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Superconvergence and Extrapolation Analysis of a Nonconforming Mixed Finite Element Approximation for Time-Harmonic Maxwell's Equations

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Abstract

In this paper, a nonconforming mixed finite element approximating to the three-dimensional time-harmonic Maxwell's equations is presented. On a uniform rectangular prism mesh, superclose property is achieved for electric field E and magnetic field H with the boundary condition $E \times \mathbf{n} = 0$ by means of the asymptotic expansion. Applying postprocessing operators, a superconvergence result is stated for the discretization error of the preprocessed discrete solution to the solution itself. To our best knowledge, this is the first global superconvergence analysis of nonconforming mixed finite elements for the Maxwell's equations. Furthermore, the approximation accuracy will be improved by extrapolation method.

Keywords: nonconforming mixed finite element; superconvergence; extrapolation; time-harmonic Maxwell's equations

AMS subject classifications: 65M50, 65M60

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1 Introduction

Maxwell's equations describe the evolutions of the electromagnetic fields. Great interests have been shown on the study of Maxwell's equations by finite element methods, see e.g. [1, 4, 5, 8, 9, 10, 18, 19, 20, 22, 23, 24, 25].

Superconvergence of FEMs is a phenomenon that the convergence rate exceeds what general cases can provide, which can be achieved for smoother solutions with structured meshes. Many studies have been conducted for superconvergence analysis, see e.g. [7, 11, 16]. In 1994, Monk initiated the investigation on superconvergence for Maxwell's equations [21]. Recently, Lin and his collaborators systematically developed global superconvergence for the Maxwell's equations using the integral identity methods and postprocessing interpolations. All these works are for conforming finite elements, for example, ECHL element [13] and Nedelec element [12, 16, 29].

However, in the context of nonconforming mixed finite elements for the Maxwell's equations, to our best knowledge, there is no any result for global superconvergence, especially in three-dimensional case. A nonconforming mixed finite element approximation of time-harmonic Maxwell's equations with absorbing boundary condition was developed from the Rannacher-Turek nonconforming element by Douglas etc. in [6]. The $O(h^{\frac{1}{2}})$ convergence rate for electric field E and magnetic field H in L^2-norm was presented. Unfortunately, the superclose and global superconvergence property cannot be achieved for this element, see the reasons in [14]. The Rannacher-Turek nonconforming element and its variational forms have been studied in many papers, see e.g. [2, 3, 14, 28, 30]. One of the variational forms of this element is the EQ_1^{rot} element. In [14], the superclose property was obtained for this EQ_1^{rot} element for the elliptic problem based on the integral identity formulations. The authors also constructed a postprocess interpolation which resulted in a superconvergence of the order $O(h^2)$ in H^1-seminorm for the discretization error of the preprocessed discrete solution to the solution itself. In [17], asymptotic expansions of the nonconforming finite element EQ_1^{rot} for the low-frequency time-harmonic Maxwell's equations were presented. Superclose and Superconvergence results of $O(h^2)$ error were obtained in the discrete $H(curl)$ space.

In this paper, a nonconforming mixed finite element will be developed from the EQ_1^{rot} element for the three-dimensional time-harmonic Maxwell's equations. Superclose property can be demonstrated through asymptotic expansion method [13], which is much simpler than the integral identity method when analyzing the superclose properties for nonconforming finite elements. Superconvergence results can also be derived by a postprocessing procedure. Furthermore, extrapolation operators will be constructed to improve the accuracy based on the asymptotic expansion formulations.

This paper is organized as follows. In section 2, based on the mixed formulation of time-harmonic Maxwell's equations, a new nonconforming mixed finite element method is proposed. In section 3, asymptotic expansion behavior is demonstrated for interpolation error and nonconforming error of the nonconforming mixed finite element. In section 4, superclose and superconvergence analysis of electric field E and magnetic field H is derived, respectively. In section 5, the accuracy is further improved by extrapolation method. To end this paper, we make some concluding remarks in section 6.

2 Time-Harmonic Maxwell's Equations and a Nonconforming Mixed Finite Element Approximation

Let Ω be a bounded cubic domain in $\mathcal{R}^3$ with boundary $\partial\Omega = \bigcup_{i=1}^{6} \partial\Omega_i$, where $\partial\Omega_i, i = 1, \cdots, 6$ is the front, back, right, left, upper and lower face of Ω, respectively, and denote the unit outward normal $\mathbf{n}$. Consider the following Maxwell's equations [6, 19]:

$$\sigma E - \mathbf{curl} H = F, \qquad \text{in } \Omega, \tag{1}$$
$$i\omega\mu H + \mathbf{curl} E = 0, \qquad \text{in } \Omega, \tag{2}$$
$$E \times \mathbf{n} = 0, \qquad \text{on } \partial\Omega. \tag{3}$$

where ω is a given angular frequency, and assume that σ, μ are constants. The existence and uniqueness results for the solution of equations (1)–(3) was given in [26].

A weak formulation of system (1)-(3) will be as follows: Find $(E, H) \in \mathbf{H_0}(\mathbf{curl}, \Omega) \times (L^2(\Omega))^3$ such that

$$(\sigma E, \phi) - (H, \mathbf{curl}\phi) = (F, \phi), \qquad \forall \phi \in \mathbf{H_0}(\mathbf{curl}, \Omega), \tag{4}$$
$$(i\omega\mu H, \psi) + (\mathbf{curl} E, \psi) = 0, \qquad \forall \psi \in (L^2(\Omega))^3. \tag{5}$$

where

$$\mathbf{H_0}(\mathbf{curl},\Omega) = \{\phi \in (L^2(\Omega))^3 \big| \mathbf{curl}\ \phi \in (L^2(\Omega))^3, \phi \times \mathbf{n}|_{\partial\Omega} = 0\}.$$

Now we propose a new nonconforming mixed finite element. Let $\mathcal{J}_h$ be a uniform cube partition of domain Ω, $2h_x, 2h_y, 2h_z$ is the length, width and height of element K, $h_K = \max\{h_x, h_y, h_z\}_K, h = \max\limits_K\{h_K\}$, (x_K, y_K, z_K) be the center of element K. Let $\hat{K} = [-1, 1]^3$ be the reference element, then there exists an affine mapping $F_K : \hat{K} \longrightarrow K$

$$\begin{cases} x = x_K + h_x\hat{x}, \\ y = y_K + h_y\hat{y}, \\ z = z_K + h_z\hat{z}. \end{cases} \tag{6}$$

Let $\hat{Q} = \hat{Q}_x \times \hat{Q}_y \times \hat{Q}_z$, where

$$\begin{aligned} \hat{Q}_x &= \mathrm{Span}\{1, \hat{y}, \hat{z}, \hat{y}^2, \hat{z}^2\}, \\ \hat{Q}_y &= \mathrm{Span}\{1, \hat{z}, \hat{x}, \hat{z}^2, \hat{x}^2\}, \\ \hat{Q}_z &= \mathrm{Span}\{1, \hat{x}, \hat{y}, \hat{x}^2, \hat{y}^2\}. \end{aligned}$$

Denote $\hat{g}_i, i = 1, \cdots, 6$, be the front, back, right, left, upper and lower face of the reference element. For $\hat{\phi} \in \hat{Q}(\hat{K})$, the local interpolation operator on the reference element is defined as : $\hat{\pi} : \mathbf{H_0}(\mathbf{curl}, \hat{K}) \longrightarrow \hat{Q}(\hat{K})$

$$\frac{1}{|\hat{g}_i|}\int_{\hat{g}_i}(\hat{\pi}\hat{\phi} - \hat{\phi})d\hat{s} = 0, i = 1, 2, \cdots, 6, \quad \frac{1}{|\hat{K}|}\int_{\hat{K}}(\hat{\pi}\hat{\phi} - \hat{\phi})d\hat{x}d\hat{y}d\hat{z} = 0. \tag{7}$$

In every element K, the local interpolation operator is defined as

$$\pi_K\phi = (\hat{\pi}\hat{\phi}) \circ F_K^{-1},$$

so the interpolation operator π_h in the domain Ω is defined as

$$\pi_h\phi\big|_K = \pi_K\phi.$$

Note that (7) provides fifteen degrees of freedom needed to determine on an element in $\hat{Q}(\hat{K})$.

