Computational Optimization Problems in Practical Finance

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Is computational finance $\subset$ computational optimization?

No, but {computational optimization}

$\cap$

{computational finance}

=

a significant set

Objectives of this talk…

1. Entertainment

2. Intro to some of the practical problems of computational finance

3. Illustration of the important role that optimization can play (but be careful! Look out for solution sensitivity to problem parameters, robustness, conditioning of problem).

5 Computational Finance Problems (with optimization solutions)

Problem 1: The Implied Volatility Surface Problem

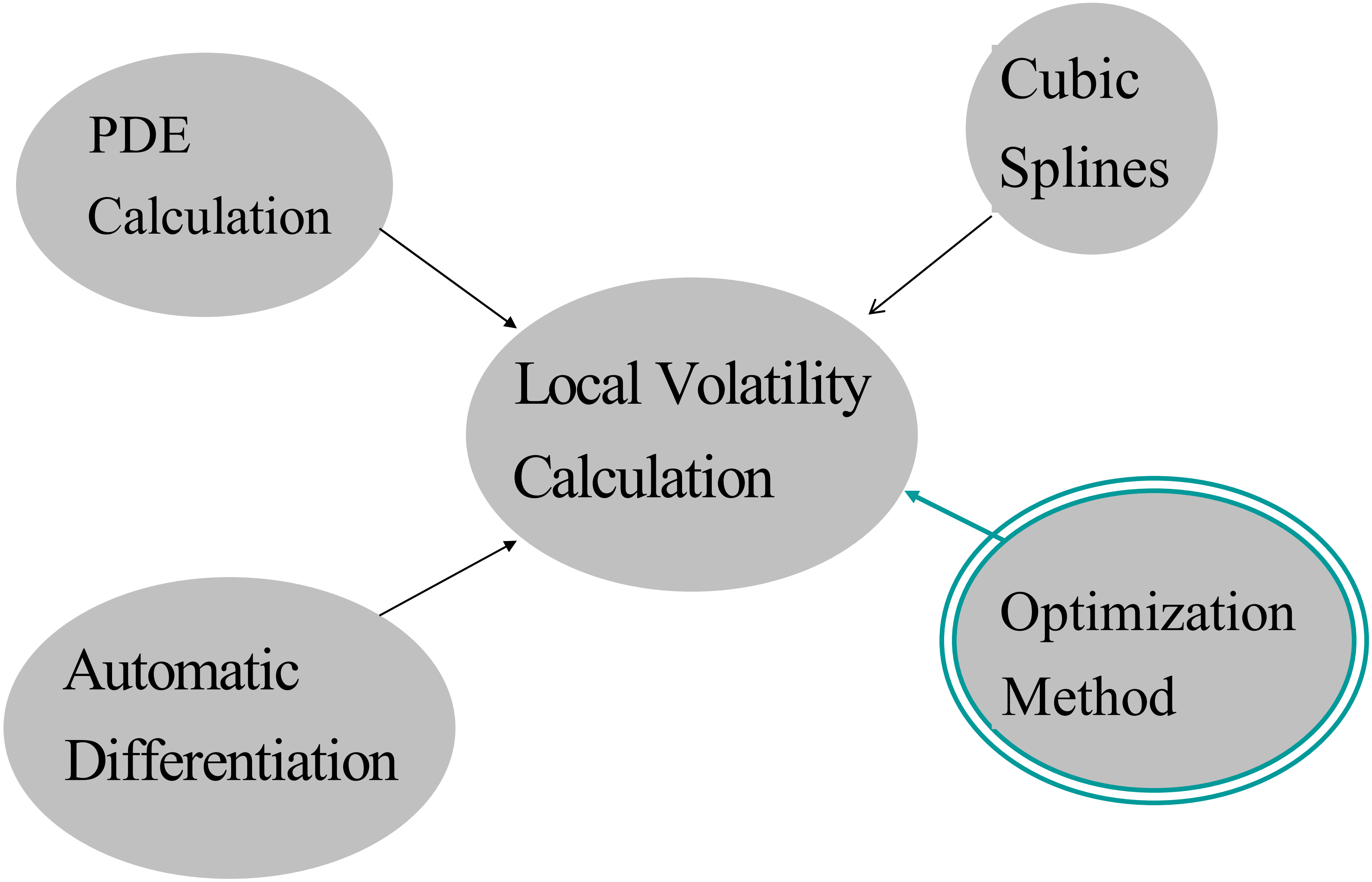
Problem 2: The Incomplete Market Hedging Problem (A: local)

[Problem 3: The Incomplete Market Hedging Problem (B: global)]

Problem 4: The Portfolio of Derivatives Hedging Problem

Problem 5: The Optimal VaR/CVaR Problem

Problem 1: The Implied Volatility Surface Problem


PDE
Calculation
Cubic
Splines
Local Volatility
Calculation
Automatic
Differentiation
Optimization
Method

Elementary (solved) questions

1. How to fairly price (vanilla) options?
2. How to determine the volatility parameter σ (needed for 1)?
   Useful for pricing other (exotic options), hedging, …

Background: **Vanilla put option** – The buyer has the option (not the obligation) to sell the underlying at strike price K at time (maturity) T.

Vanilla call option – The buyer has the option (not the obligation) to buy the underlying at strike price K at time (maturity) T.

The answer to 1:

Assuming geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

unobservable

+ complete market, no arbitrage, constant (future) volatility..
the unique fair price is given by

W : Brownian motion
μ : constant, the drift
σ : constant, the volatility

Black-Scholes Solution

$$\frac{\partial v}{\partial t} + (r - q)s\frac{\partial v}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial^2 s} = rv$$

r : risk-free interest rate

q : dividend rate

- $\exists$ natural boundary conditions

No μ!

Solving The B-S Equation

Given volatility, B-S is easy to solve:

e.g.,

•evaluating a discretized PDE backwards in time
•evaluating a binomial/trinomial tree
•explicit soln (using l cumulative normal dist'n lookup)

•**The problem**: how to get σ

Implied Volatility

The answer: assume todays vanilla options are well-priced by the market and solve the inverse problem!

$$F(\sigma) - value = 0$$

Known, trusted

But this leads to a non-constant σ (i.e., different data points yield diff. answers)

In fact, it appears

$$\sigma = \sigma(S_t, t)$$

Option pricing model: 1-Factor Continuous Diffusion Approach

$$\frac{dS_t}{S_t} = \mu(S_t, t)dt + \sigma\ (S_t, t)dW_t$$

W: standard Brownian motion

μ, σ: deterministic functions

$\sigma(s, t)$: local volatility function

Fair price for vanilla option :

Generalized Black - Scholes:

$$\frac{\partial v}{\partial t} + (r - q)s\frac{\partial v}{\partial s} + \frac{1}{2}\sigma(s,t)^2 s^2 \frac{\partial^2 v}{\partial^2 s} = rv$$

No μ!

Evaluation

Given the vol surface $\sigma = \sigma(S_t, t)$

Numerical approaches can be used to solve the generalized B-S equation.
But,

How to get $\sigma = \sigma(S_t, t)$?

Optimization answer 1 (bad)

Take todays (trusted) prices and invert GBS model to extract vol surface (generalize the 1-D case):

$$\min \left\| F(\sigma) - values \right\|_2$$

GBS → F; unknown surface → σ; trusted prices → $values$

Why bad? 1. too curvaceous (can be smoothed but…)

2. too many optimization variables (number of grid points when $\sigma = \sigma(S_t, t)$ is discretized (too few values))

Optimization answer 2 (better)

Add as smoothing (regularization) term:

$$\min \left\| F(\sigma) - values \right\|_2 + \text{smoothing term}$$

But:

1. Still thousands of variables (nonlinear obj fcn)

2. How to balance the 2 objectives

Optimization answer 3 (best)

Model the implied vol surface by a bi-cubic spline **form,** with *p* unknown knot values: $\overline{\sigma} = (\overline{\sigma}_1, \ldots, \overline{\sigma}_p)$

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To determine $\bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_p)$ solve

$$\min f(\bar{\sigma}) \equiv \left\| F(v(\bar{\sigma})) \right\|^2$$

The optimization solution

The problem $$\min f(\bar{\sigma}) \equiv \left\| F(v(\bar{\sigma})) \right\|^2$$

Is nonlinear least-squares (as before). But the number of unknowns is the number of knot points *p* which can be chosen

$$p \leq \text{\#option values available}$$

A bit of 'art' needed here

Smoothness: built- in!

Example…

Problem 1 moral:

Design an optimization approach, if possible, so that the number of optimization variables is small but appropriate to the available information.

Emphasize smoothness, not just ‘matching’ the data.

