Matrix Structures and Image Restoration: Boundary Conditions, Re-Blurring, and Regularizing Multigrid-Type Algorithms

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We consider the de-blurring problem of noisy and blurred images in the case of space invariant point spread functions (PSFs). The use of appropriate boundary conditions (see [2,10,12]) leads to linear systems with structured coefficient matrices related to space invariant operators like Toeplitz, circulants, trigonometric matrix algebras etc. We can obtain an effective and fast solver by combining the optimally convergent algebraic multigrid described in [11,1] with the Tikhonov regularization (see [3]). A completely alternative proposal is to apply the latter algebraic multigrid (which is designed ad hoc for structured matrices) with the low-pass projectors typical of the classical geometrical multigrid employed in a PDEs context. Thus, using an appropriate smoother, we obtain an iterative regularizing method (see [9]). Unfortunately, the normal equations approach used in connection with popular regularization processes (Tikhonov, CGNE, Landweber etc.) spoils the structure of matrix algebra and the modelistic features of the most precise boundary conditions i.e. reflective [10] and anti-reflective [12,5]. A remedy both for the computational and modelistic problems is to replace the transposition operation ($A \to A^T$) by the correlation operation ($A \to A'$, see [4]): we called this idea "re-blurring" (for a comprehensive discussion on this subject see [8,7,6,4]).

We now give more details on the iterative regularizing multigrid method proposed and discussed in [9]. The main steps are:

(1) projection in a subspace where it is easier to distinguish between the signal and the noise,
(2) application of an iterative regularizing method is the projected subspace.

In fact, any iterative regularizing method like conjugate gradient (CG), conjugate gradient for normal equation (CGNE), Landweber etc., can be used as smoother in our multigrid algorithm. The projector is chosen according to [11,1] in order to maintain the same algebraic structure at each recursion level and having a low-pass filter property, which is very useful in order to reduce the noise effects. In this way, we obtain a better restored image with a flatter restoration error curve and also in less time than the auxiliary method used as smoother.

Like any multigrid algorithm, the resulting technique is parameterized in order to have more degrees of freedom: a simple choice of the parameters allows to devise a powerful regularizing method whose main features are the following:

a) it is used with early stopping (as the CG, CGNE, and the Landweber method) and its cost per iteration is about 1/3 of the cost of the method used as smoother (CG, Landweber, CGNE);
b) it can be adapted to work with the re-blurring approach and with all the boundary conditions used in literature (Dirichlet [2], periodic [2], Neumann or reflective [10] or anti-reflective [12]) since the basic algebraic multigrid considered in [1] is an optimally convergent method for any of the involved structures (Toeplitz, circulant, cosine-algebra or sine-algebra) which naturally arise from the chosen boundary conditions;
c) the minimal relative restoration error with respect to the true image is significantly lower with regard to all the best known techniques directly applied to the system $A\mathbf{f} = \mathbf{g}$ (Riley, CG, preconditioned CG, etc.) with optimal parameters and the associated curve of the relative restoration errors with respect to the iterations is “flatter” (therefore the quality of the reconstruction is not critically dependent on the choice of the iteration where the process has to be stopped);
d) when it is applied to the normal equation $A^T A\mathbf{f} = A^T\mathbf{g}$ we observe that the minimal relative restoration error is slightly lower than Tikhonov, CGNE, Landweber with optimal parameters and the convergence is substantially faster when compared with all the best known iterative techniques (CGNE, Landweber, etc.) with optimal parameters; moreover, the associated curve of the relative errors with respect to the iterations is, at least in out set of experiments, “flatter”;
e) when it is applied to the system $A\mathbf{f} = \mathbf{g}$ the minimal relative error is comparable with regard to all the best known techniques for the normal equations $A^T A\mathbf{f} = A^T\mathbf{g}$, but in this case the convergence is much faster;
f) it can be combined with nonnegativity constraints (by using a simple projection at every step): in that case we observed a substantial gain in the total cost and in the precision when compared with the very precise but extremely slowly convergent projected CGNE and Landweber; finally, in principle, it can be used in connection with the re-blurring approach (i.e. A' in place of A^T) and with edge preserving procedures such as Total Variation, Bayesian methods etc.

As direct consequence of c) and d), the choice of the exact iteration where to stop is less critical than in other regularizing iterative methods while, as a consequence of e) and f), we can choose multigrid procedures which are extremely more efficient than classical techniques without losing accuracy in the restored image. Several numerical experiments show the effectiveness of our proposals. A theoretical analysis of multigrid methods is usually a difficult task and a first largely used approach considers a two grid method. In the same way, to proving the regularizing properties of our multigrid methods, we provide some estimations on the filter factor of the two level strategy.

Finally, we propose a possible generalization where the multigrid regularization is applied as a one-step method: now the only parameter to choose is the number of recursive calls which works, in some sense, like a threshold parameter.

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