{n}$ are the original image, the observed image, recovered image and the noise vector, respectively. Three original images are used to test the proposed algorithms. They are shown in Figures 4.1(a)-4.3(a). The "Cameraman" image is blurred by a Gaussian PSF with radius of 3 and variance of 2; the "Barbara" image are blurred by out-of-focus PSFs with radii of 3 and 7 respectively, and the "CarNo" image are blurred by separable PSFs generated by horizontal motions with 9 and 15 pixels respectively. Then a Gaussian noise is added to each blurred image, the resulting observed images are shown in Figures 4.1-4.3.



(a) (b)

(c)

FIG. 4.1. *(a) The original image: "Cameraman"; the observed images: (b) Gaussian PSF of radius = 3 and BSNR = 40dB; (c) Gaussian PSF of radius = 3 and BSNR = 30dB.*

In Tables 4.1-4.3, we summarize the results for different algorithms. The GTV algorithm is used as the blurring matrix can be diagonalized by the discrete transform matrix, see Section 3.1. The GHTV algorithm is used as we consider the blurring matrix cannot be diagonalized by the discrete transform matrix, and we employ the Hutchinson estimator in the generalized cross validation results. In Tables 4.1-4.2, we show the restoration results by using the GTV and GHTV algorithms. We see from the tables that the performance of GTV is about the same as that of GHTV in terms of SNR, relative error and computational time.

(a) (b)

(c)

FIG. 4.2. *(a) The original image: "Barbara"; the observed images: (b) out-of-focus PSF of radius = 3 and BSNR = 40dB; (c) out-of-focus PSF of radius = 7 and BSNR = 30dB.*

In Table 4.3, we test different values of α_1 and α_2 and select them by maximizing the SNR of the restored image with respect to the original image. The number of tested values of α_1 and α_2 is nine thousands in total for the selection of α_1 and α_2 in the image restoration model. It is clear from Tables 4.1-4.3 that both SNRs and the relative errors of the restored images are better than those by the proposed algorithms. As the original image is assumed to be known in the selection of regularization parameters, it is not practical in the image restoration setting. Here the purpose is to set up benchmark results for comparison. We also see in Table 4.3 that the evaluation

(a) (b)

(c)

FIG. 4.3. *(a) The original image: "CarNo"; the observed images: (b) motion PSF of length = 9 and BSNR = 40dB; (c) motion PSF of length = 15 and BSNR = 30dB.*

of different values of α_1 and α_2 in the selection of regularization parameters takes a a lot of computational time even the original image is used. In Figures 4.4-4.9, we show the restored images of using different methods. We see from the figures that the visual quality of the restored images by the GTV and GHTV algorithms are quite acceptable compared with those restored images by the "Benchmark" method. According to these restoration results, the performance of the proposed algorithms is quite competitive.

Next we report the GCV values of each iteration of the proposed image restoration

Image	Blur/BSNR	SNR	Relative Errors	Time (seconds)
Cameraman	Gaussian(3)/40	21.67	0.0825	13.47
Cameraman	Gaussian(3)/30	19.57	0.1051	12.92
Barbara	Out-of-focus(3)/40	23.89	0.0639	12.77
Barbara	Out-of-focus(7)/30	19.68	0.1038	13.17
CarNo	Motion(9)/40	26.92	0.0451	19.39
CarNo	Motion(15)/30	22.14	0.0782	20.53

TABLE 4.1

The restoration results using the GTV algorithms.

Image	Blur/BSNR	SNR	Relative Errors	Time (seconds)
Cameraman	Gaussian(3)/40	21.56	0.0836	13.63
Cameraman	Gaussian(3)/30	19.59	0.1049	14.05
Barbara	Out-of-focus(3)/40	23.77	0.0648	13.70
Barbara	Out-of-focus(7)/30	19.69	0.1036	14.73
CarNo	Motion(9)/40	26.92	0.0451	20.05
CarNo	Motion(15)/30	22.15	0.0780	21.58

TABLE 4.2

The restoration results using the GHTV algorithms.