Next let $\hat{S} = \hat{S}_x \times \hat{S}_y \times \hat{S}_z$, where

$$\begin{aligned} \hat{S}_x &= \mathrm{Span}\{1, \hat{y}, \hat{z}\}, \\ \hat{S}_y &= \mathrm{Span}\{1, \hat{z}, \hat{x}\}, \\ \hat{S}_z &= \mathrm{Span}\{1, \hat{x}, \hat{y}\}. \end{aligned}$$

and define a local interpolation on the reference element $\hat{I} : (L^2(\hat{K}))^3 \longrightarrow \hat{S}(\hat{K})$ as: for $\hat{\psi} = (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3) \in \hat{S}(\hat{K})$

$$\int_{\hat{K}} (\hat{I}\hat{\psi}_l - \hat{\psi}_l) d\hat{x} d\hat{y} d\hat{z} = 0, \quad \int_{\hat{K}} curl(\hat{I}\hat{\psi}_l - \hat{\psi}_l) d\hat{x} d\hat{y} d\hat{z} = 0, l = 1, 2, 3. \quad (8)$$

Here, $curl$ in (8) is the two-dimensional defined as usual:

$$curl\hat{\psi}_1 = (\frac{\partial \hat{\psi}_1}{\partial \hat{z}}, -\frac{\partial \hat{\psi}_1}{\partial \hat{y}}), \quad curl\hat{\psi}_2 = (\frac{\partial \hat{\psi}_2}{\partial \hat{x}}, -\frac{\partial \hat{\psi}_2}{\partial \hat{z}}), \quad curl\hat{\psi}_3 = (\frac{\partial \hat{\psi}_3}{\partial \hat{y}}, -\frac{\partial \hat{\psi}_3}{\partial \hat{x}}).$$

In every element K, the local interpolation operator I_K is defined as

$$I_K \psi = (\hat{I}\hat{\psi}) \circ F_K^{-1},$$

so the interpolation operator I_h in the domain Ω is defined as

$$I_h \psi\big|_K = I_K \psi.$$

Note that (8) provides nine degrees of freedom needed to determine on an element in $\hat{S}(\hat{K})$ and that

$$\mathbf{curl}\hat{Q} = \hat{S}.$$

The nonconforming mixed finite element space will be defined as

$$\begin{aligned} V_h &= \{\phi \in [L^2(\Omega)]^3 : \phi\big|_K \circ F_K \in \hat{Q}, K \in \mathcal{J}_h, \phi \times \mathbf{n}\big|_{\partial\Omega} = 0\}, \\ W_h &= \{\psi \in [L^2(\Omega)]^3 : \psi\big|_K \circ F_K \in \hat{S}, K \in \mathcal{J}_h\}. \end{aligned}$$

The discrete mixed finite element formulations of equation (4)-(5) show as follows: Find $(E_h, H_h) \in V_h \times W_h$ such that

$$(\sigma E_h, \phi)_h - (H_h, \mathbf{curl}\phi)_h = (F, \phi) \quad \forall \phi \in V_h, \quad (9)$$

$$(i\omega\mu H_h, \psi)_h + (\mathbf{curl}E_h, \psi)_h = 0 \qquad \forall \psi \in W_h, \quad (10)$$

with the discrete inner product in $(L^2(\Omega))^3$ is defined by

$$(u_h, v_h)_h = \sum_K \int_K u_h v_h dxdydz, \quad \forall u_h, v_h \in V_h \text{ or } W_h.$$

and norm

$$\|u_h\|_0 = (\sum_K \int_K u_h^2 dxdydz)^{\frac{1}{2}}.$$

A traditional error estimation

$$\|E - E_h\|_0 + \|H - H_h\|_0 \leq Ch(\|E\|_2 + \|H\|_2), \tag{11}$$

can be derived with the boundary condition (3). The error estimation technique can be found in [27]. From the discussion in the next three sections, we can see that this new nonconforming mixed element has a global superconvergence property by constructing a postprocess operator which has an $O(h^2)$ accuracy, and has $O(h^4)$ accuracy by extrapolation method.

3 Asymptotic Expansion Formulations

In this section, the main aim is to derive asymptotic expansion formulations of interpolation error and nonconforming error.

$\forall \phi \in \mathbf{H_0}(\mathbf{curl}, \Omega), \psi \in (L^2(\Omega))^3$,

$$D((E, H); (\phi, \psi)) \triangleq (\sigma E, \phi) - (H, \mathbf{curl}\phi) + (i\omega\mu H, \psi) + (\mathbf{curl}E, \psi),$$

and $\forall \phi_h \in V_h, \psi_h \in W_h$,

$$D_h((E_h, H_h); (\phi_h, \psi_h)) \triangleq (\sigma E_h, \phi_h)_h - (H_h, \mathbf{curl}\phi_h)_h + (i\omega\mu H_h, \psi_h)_h + (\mathbf{curl}E_h, \psi_h)_h.$$

From (4)-(5) and (9)-(10), $\forall \phi \in V_h, \psi \in W_h$, we have

$$D_h((E - E_h, H - H_h); (\phi, \psi)) = - < H, \phi \times \mathbf{n} >_h,$$

where $< H, \phi \times \mathbf{n} >_h = \sum\limits_K \int_{\partial K} H\phi \times \mathbf{n} ds$. And

$$\begin{aligned} & D_h((E_h - \pi_h E, H_h - I_h H); (\phi, \psi)) \\ & =< H, \phi \times \mathbf{n} >_h + D((E - \pi_h E, H - I_h H); (\phi, \psi)). \end{aligned} \tag{12}$$

Next, some useful asymptotic expansion formulations of (12) will be shown, and more details can be seen in [13]. Let $E = (E_1, E_2, E_3), H = (H_1, H_2, H_3), \phi = (\phi_1, \phi_2, \phi_3), \psi = (\psi_1, \psi_2, \psi_3)$. For different $\hat{E}_i$ and $\hat{H}_i$, $i = 1, 2, 3$ on the reference element $\hat{K}$, their interpolations $\hat{\pi}_{\hat{K}}\hat{E}_i, \hat{I}_{\hat{K}}\hat{H}_i, i = 1, 2, 3$ are shown in the following two tables respectively