Problem 2: The Incomplete Market Hedging Problem (A: local)

2 philosophical points

1. Many hedging strategies **assume a complete market** (which implies continuous hedging). Then, after all the theorems: '**in practise' we will (of course) only hedge at discrete times** '(which implies an incomplete market).

So, perhaps better to assume reality to begin with (but of course fewer theorems, fewer papers,…)

2. Least-squares minimization has many advantages, especially theoretical (more theorems!). But absolute-value minimization pays less attention to outliers and can yield better 'average case' results.

The Setting….

- *T>0:* Expiry of a European option

- Discrete hedging dates: $0 = t_0 < t_1 < \ldots < t_M = T$
  Hence, incomplete market

- (Ω, F, P) prob. space with filtration $(F_k)_{k=\overline{0,M}}$

- trivial: $F_0 = \{\varnothing, \Omega\}$

- $(X_k)_{k=\overline{0,M}} : F_k -$ measurable discounted asset price process

- Bond price *B=1*

- H: an $F_T -$ measurable random payoff for an option

…The Setting….

- **Hedging** portfolio value at t_k : $V_k = \varepsilon_k X_k + \eta_k$
    - ε_k : Units of underlying held at t_k
    - η_k :Units of bonds held at t_k

Where, $(\varepsilon_k)_{k=\overline{0,M}}$ and $(\eta_k)_{k=\overline{0,M}}$

denote a hedging strategy

2 Definitions:

•**Accumulated Gain** (change in value of the hedging portfolio due to change in stock price before any changes in the portfolio):

$$G_k = \sum_{j=0}^{k-1} \varepsilon_j (X_{j+1} - X_j), \quad 1 \le k \le M, \quad G_0 = 0$$

•**Cumulative Cost :** $C_k = V_k - G_k, \quad 0 \le k \le M$

(Self-financing if $C_0 = C_1 = \ldots = C_M$

i.e., $(\varepsilon_{k+1} - \varepsilon_k) X_{k+1} + \eta_{k+1} - \eta_k = 0$)

Local risk minimization

- The cost at *M* is *H* ; let $V_M = H \ (\eta_M = H)$

- General idea: Choose a hedging strategy so that

$$C_{k+1} - C_k \approx 0, \quad 0 \le k \le M - 1$$

- E.g., local quadratic risk minimization:

$$\text{for } k = M-1, M-2, \ldots, 0, \\ \min \mathrm{E}((C_{k+1} - C_k)^2 \mid F_k)$$

An alternative → L1 Incremental Cost

Quadratic measure may be less than ideal:

Larger incremental costs heavily weighted

Not in monetary units

L1-measure:

$$V_M = H, \text{ for } k = M-1, M-2, \ldots, 0$$

$$\min \; E(|C_{k+1} - C_k| \quad | F_k)$$

2 methods: local L1 minimization, local L2 minimization

•**M**ethod 1:

$$\min \ E(|C_{k+1} - C_k| \ \ | F_k)$$

•**M**ethod 2:

$$\min \ E((C_{k+1} - C_k)^2 \ \ | F_k)$$

Implementation

Suppose stock price modeled using a binomial tree with *N* periods

Hedging can take place at $M << N$ times at dates

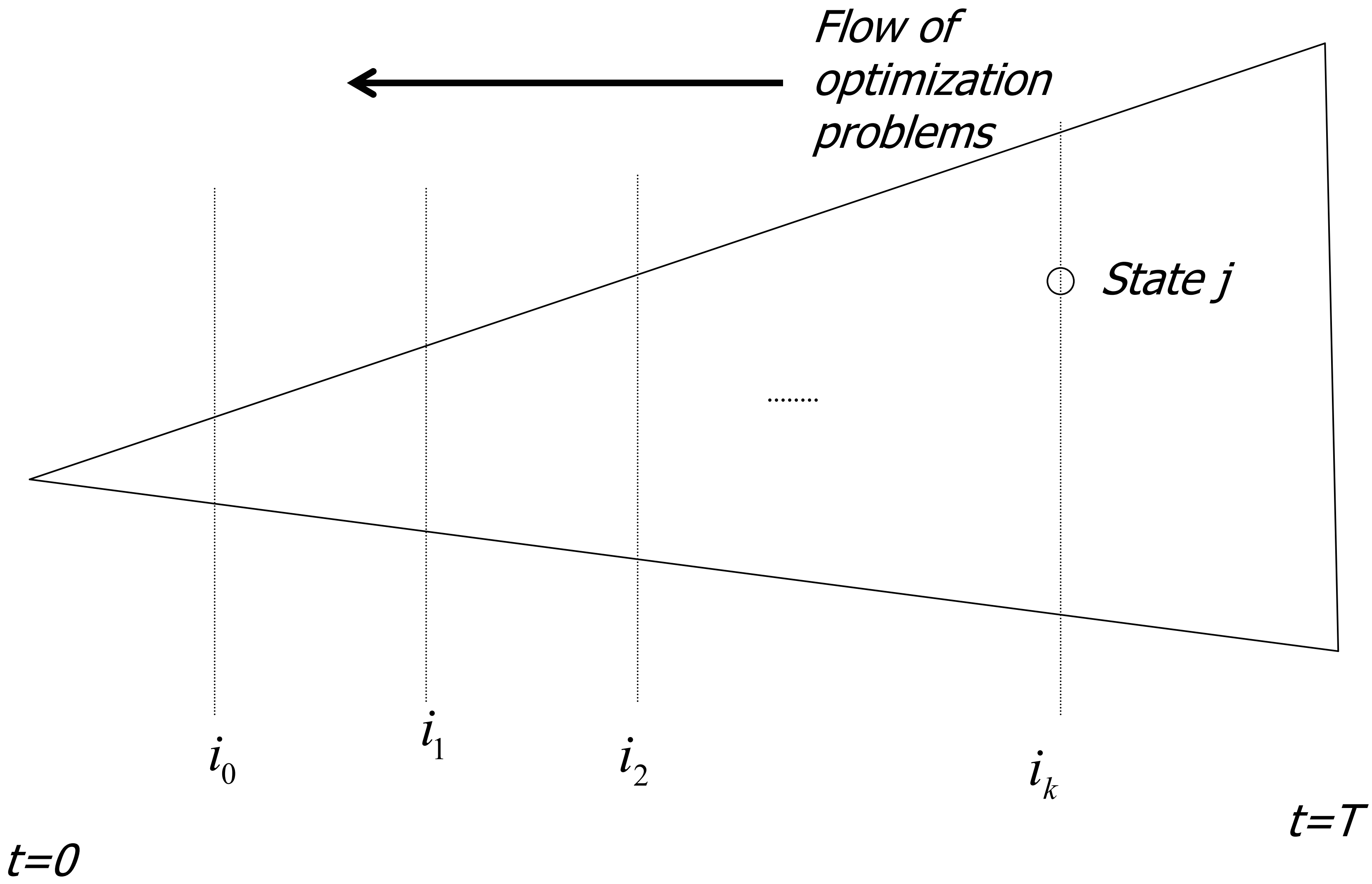
$$0 = i_0 < \ldots < i_{M-1} < N := i_M$$

Hence, at time i_k there are $n_k = i_k + 1$
possible states for the stock price.

Given state *j* at time i_k the stock price can only move to

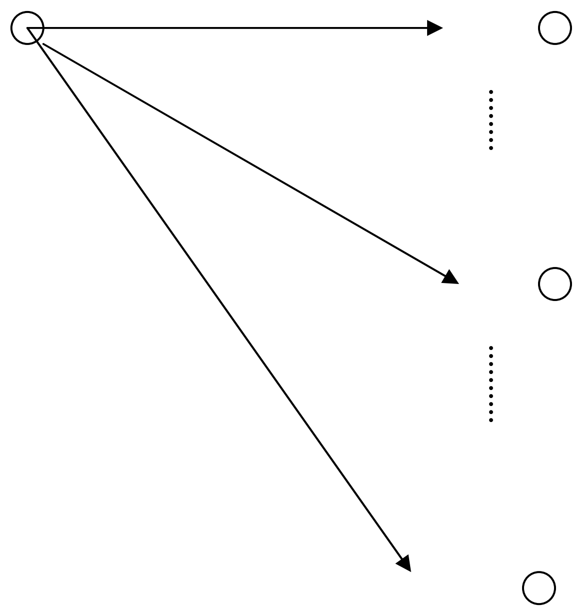
$\alpha_k = i_{k+1} - i_k + 1$ possible states