Image	Blur/BSNR	SNR	Relative Errors	Time (seconds)
Cameraman	Gaussian(3)/40	23.01	0.0707	11.7×10^4
Cameraman	Gaussian(3)/30	20.64	0.0929	9.0×10^4
Barbara	Out-of-focus(3)/40	25.74	0.0517	11.4×10^4
Barbara	Out-of-focus(7)/30	21.28	0.0863	11.8×10^4
CarNo	Motion(9)/40	27.54	0.0420	13.5×10^4
CarNo	Motion(15)/30	22.86	0.0719	26.1×10^4

TABLE 4.3

The benchmark restoration results.

algorithms. We use the case for the restoration of a blurred and noisy "CarNo" image in Figure 4.3(b) as an example. In Tables 4.1 and 4.2, we set the maximum number $k_{\max}$ of iterations to be 50 and the growth factor θ to be 1.2. We show the GCV values of each iteration and their corresponding SNRs and relative errors of the restored images in Figures 4.10 and 4.11. We find from the figures that when the GCV values decrease, the corresponding SNRs increase and relative errors decrease in general for both GTV and GHTV algorithms. These results indicate that the evaluation of GCV values in the proposed algorithms enable us to perform image restoration properly. More precisely, when we use the minimum GCV value in the curve, the corresponding restored image has quite high SNR and small relative error. In the proposed algorithm, the restored image is continuously improved with respect to the iterations, and the regularization parameter α (or α_1) is also updated according to the generalized cross-validation technique and the iterate of the image solution. Moreover, we observe that there are some oscillations around the minimum GCV value in the curve. We guess that the regularization parameter α_2 is increasing with respect to the iterations, therefore the GCV values around the minimum point will be quite sensitive to the change of α_2. We remark that the other image restoration examples in Figures 4.4-4.9 also have the similar phenomenon.

Finally, we test the effect of the growth factor θ. In Figures 4.12-4.13, we show the GCV values of each iteration and their corresponding SNRs and relative errors of the restored images for $\theta = 1.1$ and $\theta = 1.4$ respectively. By comparing with the curves in Figure 4.10 ($\theta = 1.2$), we see in these three figures that we can use the minimum GCV value to obtain high quality image restoration results. Indeed, their corresponding SNRs and relative errors in these three figures are about the same. Because θ is small (large), the number of iterations required for obtaining the minimum GCV value is more (less). According to these experimental results, we find that it is quite safe to set the growth factor to be 1.2 in the calculation where the image restoration results are obtained in Tables 4.1 and 4.2.

5. Concluding Remarks. In this paper, we presented an effective and efficient method to solve the regularization problem in the fast total variation minimization method for image restoration. In the alternating minimization algorithm, we have employed the generalized cross-validation method for deblurring problem. Our experimental results show that the quality of restored images by the proposed method with no prior knowledge is quite well, and the computational time is also very small.

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(a) (b)

(c)