$\hat{E}_i$	1	$\hat{x}$	$\hat{y}$	$\hat{z}$	$\hat{x}\hat{y}$	$\hat{x}\hat{z}$	$\hat{y}\hat{z}$	$\hat{x}^2$	$\hat{y}^2$	$\hat{z}^2$
$\hat{\pi}_{\hat{K}}\hat{E}_3$	1	$\hat{x}$	$\hat{y}$	0	0	0	0	$\hat{x}^2$	$\hat{y}^2$	$\frac{1}{3}$
$\hat{\pi}_{\hat{K}}\hat{E}_2$	1	$\hat{x}$	0	$\hat{z}$	0	0	0	$\hat{x}^2$	$\frac{1}{3}$	$\hat{z}^2$
$\hat{\pi}_{\hat{K}}\hat{E}_1$	1	0	$\hat{y}$	$\hat{z}$	0	0	0	$\frac{1}{3}$	$\hat{y}^2$	$\hat{z}^2$
$\hat{E}_i$	$\hat{x}^3$	$\hat{y}^3$	$\hat{z}^3$	$\hat{x}^2\hat{y}$	$\hat{x}\hat{y}^2$	$\hat{x}^2\hat{z}$	$\hat{x}\hat{z}^2$	$\hat{y}^2\hat{z}$	$\hat{y}\hat{z}^2$	$\hat{x}\hat{y}\hat{z}$
$\hat{\pi}_{\hat{K}}\hat{E}_3$	$\hat{x}$	$\hat{y}$	0	$\frac{1}{3}\hat{y}$	$\frac{1}{3}\hat{x}$	0	$\frac{1}{3}\hat{x}$	0	$\frac{1}{3}\hat{y}$	0
$\hat{\pi}_{\hat{K}}\hat{E}_2$	$\hat{x}$	0	$\hat{z}$	0	$\frac{1}{3}\hat{x}$	$\frac{1}{3}\hat{z}$	$\frac{1}{3}\hat{x}$	$\frac{1}{3}\hat{z}$	0	0
$\hat{\pi}_{\hat{K}}\hat{E}_1$	0	$\hat{y}$	$\hat{z}$	$\frac{1}{3}\hat{y}$	0	$\frac{1}{3}\hat{z}$	0	$\frac{1}{3}\hat{z}$	$\frac{1}{3}\hat{y}$	0

$\hat{H}_i$	1	$\hat{x}$	$\hat{y}$	$\hat{z}$	$\hat{x}\hat{y}$	$\hat{x}\hat{z}$	$\hat{y}\hat{z}$	$\hat{x}^2$	$\hat{y}^2$	$\hat{z}^2$
$\hat{I}_{\hat{K}}\hat{H}_3$	1	$\hat{x}$	$\hat{y}$	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\hat{I}_{\hat{K}}\hat{H}_2$	1	$\hat{x}$	0	$\hat{z}$	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\hat{I}_{\hat{K}}\hat{H}_1$	1	0	$\hat{y}$	$\hat{z}$	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\hat{H}_i$	$\hat{x}^3$	$\hat{y}^3$	$\hat{z}^3$	$\hat{x}^2\hat{y}$	$\hat{x}\hat{y}^2$	$\hat{x}^2\hat{z}$	$\hat{x}\hat{z}^2$	$\hat{y}^2\hat{z}$	$\hat{y}\hat{z}^2$	$\hat{x}\hat{y}\hat{z}$
$\hat{I}_{\hat{K}}\hat{H}_3$	$\hat{x}$	$\hat{y}$	0	$\frac{1}{3}\hat{y}$	$\frac{1}{3}\hat{x}$	0	$\frac{1}{3}\hat{x}$	0	$\frac{1}{3}\hat{y}$	0
$\hat{I}_{\hat{K}}\hat{H}_2$	$\hat{x}$	0	$\hat{z}$	0	$\frac{1}{3}\hat{x}$	$\frac{1}{3}\hat{z}$	$\frac{1}{3}\hat{x}$	$\frac{1}{3}\hat{z}$	0	0
$\hat{I}_{\hat{K}}\hat{H}_1$	0	$\hat{y}$	$\hat{z}$	$\frac{1}{3}\hat{y}$	0	$\frac{1}{3}\hat{z}$	0	$\frac{1}{3}\hat{z}$	$\frac{1}{3}\hat{y}$	0

When

$$h_1 \equiv h_x, h_2 \equiv h_y, h_3 \equiv h_z, \forall K \in \mathcal{J}_h, \tag{13}$$

we call the mesh $\mathcal{J}_h$ is uniform [13].

Theorem 3.1. *Assume that $E \in (H^4(\Omega))^3 \bigcap \mathbf{H_0}(\mathbf{curl}, \Omega)$ and mesh $\mathcal{J}_h$ is uniform, we have*

$$(\sigma(E - \pi_h E), \phi)_h = O(h^4)\|E\|_4\|\phi\|_0, \quad \forall \phi \in V_h. \tag{14}$$

Proof: From

$$(\sigma(E - \pi_h E), \phi)_h = \sum_K \int_K \big(\sigma(E_1 - \pi_K E_1) \cdot \phi_1$$
$$+\sigma(E_2 - \pi_K E_2) \cdot \phi_2 + \sigma(E_3 - \pi_K E_3) \cdot \phi_3\big) dxdydz,$$

Here we only consider one term. Let bilinear form

$$B(\hat{E}_1, \hat{\phi}_1) = \int_{\hat{K}} (\hat{E}_1 - \hat{\pi}_{\hat{K}}\hat{E}_1) \cdot \hat{\phi}_1 d\hat{x}d\hat{y}d\hat{z},$$

then we have

$$|B(\hat{E}_1,\hat{\phi}_1)| \le C\|\hat{E}_1\|_{4,\hat{K}}\|\hat{\phi}_1\|_{0,\hat{K}}.$$

Here and after let $\hat{\phi}_1 = (1,\hat{y},\hat{z},\hat{y}^2,\hat{z}^2)$. By calculating,

$$\begin{aligned}
&B(\hat{y}^3,\hat{\phi}_1) = \int_{\hat{K}}(\hat{y}^3-\hat{y})(1,\hat{y},\hat{z},\hat{y}^2,\hat{z}^2)d\hat{x}d\hat{y}d\hat{z}\\
&= -\frac{16}{15}(0,1,0,0,0) = -\frac{1}{45}\int_{\hat{K}}\hat{E}_{1\hat{y}\hat{y}\hat{y}}\hat{\phi}_{1\hat{y}}d\hat{x}d\hat{y}d\hat{z}, \quad \hat{E}_1=\hat{y}^3,\\
&B(\hat{z}^3,\hat{\phi}_1) = -\frac{1}{45}\int_{\hat{K}}\hat{E}_{1\hat{z}\hat{z}\hat{z}}\hat{\phi}_{1\hat{z}}d\hat{x}d\hat{y}d\hat{z}, \quad \hat{E}_1=\hat{z}^3,\\
&B(\hat{P}_3/\{\hat{y}^3,\hat{z}^3\},\hat{\phi}_1) = 0.
\end{aligned}$$

Let

$$B(\hat{E}_1,\hat{\phi}_1) = -\frac{1}{45}\int_{\hat{K}}\hat{E}_{1\hat{y}\hat{y}\hat{y}}\hat{\phi}_{1\hat{y}}d\hat{x}d\hat{y}d\hat{z} - \frac{1}{45}\int_{\hat{K}}\hat{E}_{1\hat{z}\hat{z}\hat{z}}\hat{\phi}_{1\hat{z}}d\hat{x}d\hat{y}d\hat{z} + G(\hat{E}_1,\hat{\phi}_1),$$

we find

$$G(\hat{P}_3,\hat{\phi}_1) = 0.$$

Therefore, by Bramble–Hilbert Lemma [see Lemma 2.6, [13]], we have

$$B(\hat{E}_1,\hat{\phi}_1) = -\tfrac{1}{45}\int_{\hat{K}}(\hat{E}_{1\hat{y}\hat{y}\hat{y}}\hat{\phi}_{1\hat{y}} + \hat{E}_{1\hat{z}\hat{z}\hat{z}}\hat{\phi}_{1\hat{z}})d\hat{x}d\hat{y}d\hat{z} + O(1)|\hat{E}_1|_{4,\hat{K}}\|\hat{\phi}_1\|_{0,\hat{K}}.$$