….Implementation…



…Implementation

S_k^j



$$S_{k+1}^{j} = u^{\alpha_k - 1} S_k^j$$

$$S_{k+1}^{j+i-1} = u^{j} d^{\alpha_k - 1 - j} S_k^j$$

$$S_{k+1}^{j+\alpha_k - 1} = d^{\alpha_k - 1} S_k^j$$

$$X_k^j = \frac{S_k^j}{B_k^j}$$

…Implementation…

So at each hedging time t_k for each state *j*
The optimization problem to solve is

$$\min \; E(|C_{k+1} - C_k| \quad | X_k = X_k^j)$$

For k=M-1,…,0

For each state j

$$\min_{\varepsilon_k^j, \eta_k^j} \sum_{l=0}^{\alpha_k - 1} p_l \, | X_{k+1}^{j+1} (\varepsilon_{k+1}^{j+1} - \varepsilon_k^j) + (\eta_{k+1}^{j+1} - \eta_k^j) |$$

…Implementation…

With a bit of manipulation…

For k=M-1,…,0

For each state j

$$\min_{z^j \in R^2} \left\| A^j z - b^j \right\|_1$$

Where matrix A^j is α_k-by-2

Computational results

- **A**ssume $\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t \text{ where } Z_t$ is a standard Brownian motion
- **A** binomial tree is calibrated to this process, e.g., CRR
- **A**ssume

$$T = 1,\ S_0 = 100$$
$$\mu = .2,\ \sigma = .2$$
$$r = .1,\ \text{\#periods=600}$$

- **C**onsider European *put* options with different strike prices

Performance measures

(Discounted) **incremental cost** (risk**):**

$$\frac{1}{M}\sum_{k=0}^{M-1} | C_{k+1} - C_k |$$

Multiple Rebalancing Times

Less frequent →

Rebalance every K periods

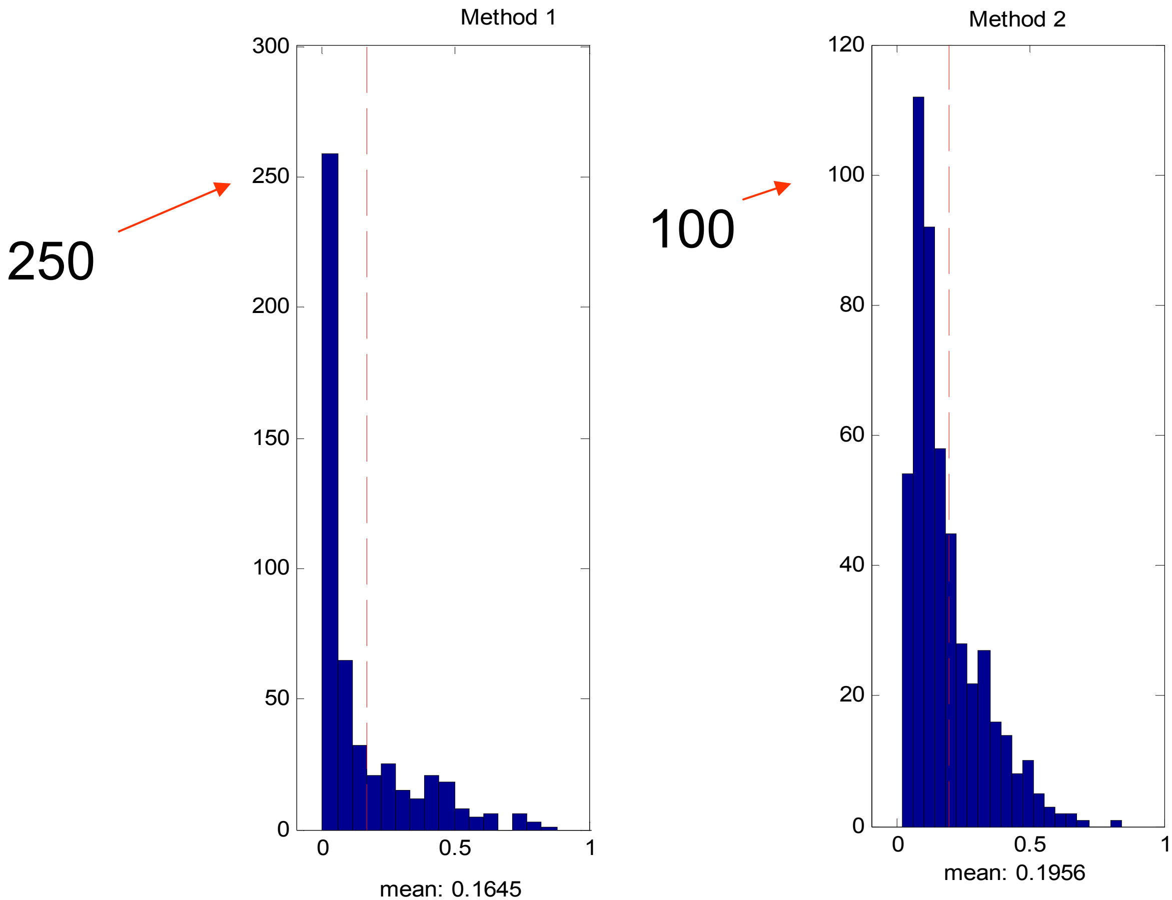
Strike	Md	1	10	50	100	300	600
95	1	0.0047	0.0343	0.1645	0.2920	0.7459	1.0886
	2	0.0047	0.0394	0.1956	0.3600	1.0762	1.9194
100	1	0.0060	0.0458	0.2425	0.4457	1.2655	1.8960
	2	0.0060	0.0474	0.2518	0.4750	1.4841	2.7052
105	1	0.0072	0.0648	0.3238	0.5920	1.9134	3.0545
	2	0.0072	0.0612	0.3042	0.5820	1.9053	3.5402

In

$s_0 = 100$

Average Incremental Cost, Risk (500 Simulations)

Histogram of incremental Costs(risks): strike =95, monthly rebal.



500 simulations

Strike = 95 and Monthly Rebalancing

Total Cost:

Method 1: **70%** less than mean.   **55%** less than ½ mean

Method2: **51%** less than mean. **12%** less than ½ mean

Incremental Cost:

Method 1: **69%** less than mean.   **58**% less than ½ mean

Method 2: **63%** less than mean.   **30%** less than ½ mean.

Problem 2 Moral

1. Many hedging strategies **assume a complete market** (which implies continuous hedging). Then, after all the theorems: '**in practise' we will (of course) only hedge at discrete times** '(which implies an incomplete market).

So, perhaps better to assume reality to begin with (but of course fewer theorems, fewer papers,…)

2. Least-squares minimization has many advantages, especially theoretical (more theorems!). But absolute-value minimization pays less attention to outliers and can yield better 'average case' results.

[Problem 3: The Incomplete Market Hedging Problem (B: global)]

Problem 4: The Portfolio of Derivatives Hedging Problem

Philosophical motivating points

1. Derivative portfolio hedging problems are often ill-posed

2. Hedge risk minimization can be preferable to hedging by sensitivities

3. Watch out for stochastic vol

The setting and the problem

The problem: Effectively hedge a large portfolio of derivative instruments

Formalize:

Risk factors: $S \in \Re^d$

Hedging instruments: $\{V_1, \cdots, V_n\}$, $V_i(S,t)$ value at time t

Value of ***hedging*** *portfolio:* $\pi(x,S,t) = Vx$ where $V = [V_1, \cdots, V_n]$

Value of ***target*** *portfolio at time* t*:* $\pi^0(S,t)$

Sensitivities of hedging instruments

$$\frac{\partial V}{\partial t} = \left[\frac{\partial V_1}{\partial t}, \cdots, \frac{\partial V_n}{\partial t} \right] \in \Re^{1 \times n}$$

n

$$\frac{\partial V}{\partial S} = \left[\frac{\partial V_1}{\partial S}, \cdots, \frac{\partial V_n}{\partial S} \right] \in \Re^{d \times n} =$$

d

$$\Gamma = \left[\Gamma_1, \cdots, \Gamma_n \right] \in \Re^{d \times n}, \text{ where } \Gamma_i = \left[\frac{\partial^2 V_i}{\partial S_1^2}, \cdots, \frac{\partial^2 V_i}{\partial S_d^2} \right]^T$$

=

n

d

* for simplicity of presentation we assume here that each hedging instrument V_i depends on exactly one risk factor

The Hedge Risk Minimization Approach

We measure risk as the expected quadratic replicating error at time *t*:

$$\min_{x \in \Re^n} risk(x) \equiv \mathrm{E}\left\{ [\sum_{1=1}^{n} x_i V_i(S,t) - \pi^0(S,t)]^2 \right\}$$