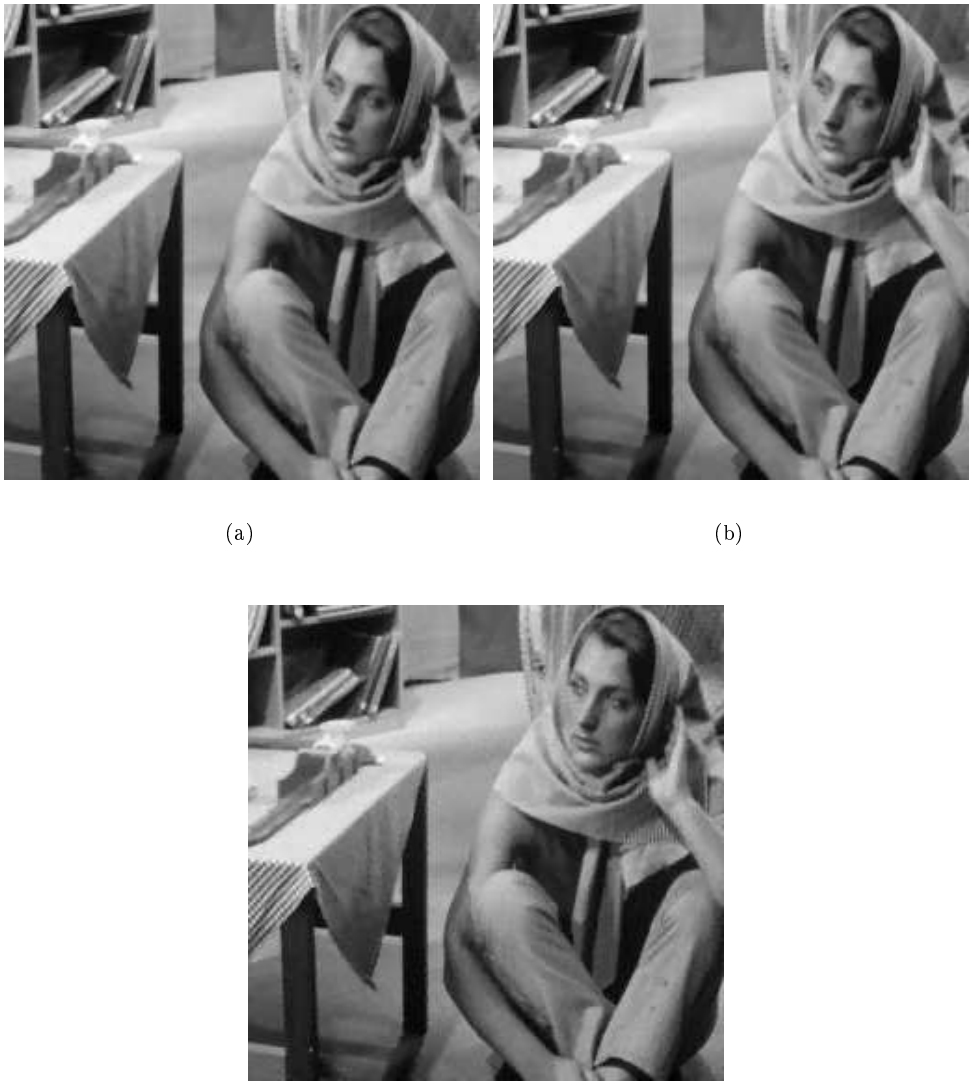
FIG. 4.4. *The restored images for Figure 4.1(b) by different methods: (a) the GTV algorithm; (b) the GHTV algorithm; (c) the "Benchmark" method.*

(a)

(b)

(c)

FIG. 4.5. *The restored images for Figure 4.1(c) by different methods: (a) the GTV algorithm; (b) the GHTV algorithm; (c) the "Benchmark" method.*



(a) (b)

(c)

FIG. 4.6. *The restored images for Figure 4.2(b) by different methods: (a) the GTV algorithm; (b) the GHTV algorithm; (c) the "Benchmark" method.*

(a)

(b)

(c)

FIG. 4.7. *The restored images for Figure 4.2(c) by different methods: (a) the GTV algorithm; (b) the GHTV algorithm; (c) the "Benchmark" method.*

(a)

(b)

(c)

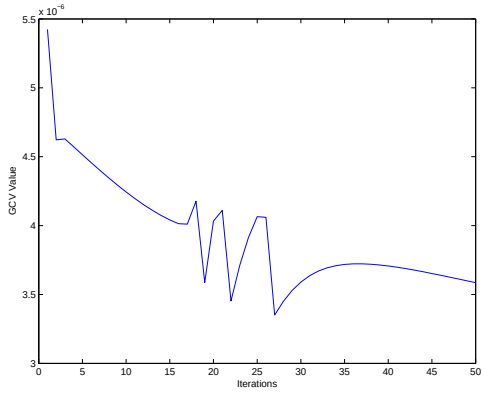
FIG. 4.8. *The restored images for Figure 4.3(b) by different methods: (a) the GTV algorithm; (b) the GHTV algorithm; (c) the "Benchmark" method.*

(a)