Hence, if the partition is uniform

$$\begin{aligned}
&\int_\Omega \sigma(E_1-\pi_K E_1)\cdot\phi_1 dxdydz\\
&= \sigma\sum_K\Big(-\frac{h_y^4}{45}\int_K E_{1yyy}\phi_{1y}dxdydz - \frac{h_z^4}{45}\int_K E_{1zzz}\phi_{1z}dxdydz\Big) + O(h^4)|E_1|_4\|\phi_1\|_0\\
&= \sigma\Big(\frac{h_y^4}{45}\int_\Omega E_{1yyyy}\phi_1 dxdydz + \frac{h_z^4}{45}\int_\Omega E_{1zzzz}\phi_1 dxdydz\Big) + O(h^4)|E_1|_4\|\phi_1\|_0. \quad (15)
\end{aligned}$$

Here we use integration by parts and the continuity of ϕ_1 on the interface of two adjoint element. We also need the boundary condition $E\times\mathbf{n}=0$, which implies E_1, E_3 equal to 0 on $\partial\Omega_1, \partial\Omega_2$, E_3, E_2 equal to 0 on $\partial\Omega_3, \partial\Omega_4$, and E_2, E_1 equal to 0 on $\partial\Omega_5, \partial\Omega_6$. That means that all the tangential directional of E_i equal to 0 on the corresponding faces.

Similarly,

$$
\begin{aligned}
&\int_\Omega \sigma(E_2-\pi_K E_2)\cdot\phi_2 dxdydz\\
&=\sigma\big(\frac{h_x^4}{45}\int_\Omega E_{2xxxx}\phi_2 dxdydz+\frac{h_z^4}{45}\int_\Omega E_{2zzzz}\phi_2 dxdydz\big)\\
&+O(h^4)|E_2|_4\|\phi_2\|_0, \qquad (16)\\
&\int_\Omega \sigma(E_3-\pi_K E_3)\cdot\phi_3 dxdydz\\
&=\sigma\big(\frac{h_x^4}{45}\int_\Omega E_{3xxxx}\phi_3 dxdydz+\frac{h_y^4}{45}\int_\Omega E_{3yyyy}\phi_3 dxdydz\big)\\
&+O(h^4)|E_3|_4\|\phi_3\|_0. \qquad (17)
\end{aligned}
$$

Combining (15)–(17), we can finish the proof. □

Theorem 3.2. *Assume that $H\in(H^4(\Omega))^3$ and mesh $\mathcal{J}_h$ is uniform, we have*

$$
(i\omega\mu(H-I_hH),\psi)_h=O(h^4)\|H\|_4\|\psi\|_{1h},\quad \forall\psi\in W_h. \qquad (18)
$$

where $\|\psi\|_{1h}=\sum\limits_K\sum\limits_{|\alpha|\le 1}(\int_K(D^\alpha\psi)^2)^{1/2}$.

Proof:

Now we consider the asymptotic expansion for

$$
\begin{aligned}
(i\omega\mu(H-I_KH),\psi)_h&=\sum_K\int_K(i\omega\mu(H_1-I_KH_1)\cdot\psi_1\\
&+i\omega\mu(H_2-I_KH_2)\cdot\psi_2+i\omega\mu(H_3-I_KH_3)\cdot\psi_3)dxdydz.
\end{aligned}
$$

Here and after denote $\hat\psi_3=(1,\hat x,\hat y)$. Let bilinear form

$$
B(\hat H_3,\hat\psi_3)=\int_{\hat K}(\hat H_3-\hat I_{\hat K}\hat H_3)\cdot\hat\psi_3 d\hat x d\hat y d\hat z.
$$

Then we have

$$
|B(\hat H_3,\hat\psi_3)|\le C\|\hat E_3\|_{4,\hat K}\|\hat\psi_3\|_{0,\hat K}.
$$

By calculating

$$\begin{aligned}
&B(\hat{x}^3,\hat{\psi}_3)=\int_{\hat{K}}(\hat{x}^3-\hat{x})(1,\hat{x},\hat{y})d\hat{x}d\hat{y}d\hat{z}\\
&=-\frac{16}{15}(0,1,0)=-\frac{1}{45}\int_{\hat{K}}\hat{H}_{3\hat{x}\hat{x}\hat{x}}\hat{\psi}_{3\hat{x}}d\hat{x}d\hat{y}d\hat{z},\quad \hat{H}_3=\hat{x}^3,\\
&B(\hat{y}^3,\hat{\psi}_3)=-\frac{1}{45}\int_{\hat{K}}\hat{H}_{3\hat{y}\hat{y}\hat{y}}\hat{\psi}_{3\hat{y}}d\hat{x}d\hat{y}d\hat{z},\quad \hat{H}_3=\hat{y}^3,\\
&B(\hat{P}_3/\{\hat{x}^3,\hat{y}^3\},\hat{\psi}_3)=0.
\end{aligned}$$

Let

$$B(\hat{H}_3,\hat{\psi}_3)=-\frac{1}{45}\int_{\hat{K}}(\hat{H}_{3\hat{x}\hat{x}\hat{x}}\hat{\psi}_{3\hat{x}}+\hat{H}_{3\hat{y}\hat{y}\hat{y}}\hat{\psi}_{3\hat{y}})d\hat{x}d\hat{y}d\hat{z}+G_3(\hat{H}_3,\hat{\psi}_3).$$

We have

$$G_3(\hat{P}_3,\hat{\psi}_3)=0.$$

Therefore,

$$B(\hat{H}_3,\hat{\psi}_3)=-\frac{1}{45}\int_{\hat{K}}(\hat{H}_{3\hat{x}\hat{x}\hat{x}}\hat{\psi}_{3\hat{x}}+\hat{H}_{3\hat{y}\hat{y}\hat{y}}\hat{\psi}_{3\hat{y}})d\hat{x}d\hat{y}d\hat{z}+O(1)|\hat{H}_3|_{4,\hat{K}}\|\hat{\psi}_3\|_{0,\hat{K}}.$$

Hence,

$$\begin{aligned}
&\int_{\Omega}i\omega\mu(H_3-I_hH_3)\psi_3dxdydz\\
&=i\omega\mu\sum_K(-\frac{h_x^4}{45}\int_K H_{3xxx}\psi_{3x}dxdydz-\frac{h_y^4}{45}\int_K H_{3yyy}\psi_{3y}dxdydz)\\
&+O(h^4)|H_3|_4\|\psi_3\|_0. \qquad (19)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\Omega}i\omega\mu(H_2-I_hH_2)\psi_2dxdydz\\
&=i\omega\mu\sum_K(-\frac{h_x^4}{45}\int_K H_{2xxx}\psi_{2x}dxdydz-\frac{h_z^4}{45}\int_K H_{2zzz}\psi_{2z}dxdydz)\\
&+O(h^4)|H_2|_4\|\psi_2\|_0, \qquad (20)
\end{aligned}$$

$$\int_\Omega i\omega\mu(H_1 - I_h H_1)\psi_1 dxdydz$$
$$= i\omega\mu\sum_K(-\frac{h_y^4}{45}\int_K H_{1yyy}\psi_{1y}dxdydz - \frac{h_z^4}{45}\int_K H_{1zzz}\psi_{1z}dxdydz)$$
$$+O(h^4)|H_1|_4\|\psi_1\|_0. \tag{21}$$

For (19)–(21), using Cauchy-Schwartz inequality, we can get (18). Now we can finish the proof. □

Theorem 3.3. *Assume that* $H \in (H^4(\Omega))^3$ *and mesh* $\mathcal{J}_h$ *is uniform, we have*

$$((H - I_h H), \mathbf{curl}\phi)_h = O(h^4)\|H\|_4\|\phi\|_{2h}, \quad \forall\phi \in V_h. \tag{22}$$

where $\|\phi\|_{2h} = \sum\limits_K \sum\limits_{|\alpha|\le 2} (\int_K (D^\alpha\phi)^2)^{1/2}$.