The problem is ill-posed

To see that this problem is also ill-posed, suppose first that the hedging change is specified by delta-gamma approximation:

$$V_i(S,t) - V_i(S_0,0)$$

$$= \left(\frac{\partial V_i}{\partial t}\right)\delta t + \left(\frac{\partial V_i}{\partial S}\right)^T \partial S + \frac{1}{2}(\partial S)^T (\frac{\partial^2 V_i}{\partial S^2})(\partial S)$$

$$= \left(\frac{\partial V_i}{\partial t}\right)\delta t + \left(\frac{\partial V_i}{\partial S}\right)^T \partial S + \frac{1}{2}\Gamma_i^T (\partial S)^2$$

Infinite number of hedge risk minimizers

In this delta-gamma setting, if each hedge instrument depends on a single risk factor then

$$n > 2d + 1 \;\Rightarrow\; \text{infinite number of risk minimizers}$$

In this delta-gamma setting, allowing each hedge instrument to depend on several risk factors:

$$n > d + \text{sparsity indicator*} \;\Rightarrow\; \text{infinite number of risk minimizers}$$

*defined in paper

More generally, the problem is very ill-conditioned

If we move away from the delta-gamma setting, the resulting problem can be **very ill-conditioned**. For example, assume a single risk factor, a stock price defined by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dX_t$$

Experiments defined by

$$S_0 = 100,\ \sigma = .2,\ \mu = .1,\ r = .04$$

And 21 hedging instruments: underlying + vanilla calls with maturities 1,2,3,6 months and strikes [90,95,100,105,110].

Continue example…

Hedge risk minimization becomes:

$$\min_{x \in \Re^n} risk(x) = \frac{1}{m} \left\| \underline{\mathrm{V}} x - b \right\|_2^2 \text{, where}$$

$$\underline{\mathrm{V}} = \begin{bmatrix} \mathrm{V}_1(S^1, t) & \mathrm{V}_2(S^1, t) & \cdots & \mathrm{V}_\mathrm{n}(S^1, t) \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{V}_1(S^m, t) & \mathrm{V}_2(S^m, t) & \cdots & \mathrm{V}_\mathrm{n}(S^m, t) \end{bmatrix}, \ b = \begin{bmatrix} \pi^0(S^1, t) \\ \vdots \\ \pi^0(S^m, t) \end{bmatrix}$$

Continue example…ill-conditioned matrix and (impractically) large positions

Choosing *m=20,000 and hedge horizon = 1 month,*

$$cond(\underline{V}) \cong 10^{16}$$

For different target portfolios of

100 vanilla options P_V, binary options P_{bi}, barrier options, P_{ba}, and mix (plus some Asian options) P_m

	P_v	P_{bi}	P_{ba}	P_m
$\sqrt{\text{risk}}$	7.27e-4	3.05e-3	8.04e-3	7.05e-2
$\|\|x^*\|\|_1$	1.93e+6	3.08e+6	7.47e+7	1.87e+7

The point of this example…

Minimizing hedge risk alone yields massively ill-conditioned problems, and ridiculously large holdings.

However, incorporating realistic costs and bounds can yield better problems, more practical solutions…

Adding management costs and bounds

Bounds, $l_x \le x \le u_x$, can limit extreme positions and can help control initial formulation costs.

Management costs are related to both the number of different instruments in the portfolio, and the size of the positions.

Our approach to address both problems simultaneously:

$$\min_{x \in \Re^n} \{risk(x) + \alpha \sum_{i=1}^{n} c_i |x_i| : \ l_x \le x \le u_x\}$$

Per unit cost

Balance between risk and cost

Why 1-norm penalty?

There exists a finite threshold value of αc_i
For which the optimal solution has a zero holding of instrument *i*

So, as α *increases, the number of zero holdings increases.*

control

An alternative formulation:

$$\min_{x \in \Re^n} \sum_{i=1}^{n} c_i \,|\, x_i \,|$$

$$l_x \le x \le u_x$$

$$\sqrt{\text{risk(x)}} \le \mu_r$$

control

Example results: Binary options

	Model 0	Model 1	Model 2
# active instruments	19	21	5
$\sqrt{\text{risk*}}$	3.05e-3	3.41e-2	5.00e-1
$\|\|x^*\|\|_1$	3.08+6	1.87e+3	7.63e+2

$$\sqrt{risk(0)} = \mathrm{E}(\pi^0) = \cos t_0 = 1.8e+2$$

Method 0: no constraints, Method 1: bounds only

Method 2: 1-norm + bounds

Incorporating volatility uncertainty

Note that we have assumed that the future implied vol, $\sigma_{\bar{t}}$, is the same as current implied vol. Suppose we assume this in our computation but in reality,

$$\sigma_{\bar{t}} \in N(\sigma_0, \sigma_{vol})$$

Implied vol at t=0

Standard deviation

Sensitivity to errors in future vol

Suppose it is assumed, in our computation, that future vol is the same as current vol and <u>x is chosen by solving</u>

$$\min_{x \in \Re^n} risk(x) \equiv \mathrm{E}\left\{ [\sum_{1=1}^{n} x_i V_i(S,t) - \pi^0(S,t)]^2 \right\}$$

Next assume that in reality

$$\sigma_{\bar{t}} \in N(\sigma_0, \sigma_{vol})$$

And compute

$$\prod^*(x,\sigma,S) - \prod^0(x,\sigma,S)$$

…model 0 is a disaster under vol error

√ **R**isk when $\sigma_{\bar{t}} \in N(.2,.005)$

	Model 0	**Model 1**	**Model 2**
P_v	6.28e+3	3.38e+2	3.46e+1
P_{bi}	1.46e+4	8.54e+0	5.84e-1
P_{ba}	1.73e+5	5.75e+1	7.57e+1
P_m	2.77e+5	1.36e+2	3.24e+1

Why so large! ?

