# Inner/Outer Iterations For Solving Linear Systems Of Equations 

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We consider solving the linear system of equations $A x=b$. In many situations, it is natural to rewrite the system as $M x=N x+b$. This leads to the iteration

$$
\begin{equation*}
M x^{k+1}=N x^{k}+b . \tag{1}
\end{equation*}
$$

We can replace the equation (1) by the equivalent system

$$
\begin{equation*}
M z^{k}=r^{k}, \quad x^{k+1}=x^{k}+z^{k} \tag{2}
\end{equation*}
$$

where $r^{k}=b-A x^{k}$. We have assumed that it is 'easy' to solve the system (2). Unfortunately, we may have to perform inner iterations to solve (2) so that we replace the algorithm described by (2) by

$$
\begin{equation*}
M \bar{z}^{k}=r^{k}+q^{k}, x^{k+1}=x^{k}+\bar{z}^{k} . \tag{3}
\end{equation*}
$$

The vector $q^{k}$ is the residual vector associated with the inner iteration; we continue to iterate until $\left\|q^{k}\right\| \leq \tau_{k}\left\|r^{k}\right\|$. The parameters $\left\{\tau_{k}\right\}$ play an important role in the convergence analysis, and we describe various strategies for choosing $\left\{\tau_{k}\right\}$. Our analysis will be independent of the method used for solving the system (3) and the inner iteration.

There are numerous problems that fit into the framework we have decribed. In particular, inner and outer iterations play an important role in domain decomposition, solution of non-symmetric systems where the coefficient matrix is real, positive and in the implementation of the method of Uzawa in solving the Stokes equation. Furthermore, we can apply these ideas to using Chebyshev polynomials for the acceleration of various iterative methods where fast direct methods are used for the inner iteration. We describe the results of various numerical experiments.