Proof: From

$$((H - I_K H), \mathbf{curl}\phi)_h = \sum_K \int_K [(H_1 - I_h H_1)(\phi_{3y} - \phi_{2z})$$
$$-(H_2 - I_h H_2)(\phi_{3x} - \phi_{1z}) + (H_3 - I_h H_3)(\phi_{2x} - \phi_{1y})]dxdydz.$$

Only consider the first term $\int_K (H_1 - I_h H_1)\phi_{3y}dxdydz$. Here and after, denote $\hat{\phi}_3 = (1\ \hat{x}\ \hat{y}\ \hat{x}^2\ \hat{y}^2)$, $\hat{\phi}_{3\hat{y}} = (0\ 0\ 1\ 0\ 2\hat{y})$. Let bilinear form

$$B(\hat{H}_1, \hat{\phi}_3) = \int_{\hat{K}} (\hat{H}_1 - \hat{I}_{\hat{K}}\hat{H}_1)\hat{\phi}_{3\hat{y}} d\hat{x}d\hat{y}d\hat{z},$$

then

$$|B(\hat{H}_1, \hat{\phi}_3)| \le C\|\hat{H}_1\|_4\|\hat{\phi}_3\|_0.$$

By calculating,

$$B(\hat{y}^3, \hat{\phi}_3) = \int_{\hat{K}} (\hat{y}^3 - \hat{y})(0\ 0\ 1\ 0\ 2\hat{y})d\hat{x}d\hat{y}d\hat{z} = -\frac{1}{45}\int_{\hat{K}} \hat{H}_{1\hat{y}\hat{y}\hat{y}}\hat{\phi}_{3\hat{y}\hat{y}}d\hat{x}d\hat{y}d\hat{z}, \quad \hat{H}_1 = \hat{y}^3$$
$$B(\hat{P}_3/\{\hat{y}^3\}, \hat{\phi}_3) = 0.$$

Let

$$B(\hat{y}^3, \hat{\phi}_3) = -\frac{1}{45}\int_{\hat{K}} \hat{H}_{1\hat{y}\hat{y}\hat{y}}\hat{\phi}_{3\hat{y}\hat{y}}d\hat{x}d\hat{y}d\hat{z} + G(\hat{H}_1, \hat{\phi}_3),$$

then

$$G(\hat{P}_3,\hat{\phi}_3)=0.$$

Therefore,

$$B(\hat{H}_1,\hat{\phi}_3)=-\frac{1}{45}\int_{\hat{K}}\hat{H}_{1\hat{y}\hat{y}\hat{y}}\hat{\phi}_{3\hat{y}\hat{y}}d\hat{x}d\hat{y}d\hat{z}+O(1)|\hat{H}_1|_{4,\hat{K}}\|\hat{\phi}_3\|_{0,\hat{K}}.$$

Hence, integration by parts and if the partition is uniform, we have

$$\begin{aligned}&\int_\Omega (H_1-I_hH_1)\phi_{3y}dxdydz=\sum_K\int_K (H_1-I_KH_1)\phi_{3y}dxdydz\\&=\sum_K-\frac{h_y^4}{45}\int_K H_{1yyy}\phi_{3yy}dxdydz+O(h^4)|H_1|_4\|\phi_3\|_0.\\&=O(h^4)\|H_1\|_4\|\phi_3\|_{2h}\end{aligned}\tag{23}$$

Similarly,

$$\int_\Omega -(H_1-I_hH_1)\phi_{2z}dxdydz=O(h^4)|H_1|_4\|\phi_2\|_{2h},\tag{24}$$

$$\int_\Omega -(H_2-I_hH_2)\phi_{3x}dxdydz=O(h^4)|H_2|_4\|\phi_3\|_{2h},\tag{25}$$

$$\int_\Omega (H_2-I_hH_2)\phi_{1z}dxdydz=O(h^4)|H_2|_4\|\phi_1\|_{2h},\tag{26}$$

$$\int_\Omega (H_3-I_hH_3)\phi_{2x}dxdydz=O(h^4)|H_3|_4\|\phi_2\|_{2h},\tag{27}$$

$$\int_\Omega -(H_3-I_hH_3)\phi_{1y}dxdydz\ =O(h^4)|H_3|_4\|\phi_1\|_{2h}.\tag{28}$$

Combining (23)–(28), we can finish the proof.

□

Theorem 3.4. *Assume that $E\in\mathbf{H}_0(\mathbf{curl},\Omega)$, we have*

$$(\mathbf{curl}(E-\pi_hE),\psi)_h=0,\quad \forall\psi\in W_h.\tag{29}$$

Proof: By the definition of interpolation operator π_h and $\mathbf{curl}\psi\in$ *span*$\{1\}$, we have

$$(\mathbf{curl}(E-\pi_hE),\psi)_h=((E-\pi_hE),\mathbf{curl}\psi)_h+<(E-\pi_hE)\times\mathbf{n},\psi>_h=0.$$

□

Theorem 3.5. *Assume that* $H \in (H^5(\Omega))^3$ *and mesh* $\mathcal{J}_h$ *is uniform, we have the nonconforming error*

$$\begin{aligned} < H, \phi \times \mathbf{n} >_h \quad &= \frac{h_x^2}{3} \int_\Omega (H_{1xxy}\phi_3 - H_{1xxz}\phi_2) dxdydz \\ &+ \frac{h_y^2}{3} \int_\Omega (H_{2yyz}\phi_1 - H_{2xyy}\phi_3) dxdydz \\ &+ \frac{h_z^2}{3} \int_\Omega (H_{3xzz}\phi_2 - H_{3yzz}\phi_1) dxdydz \\ &+ O(h^4)\|H\|_5 \|\phi\|_{2h}, \quad \forall \phi \in V_h. \end{aligned} \tag{30}$$

Proof: Let

$$\begin{aligned} &< H, \phi \times \mathbf{n} >_h = \sum_K \int_{\partial K} H \cdot \phi \times \mathbf{n} ds \\ &= \sum_K [(\int_{g_2} - \int_{g_1})(H_3\phi_1 - H_1\phi_3) dxdz + (\int_{g_4} - \int_{g_3})(H_2\phi_3 - H_3\phi_2) dydz \\ &+ (\int_{g_5} - \int_{g_6})(H_1\phi_2 - H_2\phi_1) dxdy]. \end{aligned}$$

Our task is to show that the summation of face-integrate is of high-order, and even has an error expansion. The technique is to approximate the integrate $\phi_i, i = 1, 2, 3$ on each face $g_j, j = 1, 2, \cdots, 6$ by its conforming part, average $\bar{\phi}_i\big|_{g_j} = \frac{1}{|g_j|}\int_{g_j} \phi_i ds$, which has continuity between elements, and we can also utilize the boundary condition $\phi \times \mathbf{n} = 0$ on $\partial\Omega$. Hence, the summation

$$\begin{aligned} &\sum_K [(\int_{g_2} - \int_{g_1})(H_3\bar{\phi}_1 - H_1\bar{\phi}_3) dxdz + (\int_{g_4} - \int_{g_3})(H_2\bar{\phi}_3 - H_3\bar{\phi}_2) dydz \\ &+ (\int_{g_5} - \int_{g_6})(H_1\bar{\phi}_2 - H_2\bar{\phi}_1) dxdy] = 0. \end{aligned}$$

