Approximate Factoring Of The Inverse

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We consider an algebraic approach to preconditioning linear systems by computing approximate factors. We associate the approximate factoring problem with a minimization problem involving sparse matrix subspaces.

Let $\mathcal{W}$ and $\mathcal{V}_1$ be sparse matrix subspaces of $\mathbb{C}^{n\times n}$ over $\mathbb{C}$ containing invertible elements and assume that nonsingular elements of $\mathcal{V}_1$ are readily invertible. To approximately factor the inverse of a large and sparse nonsingular matrix $A \in \mathbb{C}^{n\times n}$ into the product WV_1^{-1}, we consider the problem $AW \approx V_1$ with non-zero matrices $W \in \mathcal{W}$ and $V_1 \in \mathcal{V}_1$ regarded as variables both. This equation can be inspected by examining the nullspace of the linear map

$$W \mapsto (I - P_1)AW, \quad \text{with } W \in \mathcal{W}, \tag{1}$$

where P_1 is the orthogonal projection onto $\mathcal{V}_1$ [2]. We have $AW \approx V_1 = P_1AW$ if and only if $(I - P_1)AW \approx 0$. This formulation leads to the optimality criterion

$$\min_{W\in\mathcal{W}, \|W\|_F=1} \|(I - P_1)AW\|_F \tag{2}$$

for generating factors W and $V_1 = P_1AW$ in terms of the singular values of the linear map (1).

To approximately solve the minimization problem (2) we approximately compute the smallest singular values of the linear operator (1) by using the power method with sparse-sparse operations. We then consider the eigenvalue problem involving the Hermitian positive semidefinite operator

$$W \mapsto LW = P_{\mathcal{W}}A^*(I - P_1)AW$$

on the matrix subspace $\mathcal{W}$, where $P_{\mathcal{W}}$ denotes orthogonal projection onto $\mathcal{W}$. Since we are interested in the extreme eigenvalues located in the left end of the spectrum, we apply the power method to

$$\alpha I - L \text{ on } \mathcal{W}$$

with $\alpha = r\|A\|^2$ having $1/2 < r \leq 3/4$.

In our presentation, we address the choice of subspaces $\mathcal{W}$ and $\mathcal{V}_1$. We also give numerical examples of convergence when the approximate factors are used to precondition restarted GMRES and compare the results to those obtained with SPAI preconditioner [3].

References

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