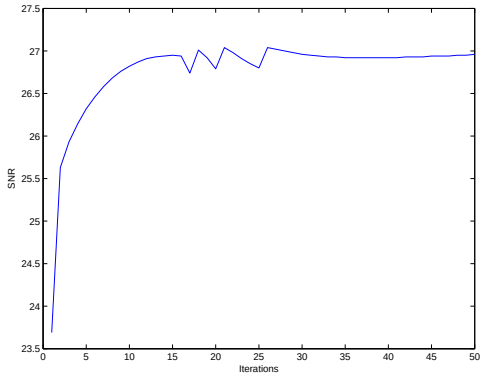
(b)

(c)

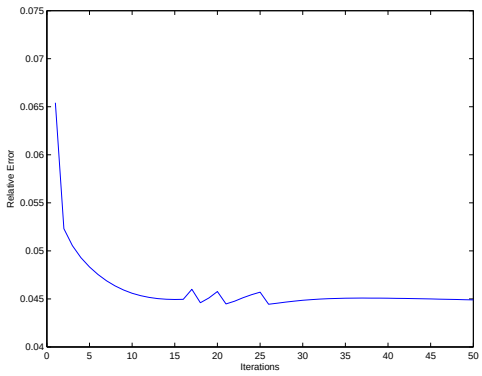
FIG. 4.9. *The restored images for Figure 4.3(c) by different methods: (a) the GTV algorithm; (b) the GHTV algorithm; (c) the "Benchmark" method.*



(a)

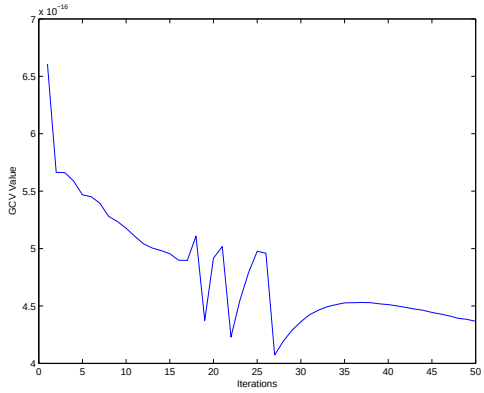


(b)

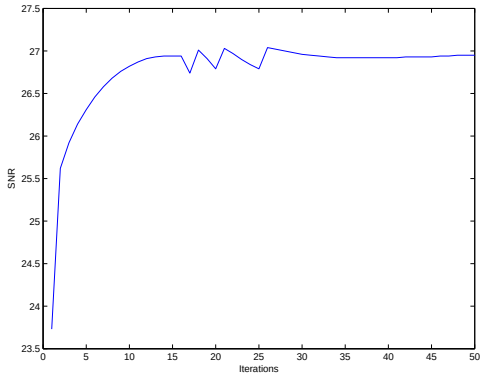


(c)

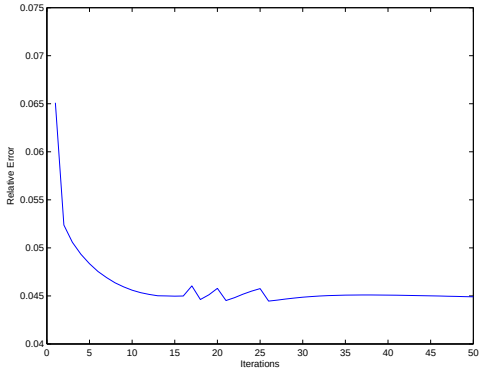
FIG. 4.10. *The GTV algorithm using* $\theta = 1.2$*: (a) GCV values v.s. iterations, (b) SNRs v.s. iterations, (c) relative errors v.s. iterations.*



(a)

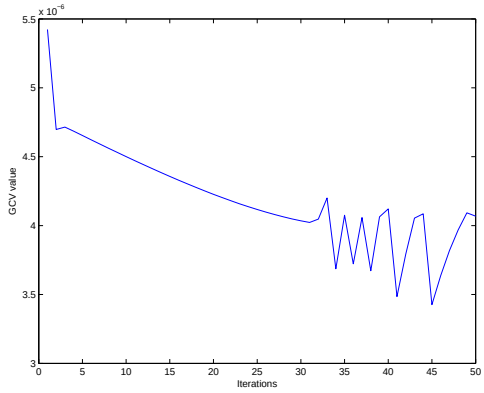


(b)