For convenience, we only consider one term, and the other is similar.

$$\sum_K (\int_{g_2} - \int_{g_1}) H_3\phi_1 dxdz = \sum_K (\int_{g_2} - \int_{g_1}) H_3(\phi_1 - \bar{\phi}_1) dxdz.$$

By the expansion [13]

$$(\phi_1 - \bar{\phi}_1)\big|_{g_j} = (z - z_K)\phi_{1z} - \big((z - z_K)^2 + \frac{h_z^2}{3}\big)\frac{\phi_{1zz}}{2}, \quad j = 1, 2,$$

we have

$$
\begin{aligned}
&(\int_{g_2} - \int_{g_1}) H_3(\phi_1 - \bar{\phi}_1) \\
&= (\int_{g_2} - \int_{g_1}) H_3((z - z_K)\phi_{1z} - \big((z - z_K)^2 + \frac{h_z^2}{3}\big)\frac{\phi_{1zz}}{2}) dxdz \\
&= \int_K [H_3((z - z_K)\phi_{1z} - \big((z - z_K)^2 + \frac{h_z^2}{3}\big)\frac{\phi_{1zz}}{2})]_y dydxdz \\
&= \int_K H_{3y}((z - z_K)\phi_{1z} - \big((z - z_K)^2 + \frac{h_z^2}{3}\big)\frac{\phi_{1zz}}{2}) dxdydz \\
&= \frac{h_z^2}{3}\int_K H_{3yz}\phi_{1z} dxdydz - \frac{4h_z^4}{45}\int_K H_{3yzz}\phi_{1zz} dxdydz + O(h^4)|H_3|_{5,K}|\phi_1|_{0,K}. \\
&= \frac{h_z^2}{3}\int_K H_{3yz}\phi_{1z} dxdydz + O(h^4)\|H_3\|_{5,K}|\phi_1|_{2,K}.
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
&\sum_K [(\int_{g_2} - \int_{g_1}) H_3(\phi_1 - \bar{\phi}_1)] \\
&= \sum_K [-\frac{h_z^2}{3}\int_K H_{3yzz}\phi_1 dxdydz + O(h^4)\|H\|_{5,K}|\phi|_{2,K}].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sum_K [(\int_{g_2} - \int_{g_1})(-H_1(\phi_3 - \bar{\phi}_3)) dxdz] \\
&= \sum_K [\frac{h_x^2}{3}\int_K H_{1xxy}\phi_3 dxdydz + O(h^4)\|H\|_{5,K}|\phi|_{2,K}].
\end{aligned}
$$

Here we need the boundary condition $\phi \times \mathbf{n} = 0$ on $\partial\Omega$.
Therefore,

$$
\begin{aligned}
&\sum_K [(\int_{g_2} - \int_{g_1})((H_3\phi_1 - H_1\phi_3)) dxdz] \\
&= \sum_K [-\frac{h_z^2}{3}\int_K H_{3yzz}\phi_1 dxdydz \\
&+ \frac{h_x^2}{3}\int_K H_{1xxy}\phi_3 dxdydz + O(h^4)\|H\|_{5,K}|\phi|_{2,K}]. \qquad (31)
\end{aligned}
$$

Similarly,

$$\sum_K [(\int_{g_4} - \int_{g_3})(H_2\phi_3 - H_3\phi_2)dydz]$$
$$= \sum_K [-\frac{h_2^2}{3}\int_K H_{2xyy}\phi_3 dxdydz + \frac{h_3^2}{3}\int_K H_{3xzz}\phi_2 dxdydz$$
$$+O(h^4)\|H\|_{5,K}|\phi|_{2,K}], \tag{32}$$
$$\sum_K [(\int_{g_5} - \int_{g_6})(H_1\phi_2 - H_2\phi_1)dxdy]$$
$$= \sum_K [-\frac{h_1^2}{3}\int_K H_{1xxz}\phi_2 dxdydz + \frac{h_2^2}{3}\int_K H_{2yyz}\phi_1 dxdydz$$
$$+O(h^4)\|H\|_{5,K}|\phi|_{2,K}]. \tag{33}$$

When the mesh $\mathcal{J}_h$ is uniform, combining (31)-(33), we can end the proof. □

4 Superclose and Superconvergence Analysis

Now based on the theorems in Section 3, we can get the principal superclose result in this paper.

Theorem 4.1. *Let (E, H) and (E_h, H_h) be the solution of (4)-(5) and (9)-(10), respectively, $\pi_h E \in V_h$ and $I_h H \in W_h$ be the interpolations of E and H, respectively. Assume $(E, H) \in (\mathbf{H_0}(\mathbf{curl}, \Omega) \cap (H^3(\Omega))^3) \times (H^3(\Omega))^3$ and the mesh $\mathcal{J}_h$ is uniform, we have*

$$\|E_h - \pi_h E\|_0 + \|H_h - I_h H\|_0 \le Ch^2(\|E\|_3 + \|H\|_3). \tag{34}$$

Proof: $\forall \phi \in \mathbf{H_0}(\mathbf{curl}, \Omega), \psi \in (L^2(\Omega))^3, \phi_h \in V_h, \psi_h \in W_h$, from the definition of $D((E, H); (\phi, \psi))$, $D_h((E_h, H_h); (\phi_h, \psi_h))$, and **Theorem** 3.1—**Theorem** 3.5, we can get

$$\|E_h - \pi_h E\|_0^2 + \|H_h - I_h H\|_0^2$$
$$\le |D_h((E_h - \pi_h E, H_h - I_h H), (E_h - \pi_h E, H_h - I_h H))|$$
$$\le | < H, (E_h - \pi_h E) \times \mathbf{n} >_h |$$
$$+|D_h((E - \pi_h E, H - I_h H), (E - \pi_h E, H - I_h H))|$$
$$= O(h^2)(\|E\|_3 + \|H\|_3)(\|E_h - \pi_h E\|_0 + \|H_h - I_h H\|_0).$$

Then (34) follows. $\square$

Based on **Theorem 4.1** and theory of constructing post-processing operator in [13, 15], there exists a post-processing operator Π_{2h}^1 such that

$$
\begin{aligned}
&1.\Pi_{2h}^1\phi \in Q_{111}(\tau), \forall \phi \in (L^2(\Omega))^3, && (35)\\
&2.\ \ \|\Pi_{2h}^1\phi - \phi\|_{0,\tau} \le Ch^2\|\phi\|_{2,\tau}, \quad \forall \phi \in (L^2(\Omega))^3, && (36)\\
&3.\ \ \|\Pi_{2h}^1\phi\|_{0,\tau} \le C\|\phi\|_{0,\tau}, \quad \forall \phi \in V_h, && (37)\\
&4.\ \ \Pi_{2h}^1\pi_h\phi = \Pi_{2h}^1\phi, \quad \forall \phi \in (L^2(\Omega))^3, && (38)
\end{aligned}
$$

where τ can be constructed by merging the adjacent 8 elements into a big element.

The postprocessing operator Π_{2h}^1 can be defined by the following procedure. Let $\tau \in \mathcal{J}_{2h}$ consist of $2 \times 2 \times 2$ elements $K_i \in \mathcal{J}_h, i = 1, 2...8$. $V_h(\tau)$ denotes the space of nonconforming finite element functions of V_h restricted onto τ. We define the local interpolation operator $\Pi_{2h}^1 : V_h(\tau) \longrightarrow Q_{111}(\tau)$ by

$$\int_{K_i} (\Pi_{2h}^1 E - E)dxdydz = 0, i = 1, 2, \cdots, 8.$$

It can be shown that the interpolation operator Π_{2h}^1 is uniquely defined.