…More model 0 under vol error

Extreme sensitivity due to **large positions** model 0 incurs combined

with **ill-conditioning** of the problem, combined with minimization using

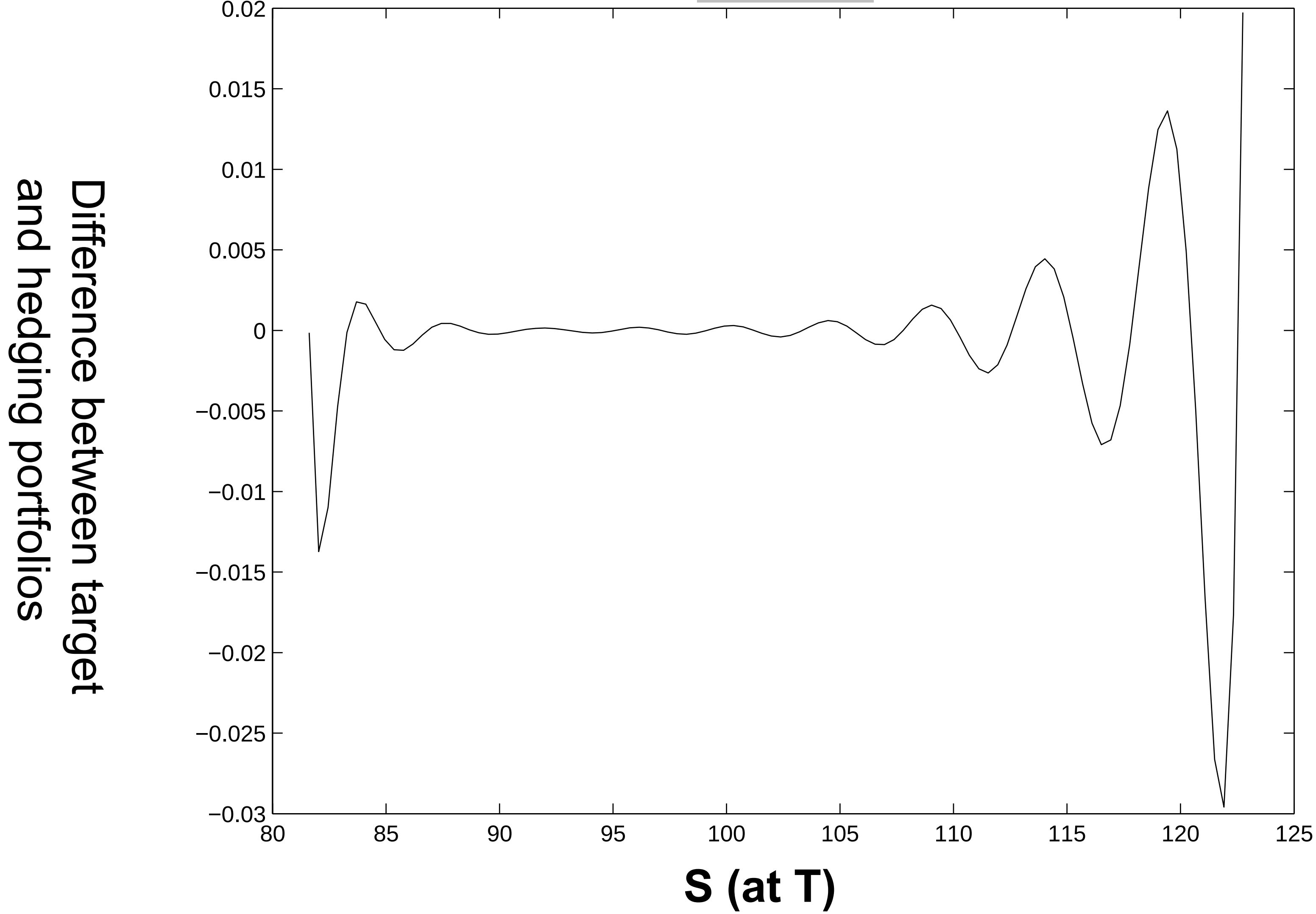
just a **single** value of σ

Minimization does do a good job reducing risk if future σ

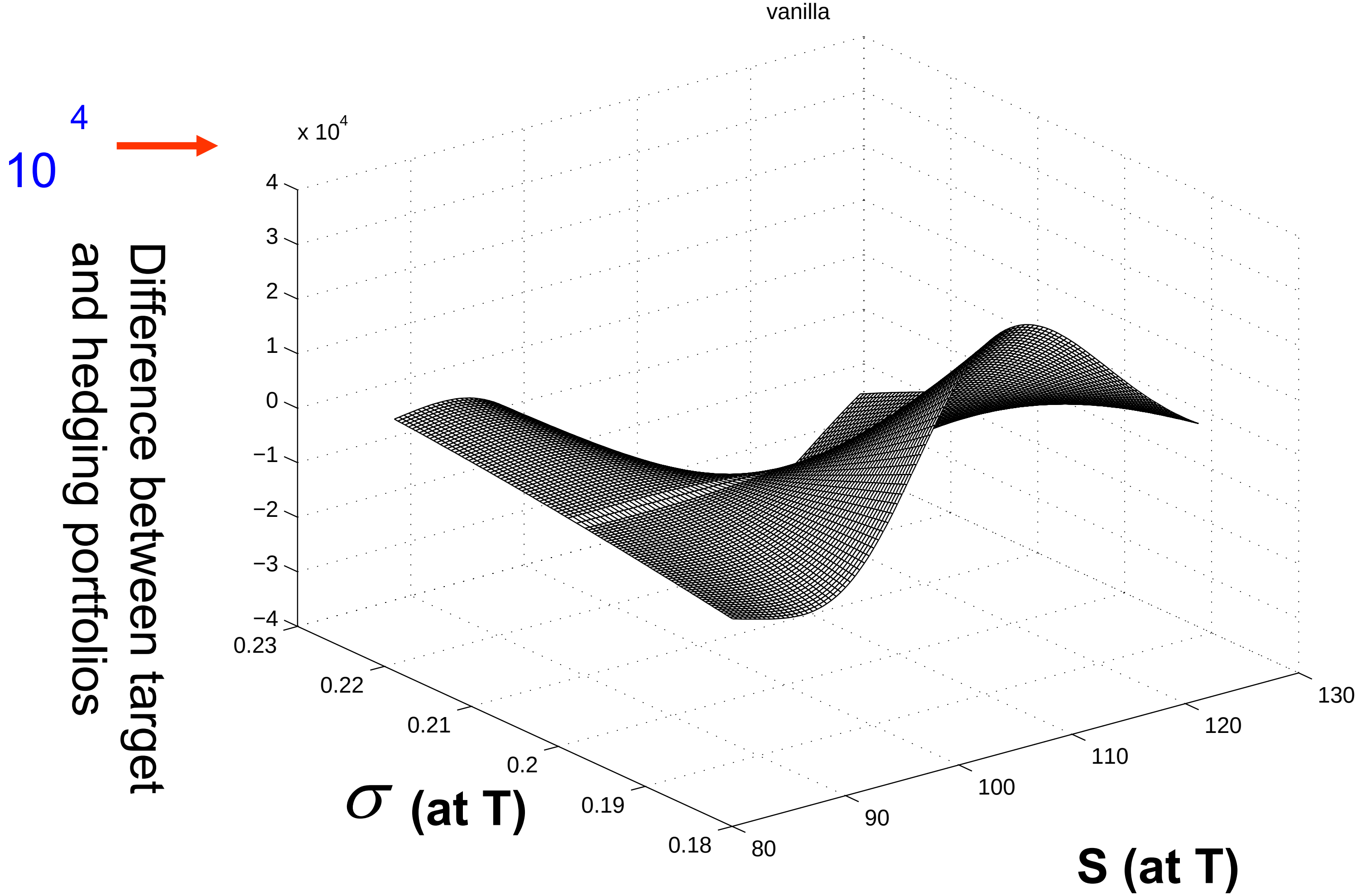
equals current value………………….

If vol is constant…

$$\sigma = .2$$



Model 0 sensitivity to vol error



…More model 0 under vol error

To help ameliorate this effect, assume volatility is stochastic: Still,

$$risk(x) = \frac{1}{m}\left\| \underline{V}x - b \right\|_2^2$$

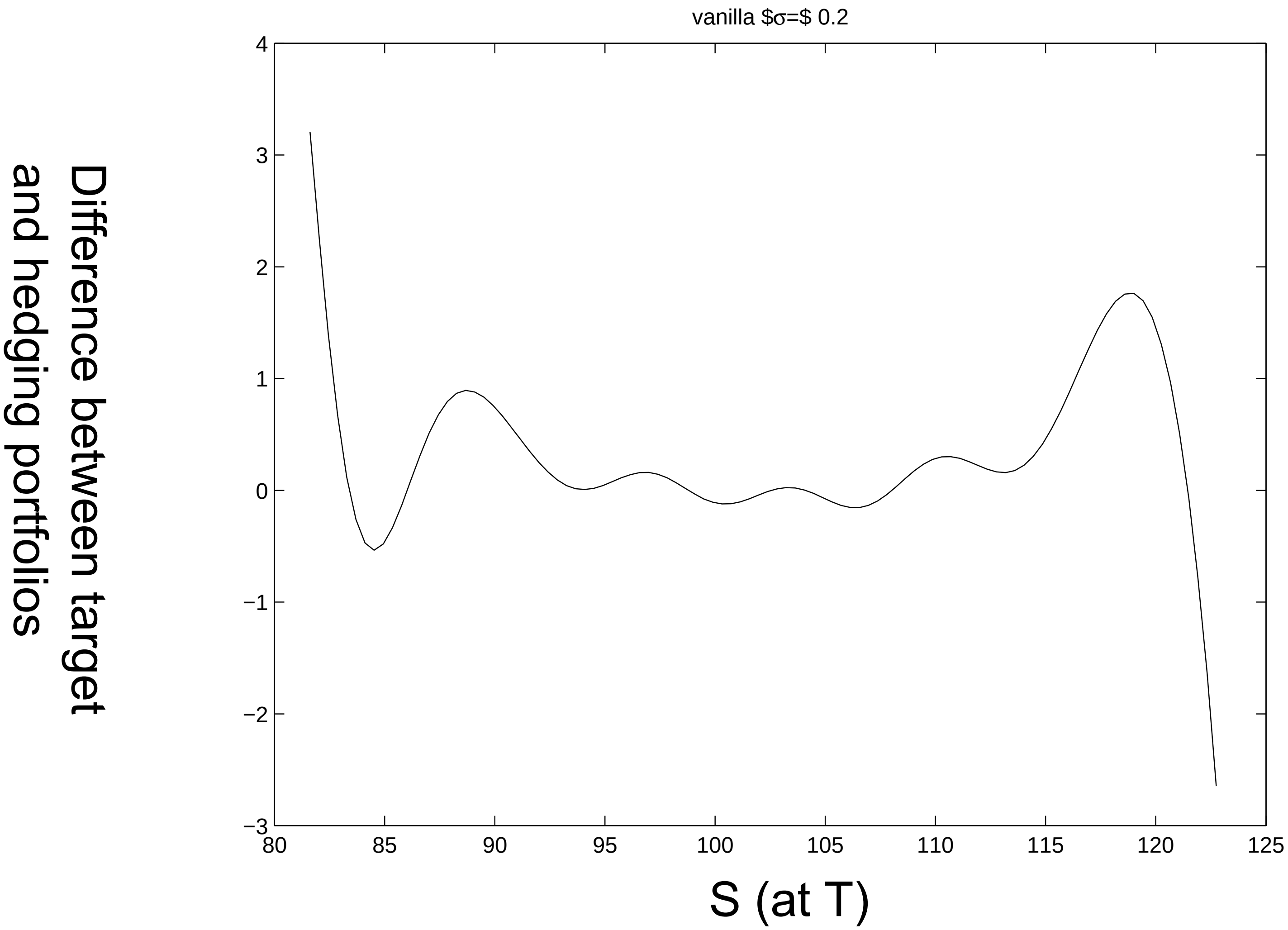
but

$$\underline{V} = \begin{bmatrix} V_1(S^1, \sigma^1, t) & V_2(S^1, \sigma^1, t) & \cdots & V_n(S^1, \sigma^1, t) \\ \vdots & \vdots & \cdots & \vdots \\ V_1(S^m, \sigma^m, t) & V_2(S^m, \sigma^m, t) & \cdots & V_n(S^m, \sigma^m, t) \end{bmatrix}, \; b = \begin{bmatrix} \pi^0(S^1, \sigma^1, t) \\ \vdots \\ \pi^0(S^m, \sigma^m, t) \end{bmatrix}$$