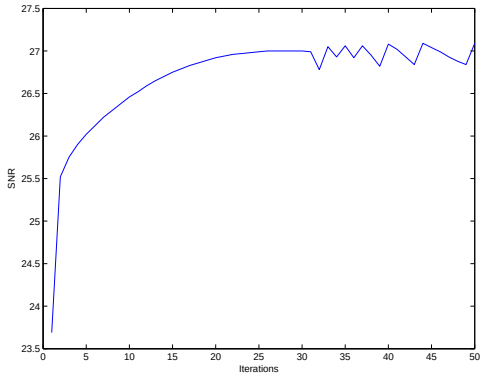


(c)

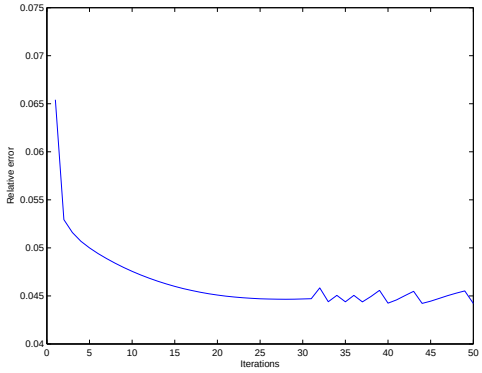
FIG. 4.11. *The GHTV algorithm using* $\theta = 1.2$*: (a) GCV values v.s. iterations, (b) SNRs v.s. iterations, (c) relative errors v.s. iterations.*



(a)

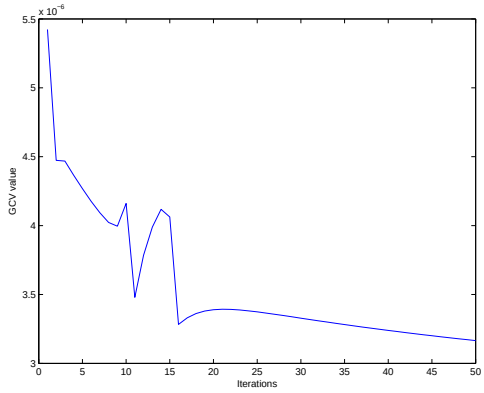


(b)

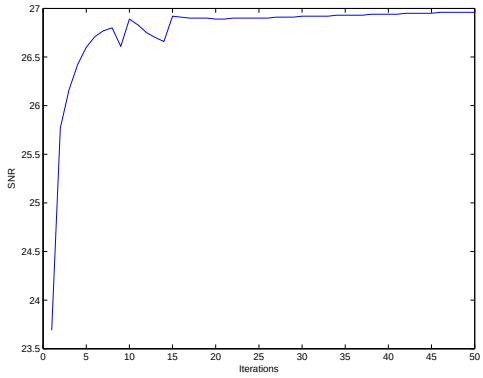


(c)

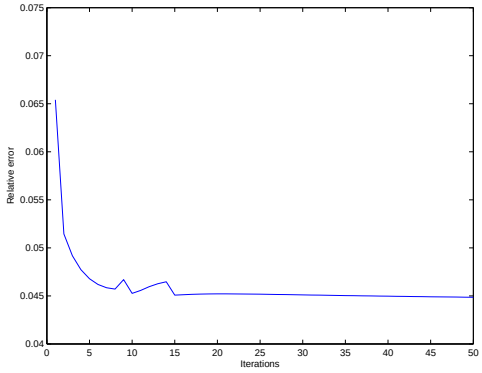
FIG. 4.12. *The GTV algorithm using* $\theta = 1.1$*: (a) GCV values v.s. iterations, (b) SNRs v.s. iterations, (c) relative errors v.s. iterations.*



(a)



(b)



(c)

FIG. 4.13. *The GTV algorithm using* $\theta = 1.4$*: (a) GCV values v.s. iterations, (b) SNRs v.s. iterations, (c) relative errors v.s. iterations.*