Here, we can also construct a post-processing operator I_{2h}^1 concerning electric fields H, such that $I_{2h}^1 H \in Q_{111}(\tau)$ with properties (35)-(38) replacing V_h with W_h in (37).

$$\int_{K_i} (I_{2h}^1 H - H)dxdydz = 0, i = 1, 2, \cdots, 8.$$

Therefore, we have the superconvergence result:

Theorem 4.2. *Let (E, H) and (E_h, H_h) be the solution of (4)-(5) and (9)-(10), respectively. Assume $(E, H) \in (\mathbf{H_0}(\mathbf{curl}, \Omega) \cap (H^3(\Omega))^3) \times (H^3(\Omega))^3$ and the mesh $\mathcal{J}_h$ is uniform, we have*

$$\|E - \Pi_{2h}^1 E_h\|_0 + \|H - I_{2h}^1 H_h\|_0 \le Ch^2(\|E\|_3 + \|H\|_3). \tag{39}$$

Proof: By (38)

$$I_{2h}^1 H_h - H = I_{2h}^1 H_h - I_{2h}^1 I_h H + I_{2h}^1 I_h H - H.$$

By (36)(37) and **Theorem 3.2**

$$\|I_{2h}^1 H_h - H\|_0 \leq C\|H_h - I_h H\|_0 + Ch^2\|H\|_2 = O(h^2)\|H\|_3.$$

Similarly,

$$\|\Pi_{2h}^1 E_h - E\|_0 = O(h^2)\|E\|_3.$$

Hence, (39) follows. □

5 Extrapolation

Extrapolation is also viewed as a superconvergence technique. In this section, we will discuss an extrapolation of the preprocessed discrete solution. In order to obtain asymptotic error expansion, we need to construct the following auxiliary equations:
Find $(\tilde{E}, \tilde{H}) \in \mathbf{H_0}(\mathbf{curl}, \Omega) \times (L^2(\Omega))^3$ such that

$$(\sigma\tilde{E}, \phi) - (\tilde{H}, \mathbf{curl}\phi) = S_h(\phi), \qquad \forall\phi \in \mathbf{H_0}(\mathbf{curl}, \Omega), \tag{40}$$

$$(i\omega\mu\tilde{H}, \psi) + (\mathbf{curl}\tilde{E}, \psi) = 0, \qquad \forall\psi \in (L^2(\Omega))^3, \tag{41}$$

with the regularity property [19] of the solution $(\tilde{E}, \tilde{H})$:

$$\|\tilde{E}\|_2 + \|\tilde{H}\|_2 \leq C(\|E\|_3 + \|H\|_3).$$

where

$$S_h(\phi) = \frac{h_x^2}{3h^2}\int_\Omega (H_{1xxy}\phi_3 - H_{1xxz}\phi_2)dxdydz$$
$$+\frac{h_y^2}{3h^2}\int_\Omega (H_{2yyz}\phi_1 - H_{2xyy}\phi_3)dxdydz$$
$$+\frac{h_z^2}{3h^2}\int_\Omega (H_{3xzz}\phi_2 - H_{3yzz}\phi_1)dxdydz.$$

Obviously, S_h has the following property

$$S_h = S_{\frac{h}{2}}.$$

Let $(\tilde{E}_h, \tilde{H}_h) \in V_h \times W_h$ be the finite element solution approximation of $(\tilde{E}, \tilde{H})$, i.e.,

$$(\sigma\tilde{E}_h, \phi)_h - (\tilde{H}_h, \mathbf{curl}\phi)_h = S_h(\phi), \quad \forall\phi \in V_h, \tag{42}$$

$$(i\omega\mu\tilde{H}_h, \psi)_h + (\mathbf{curl}\tilde{E}_h, \psi)_h = 0, \qquad \forall\psi \in W_h. \tag{43}$$

Theorem 5.1. *Assume* $(E,H)\in(\mathbf{H_0}(\mathbf{curl},\Omega)\cap(H^4(\Omega))^3)\times(H^5(\Omega))^3$, (π_hE, I_hH) *are the interpolation of* (E,H), (E_h,H_h) *and* $(\tilde{E}_h,\tilde{H}_h)$ *are the finite element approximations of* (E,H) *and* $(\tilde{E},\tilde{H})$, *respectively. Then we have the following estimations*

$$\|E_h-\pi_hE-h^2\tilde{E}_h\|_0=O(h^4)(\|E\|_4+\|H\|_5), \tag{44}$$
$$\|H_h-I_hH-h^2\tilde{H}_h\|_0=O(h^4)(\|E\|_4+\|H\|_5). \tag{45}$$

Proof: From **Theorem 3.1– 3.5** and the discrete auxiliary equation (42)-(43), we have

$$\begin{aligned}
&D_h((E_h-\pi_hE-h^2\tilde{E}_h, H_h-I_hH-h^2\tilde{h});(\phi,\psi))\\
&=D_h((E_h-E,H_h-H);(\phi,\psi))+D_h((E-\pi_hE,H-I_hH);(\phi,\psi))\\
&-h^2D_h((\tilde{E}_h,\tilde{H}_h);(\phi,\psi))\\
&=<H,\phi\times\mathbf{n}>_h-h^2D_h((\tilde{E}_h,\tilde{H}_h);(\phi,\psi))+D_h((E-\pi_hE,H-I_hH);(\phi,\psi))\\
&=O(h^4)\|H\|_5\|\phi\|_{2h}+O(h^4)(\|E\|_4+\|H\|_4)(\|\phi\|_0+\|\psi\|_{1h})\\
&=O(h^4)(\|E\|_4+\|H\|_5)(\|\phi\|_{2h}+\|\psi\|_{2h}).
\end{aligned}$$

$\forall f\in(L^2(\Omega))^3$, there exists $(u,v)\in[\mathbf{H_0}(\mathbf{curl},\Omega)\bigcap(H^2(\Omega))^3]\times(H^2(\Omega))^3$ such that

$$\begin{aligned}
&(\sigma\phi,u)-(\psi,\mathbf{curl}u)=(\phi,f), \quad \forall\phi\in\mathbf{H_0}(\mathbf{curl},\Omega),\\
&(i\omega\mu\psi,v)+(\mathbf{curl}\phi,v)=0, \qquad \forall\psi\in(L^2(\Omega))^3.
\end{aligned}$$

with the regularity property [19]: $\|u\|_2+\|v\|_2\le C\|f\|_0$.