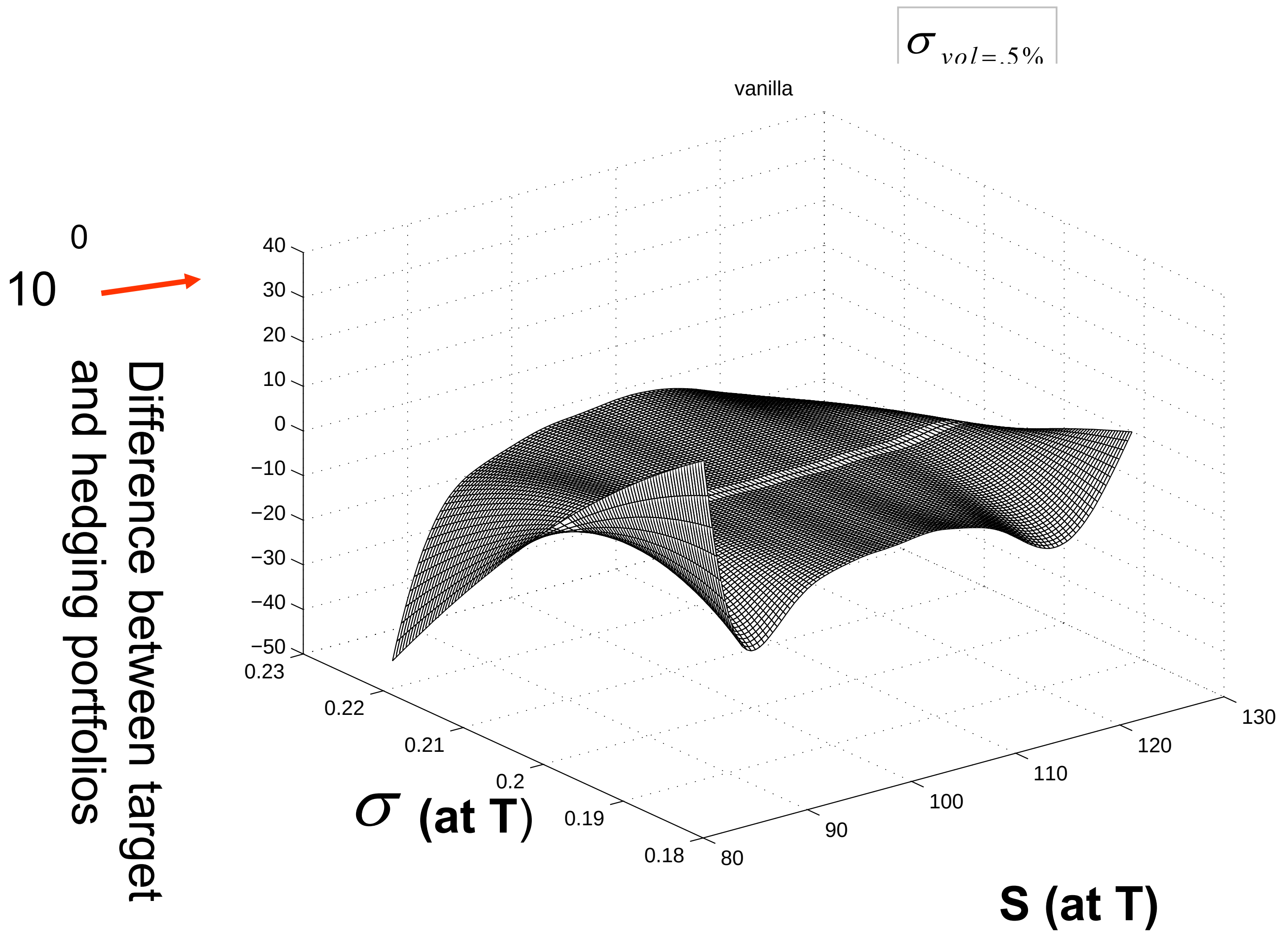
Now compute difference between target and hedging portfolios….

$$\prod^{*}(x,\sigma,S)-\prod^{0}(x,\sigma,S)$$

Not quite as good around initial vol…



..yields a much flatter variation surface



Example results when stochastic vol is included (binary option portfolio)

	Model 0	Model 1	Model 2
# active instruments	21	21	12
$\sqrt{risk^*}$	9.03e-02	1.07e-1	1.4e-1
$\|\|x^*\|\|_1$	1.32e+4	1.78e+3	7.54e+1

Problem 4 Moral

1. Hedging a portfolio of derivatives is often ill-posed
2. Adding bounds and management costs (in the 1-norm formulation) can stabilize and yield practical solutions (fewer instruments, smaller positions)
3. Further stabilizing can be achieved with incorporation of stochastic volatility

Problem 5: The Optimal VaR/CVaR Problem

The Problem

Given a set of derivative instruments (with values $V_1, V_2, \ldots, V_n$), dependent on a set $S \in \Re^d$ of risk factors, **how to choose an investment** $x \in \Re^n$ where x_i is the amount invested in instrument ***I*** *,* **to minimize the (conditional) value-at-risk**. (The worst (5%) losses.)

Some definitions

- Portfolio loss function: $f(x,S) = x^T(V^0 - V)$

- Probability density of S : $p(S)$

- Cumulative distribution function:

$$\psi(x,\alpha) = \int_{f(x,\alpha)\leq\alpha} p(S)dS$$

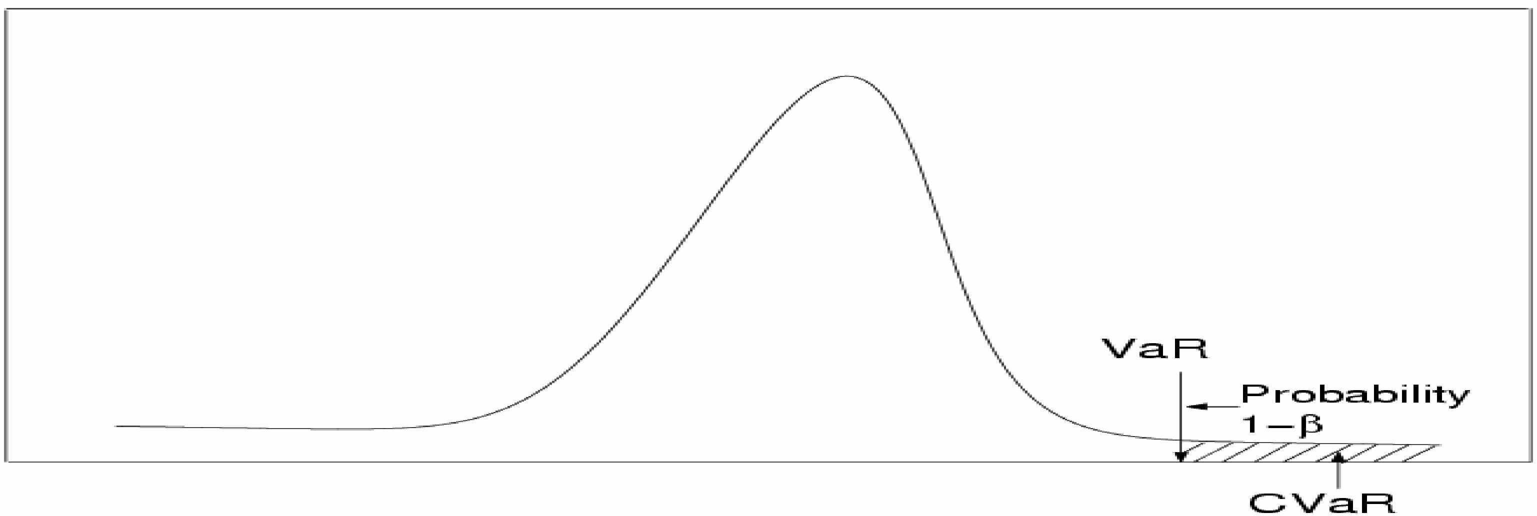
….more definitions

Value-at-risk (VaR) of a portfolio χ *for a confidence level* β,

$$\alpha_\beta(x) = \inf\{\alpha \in \Re : \psi(x,\alpha) \geq \beta\}$$

Conditional Value-at-risk (CVaR): **mean** of a the tail loss distribution

$$\phi_\beta(x) = \inf_\alpha(\alpha + (1-\beta)^{-1}\mathrm{E}(f-\alpha)^+))$$



Loss Distribution

Porfolio CVaR Optimization

Rockafellar & Uryasev: 1999,2002:

$$\min_x \phi_\beta(x) = \min_{(x,\alpha)\in X\times\Re} F_\beta(x,\alpha), \text{ where}$$

$$F_\beta(x,\alpha) = \alpha + (1-\beta)^{-1}\mathrm{E}[(f-\alpha)^+]$$

If

$f(\cdot, S)$ and X are convex then

$$\min_{(x,\alpha)\in X\times\Re} F_\beta(x,\alpha)$$

is a convex nonlinear programming problem.

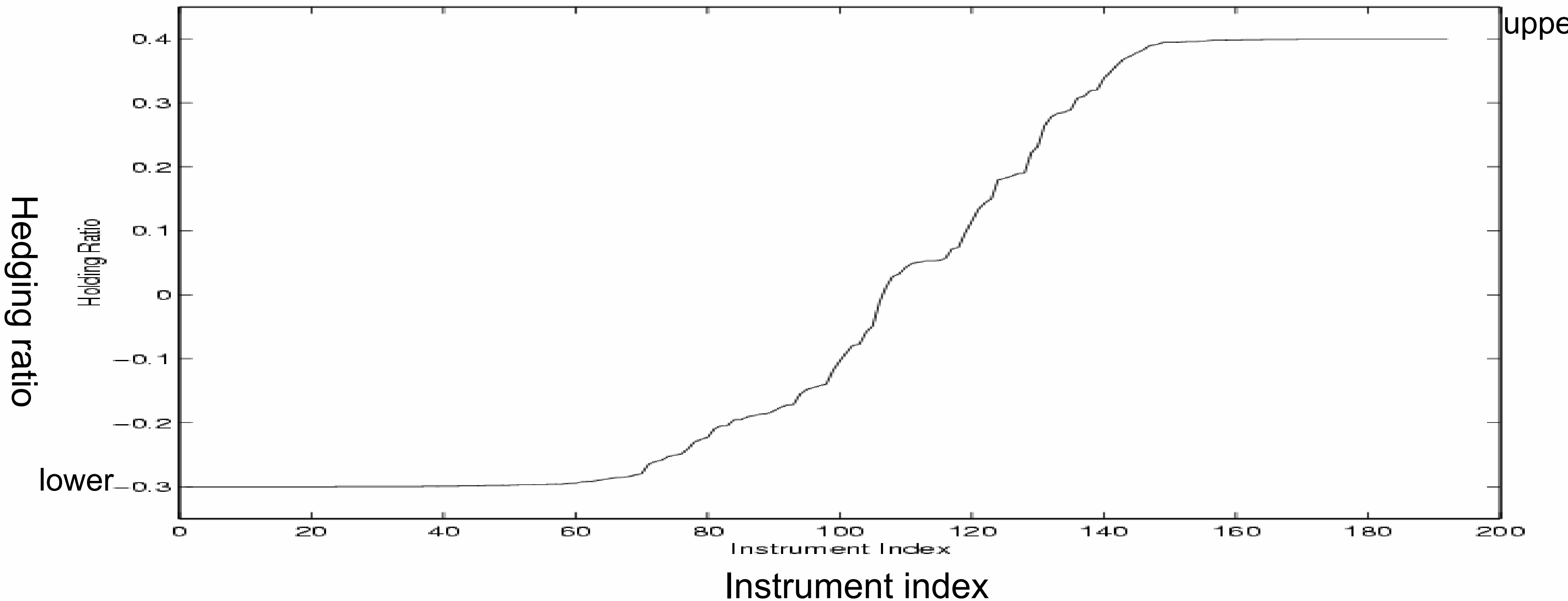
But,

Similar to the previous example,

CVaR/VaR minimization for portfolios of derivatives is ill-posed.

To see the effect of this ill-posedness, consider a typical CVaR solution:

Holdings



Hedging ratio
Holding Ratio
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
lower
uppe
0
20
40
60
80
100
120
140
160
180
200
Instrument Index
Instrument index

Properties and problems of the optimal portfolio

Properties:

1. The optimal portfolio contains all 192 instruments
2. 77% of the instruments are at their bounds

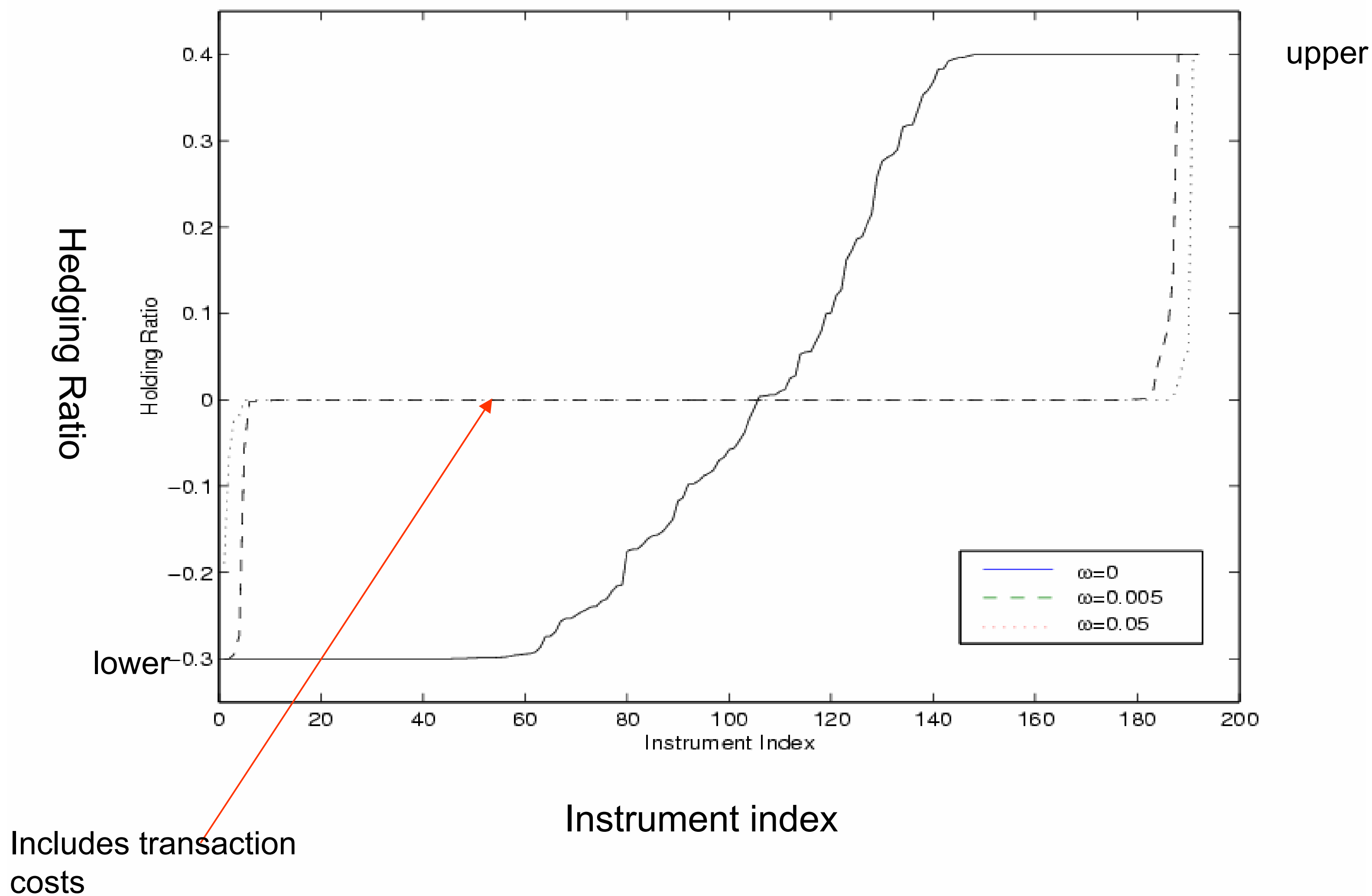
Practical Problems:

1. Large management and transaction costs
2. Magnification of the model error

A Solution: Add cost consideration to the CVaR objective:

$$\min_x (\phi_B(x) + \sum_{1=1}^{n} c_i \mid x_i \mid), \text{ where } c_i \text{ are pos. weights.}$$

Holdings



upper
lower
Hedging Ratio
Holding Ratio
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
0
20
40
60
80
100
120
140
160
180
200
Instrument Index
ω=0
ω=0.005
ω=0.05
Instrument index
Includes transaction costs

Minimizing CVaR (for portfolios of derivatives)

Can be formed as a large LP *(m simulations for* $\{(\delta V)_i\}_{i=1}^{m}$ *) :*

$$\min_{(x,y,z,\alpha)} \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} y_i$$

$$(V^0)^T x = 1$$

$$(\mathrm{E}(\delta V))^T x = \bar{r}$$

$$y_i \geq -(\delta V)_i^T x - \alpha, \; y_i \geq 0, \; i = 1, \ldots, m$$

$$l \leq x \leq u$$

LP Efficiency

	Mosek (cpu sec)			CPLEX (cpu sec)		
m	n=8	n=48	n=200	n=8	n=48	n=200
10000	11.1	61.9	1843.9	53.7	427.9	2120.8
25000	30.1	162.1	14744.6	351.44	2345.4	9907.9
50000	43.6	642.2	-	1573.82	9296.9	-

Removing the dependence on *m*

Note that

$$\bar{F}_\beta(x,\alpha) = \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} [(\delta V)_i^T x - \alpha]^+ \xrightarrow{m \to +\infty}$$

$$F_\beta(x,\alpha) = \alpha + (1-\beta)^{-1} \mathrm{E}((f(x,S) - \alpha)^+)$$

And assuming continuity of $\Psi(S,\alpha)$, $F_\beta(x,\alpha)$ is continuously differentiable.

A smooth approximation…

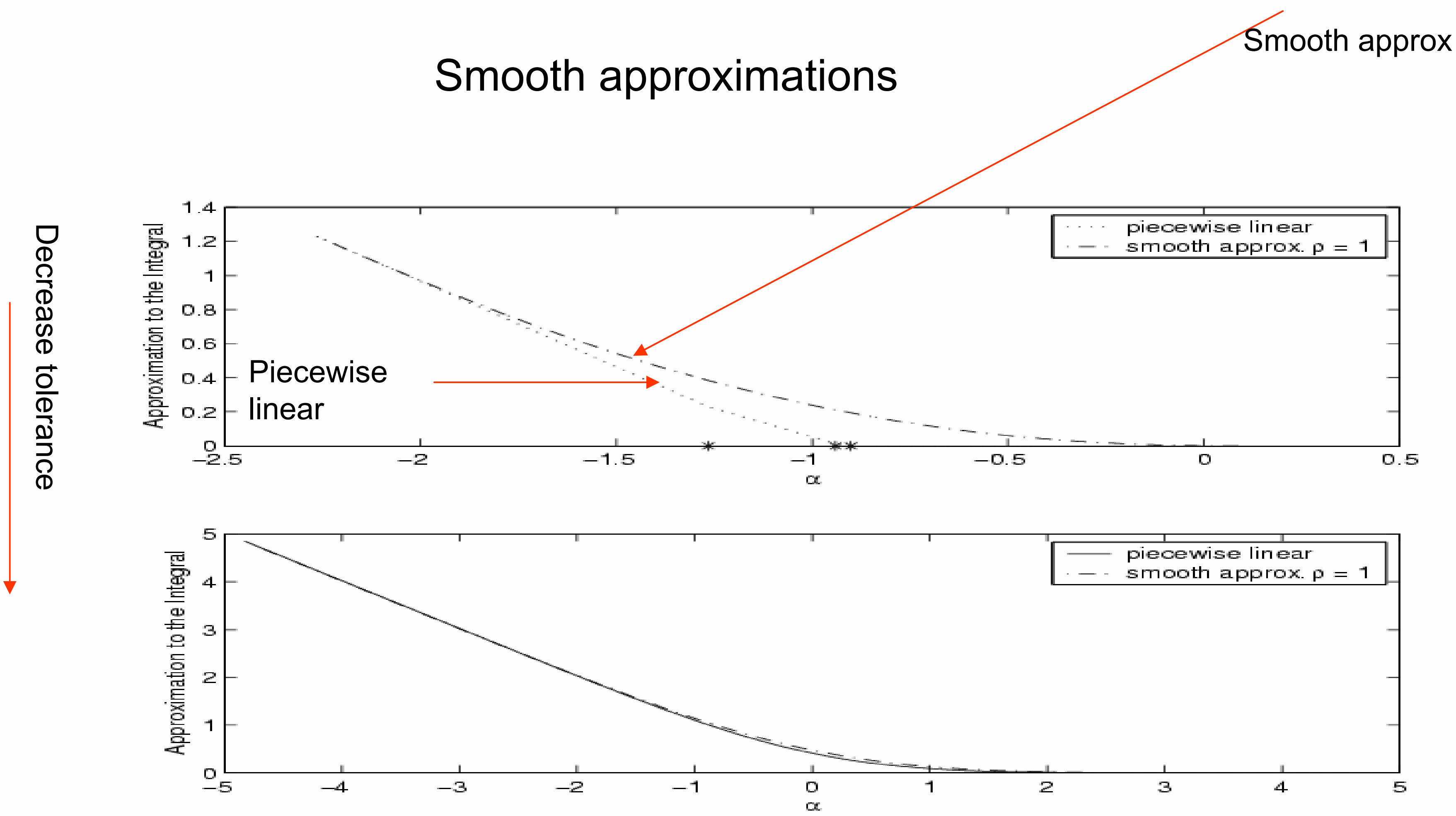
Let $p_\varepsilon(z) \approx \max(0, z)$. Given ε>0, $p_\varepsilon(z)$ is the continuously differentiable function:

$$p_\varepsilon(z) = \begin{cases} z \text{ if } z \geq \varepsilon \\ \dfrac{z^2}{4\varepsilon} + \frac{1}{2} z + \frac{1}{4} \varepsilon \text{ if } -\varepsilon \leq z \leq \varepsilon \\ 0 \text{ otherwise} \end{cases}$$

$$\tilde{F}_\beta(x, \alpha) = \alpha + \frac{1}{m(1-\beta)} \sum_{i=1}^{m} p_\varepsilon(-(\delta V)_i^T x - \alpha)$$

$\tilde{F}_\beta(x, \alpha)$ is a continuously differentiable approximation to $F_\beta(x, \alpha)$

Smooth approximations



A piecewise quadratic convex program:

This leads to the following piecewise quadratic convex program, with O(*n*) independent variables and constraints:

$$\min_{(x,\alpha)} \tilde{F}_\beta(x,\alpha) + \sum_{j=1}^{n} c_j \, |\, x_j \,|$$

$$\text{subject to } \begin{cases} (V^0)^T x = 1 \\ (E[\delta V])^T x = \bar{r} \\ l \le x \le u \end{cases}$$

Efficiency: Lp vs smoothing technique

m	n	w=0 Mosek	w=0 smth	w=.01 Mosek	w=.01 smth
25000	20	49.6	10.7	48.3	14.7
	100	826.9	82.6	687.9	177.4
	196	7484.89	875.3	2258.4	1088.8
50000	20	129.7	24.4	124.4	47.65
	100	2893.16	182.08	1068.60	412.0
	196	-	1413.3	-	1545.5

Moral of Problem 5

1. Look out for ill-posedness in the formulation of optimization problems. Correct it.

2. Optimal CVaR problems naturally lead to VERY large LPs. However, the LPs actually approximate a smooth function (as # scenarios increase.). Therefore, it can be cost effective to approximate this smooth function directly, reducing the number of constraints and the number of variables. W/o this reduction the problems quickly become intractable.

Concluding Remarks

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3. To effectively apply optimization methodology to finance, the financial ‘setting’ must be well understood!

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2. Computational finance/FE yield many interesting optimization problems to be solved (many of the discrete constraints are ‘soft’ and can be handled through the use of continuous methodologies)

3. To effectively apply optimization methodology to finance, the financial ‘setting’ must be well understood!

4. To effectively apply optimization methodology to finance, the methods/tools , strengths/weaknesses of optimization must be will understood!

Thank you for listening!

Feel free to email me with follow-up questions, etc:

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