Computing Matrix Geometric Means

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Different definitions have been given of the matrix geometric mean $G_k = G_k(A_1, \cdots, A_k)$ of k positive definite $n \times n$ matrices A_i, $i = 1, \cdots, k$. Ando, Li and Mathias [1] stated a set of properties which a "good" mean should satisfy and gave a recursive definition of mean as the common limit $G_k(A_1, \cdots, A_k)$ for $\upsilon \to \infty$ of the k matrix sequences defined by

$$A_i^{(\upsilon+1)} = G_{k-1}(A_1^{(\upsilon)}, \cdots, A_{i-1}^{(\upsilon)}, A_{i+1}^{(\upsilon)}, \cdots, A_k^{(\upsilon)}), \quad i = 1, \cdots, k \tag{1}$$

where $A_i^{(0)} = A_i$ and $G_2(A_1, A_2) = A_1(A_1^{-1}A_2)^{1/2}$. We refer to this mean as the ALM-mean. The linear convergence with rate 1/2 of the sequences (1), proven in [1], makes the computation of G_k quite expensive.

We provide a new definition of matrix geometric mean [3] which satisfies all the properties stated in [1] and, likewise the ALM-mean, is expressed in terms of the common limit of k matrix sequences. We prove that our sequences have a cubic convergence. This property leads to a dramatic acceleration, in terms of cpu time, in the real applications. We provide a geometric interpretation of our definition in terms of Riemannian geometry [2].

Since both the ALM-mean and our mean recursively reduce the computation of G_k to computing k values of G_{k-1}, one finds that the cost of computing G_k grows as $O(k!n^3)$. We propose a different approach [4] which leads to the definition of a new geometric mean whose complexity is $O(k^2n^3)$.

Finally we address the problem of computing a structured geometric mean when the input matrices belong to a space of structured matrices, say, Toeplitz matrices. In this case, the customary definitions of geometric mean apparently do not preserve the structure of the input matrices. We examine two ways of overcoming this drawback: providing different definitions, or finding out weaker structures which are preserved by the mean function.

References

[1] T. Ando, C.-K. Li, and R. Mathias. Geometric means. Linear Algebra Appl., 385:305–334, 2004.

[2] R. Bhatia and J. Holbrook. Riemannian geometry and matrix geometric means. Linear Algebra Appl., 413:594–618, 2006.

[3] D. Bini, B. Meini, F. Poloni, An effective matrix geometric mean satisfying the Ando-Li-Mathias properties, Math. Comp. 2010

[4] D. Bini, B. Iannazzo, A note on computing matrix geometric means, 2009 (submitted)