Suppose that (u_h,v_h) is the mixed nonconforming finite element solution of (u,v), respectively, then

$$\begin{aligned}
&(\sigma\phi,u_h)_h-(\psi,\mathbf{curl}u_h)_h=(\phi,f), \quad \forall\phi\in V_h,\\
&(i\omega\mu\psi,v_h)_h+(\mathbf{curl}\phi,v_h)_h=0, \qquad \forall\psi\in W_h.
\end{aligned}$$

Thus, we have

$$D_h((\phi,\psi);(u_h,v_h))=(\phi,f), \quad \forall\phi\in V_h,\psi\in W_h.$$

Take $\phi=E_h-\pi_hE-h^2\tilde{E}_h,\psi=H_h-I_hH-h^2\tilde{H}_h$, we get

$$\begin{aligned}
&(E_h-\pi_hE-h^2\tilde{E}_h,f)\\
&=D_h((E_h-\pi_hE-h^2\tilde{E}_h,H_h-I_hH-h^2\tilde{H}_h);(u_h,v_h))\\
&=O(h^4)(\|E\|_4+\|H\|_5)(\|u_h\|_{2h}+\|v_h\|_{2h}).
\end{aligned}$$

Note that

$$
\begin{aligned}
&\|u_h\|_{2h} \le \|u_h - \pi_h u\|_{2h} + \|\pi_h u - u\|_{2h} + \|u\|_2 \\
&\le Ch^{-1}\|u_h - \pi_h u\|_{1h} + C\|u\|_2 \le C\|u\|_2 \le C\|f\|_0 \\
&\|v_h\|_{2h} \le C\|f\|_0.
\end{aligned}
$$

Therefore, $\forall f \in (L^2(\Omega))^3$, there exists

$$(E_h - \pi_h E - h^2\tilde{E}_h, f) = O(h^4)(\|E\|_4 + \|H\|_5)\|f\|_0$$

Let $f = E_h - \pi_h E - h^2\tilde{E}_h$, we have

$$\|E_h - \pi_h E - h^2\tilde{E}_h\|_0 = O(h^4)(\|E\|_4 + \|H\|_5).$$

Similarly, $\forall g \in (L^2(\Omega))^3$, there exists $(w,p) \in [\mathbf{H_0}(\mathbf{curl},\Omega)\bigcap(H^2(\Omega))^3] \times (H^2(\Omega))^3$ such that

$$
\begin{aligned}
&(\sigma\phi, w) - (\psi, \mathbf{curl} w) = 0, && \forall \phi \in \mathbf{H_0}(\mathbf{curl},\Omega), \\
&(i\omega\mu\psi, p) + (\mathbf{curl}\phi, p) = (\psi, g), && \forall \psi \in (L^2(\Omega))^3.
\end{aligned}
$$

with the regularity property [19]: $\|w\|_2 + \|p\|_2 \le C\|g\|_0$. Using the same technique, we have

$$\|H_h - I_h H - h^2\tilde{H}_h\|_0 = O(h^4)(\|E\|_4 + \|H\|_5).$$

Now we can finish the proof. □

Let $\tau = \bigcup_{i=1}^{64} K_i \in \mathcal{J}_{4h}$ with $K_i \in \mathcal{J}_h$. The interpolation operator Π_{4h}^3 and I_{4h}^3 can be defined by:

$$
\begin{aligned}
&\Pi_{4h}^3 E \in Q_{333}(\tau), \qquad I_{4h}^3 H \in Q_{333}(\tau), \\
&\int_{K_i} (\Pi_{4h}^3 E - E)\,dxdydz = 0, i = 1, 2, \cdots, 64. \\
&\int_{K_i} (I_{4h}^3 H - H)\,dxdydz = 0, i = 1, 2, \cdots, 64,
\end{aligned}
$$

with the properties

$$
\begin{aligned}
&\Pi^3_{4h}\pi_h E = \Pi^3_{4h}E,\\
&\|\Pi^3_{4h}v\|_0 \le \|v\|_0, \quad \forall v \in V_h,\\
&\|\Pi^3_{4h}E - E\|_0 \le ch^4\|E\|_4, \forall E \in (H^4(\Omega))^3,\\
&I^3_{4h}I_h H = I^3_{4h}H,\\
&\|I^3_{4h}w\|_0 \le \|w\|_0, \quad \forall w \in W_h,\\
&\|I^3_{4h}H - H\|_0 \le ch^4\|H\|_4, \forall H \in (H^4(\Omega))^3.
\end{aligned}
$$

Theorem 5.2. *Under the condition of* **Theorem 5.1**, *we have the following estimations,*

$$\|\Pi^3_{4h}E_h - E - h^2\tilde{E}\|_0 = O(h^4)(\|E\|_4 + \|H\|_5), \tag{46}$$

$$\|I^3_{4h}H_h - H - h^2\tilde{H}\|_0 = O(h^4)(\|E\|_4 + \|H\|_5). \tag{47}$$

Proof: By **Theorem 5.1**, and the properties of interpolation operator Π^3_{4h}, we have

$$
\begin{aligned}
&\Pi^3_{4h}E_h - E - h^2\tilde{E} = \Pi^3_{4h}(E_h - \pi_h E - h^2\tilde{E}_h)\\
&+(\Pi^3_{4h}\pi_h E - E) + h^2(\Pi^3_{4h}\tilde{E}_h - \tilde{E})\\
&= \Pi^3_{4h}(E_h - \pi_h E - h^2\tilde{E}_h) + (\Pi^3_{4h}E - E)\\
&+h^2(\Pi^3_{4h}\tilde{E}_h - \Pi^3_{4h}\pi_h\tilde{E}) + h^2(\Pi^3_{4h}\pi_h\tilde{E} - \tilde{E})\\
&= O(h^4)(\|E\|_4 + \|H\|_5) + h^2\Pi^3_{4h}(\tilde{E}_h - \pi_h\tilde{E}) + h^2(\Pi^3_{4h}\tilde{E} - \tilde{E})\\
&= O(h^4)(\|E\|_4 + \|H\|_5).
\end{aligned}
$$

Here, we also need the superclose results and the regularity property of $\tilde{E}$. Therefore, we get (46). The same to (47). Then we can finish the proof. □

In order to use extrapolation technique, we can divide each $K_i \in \mathcal{J}_h$ into 8 small congruent element $K_{i,j,k} \in \mathcal{J}_{h/2}, (i,j,k = 1,2)$, and the corresponding nonconforming mixed finite element space is denoted by $V_{h/2} \times W_{h/2}$. Let $(E_{h/2}, H_{h/2}) \in V_{h/2} \times W_{h/2}$ and (Π^3_{4h}, I^3_{4h}) be the finite element approximation and the interpolation operator with respect to the new partition.

With the help of **Theorem 5.2**, we can improve the accuracy by applying the Richardson extrapolation.

Let $\check{E}_h = \Pi_{4h}^3 E_h, \check{H}_h = I_{4h}^3 H_h$. Compute $(E_h^{extra}, H_h^{extra})$ by the following formulas:

$$E_h^{extra} = \frac{4\check{E}_{h/2} - \check{E}_h}{3}, \tag{48}$$

$$H_h^{extra} = \frac{4\check{H}_{h/2} - \check{H}_h}{3}. \tag{49}$$

Theorem 5.3. *Under the condition of* **Theorem 5.1***, we have the following estimations to* $(E_h^{extra}, H_h^{extra})$,

$$\|E_h^{extra} - E\|_0 = O(h^4)(\|E\|_4 + \|H\|_5), \tag{50}$$

$$\|H_h^{extra} - H\|_0 = O(h^4)(\|E\|_4 + \|H\|_5). \tag{51}$$

Proof: First, we prove (50). By **Theorem 5.2**, we have

$$\begin{aligned} &4\check{E}_{h/2} - \check{E}_h - 3E = 4(\check{E}_{h/2} - E - (\frac{h}{2})^2\tilde{E}) - (\check{E}_h - E - h^2\tilde{E}) \\ &= O(h^4)(\|E\|_4 + \|H\|_5). \end{aligned}$$

Then (50) follows. The same to (51). Now we can finish the proof. □

6 Conclusion

In this paper, we propose a new nonconforming mixed finite element approximation to the three-dimensional time-harmonic Maxwell's equations. By asymptotic expansion of interpolation error and nonconforming error, we can make a conclusion that this mixed finite element has superclose property. Global superconvergence property is also achieved by constructing a postprocess operator, which means it can improve the approximation order from $O(h)$ to $O(h^2)$ with boundary condition $E \times \mathbf{n} = 0$. Furthermore, by constructing extrapolation operators, we can present $O(h^4)$ approximations in the sense of L^2-norm.

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