A New Preconditioning Technique For Linear Equations Derived From The Elimination Of Redundant Unknowns In Singular Systems

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Introduction to folded preconditioning: We consider solving a system of linear equations

$$A_L \boldsymbol{x}_L = \boldsymbol{b}_L, \tag{1}$$

iteratively for $\boldsymbol{x}_L \in \mathbb{C}^l$, where $A_L \in \mathbb{C}^{l\times l}$ is a sparse and singular matrix (i.e., $\mathrm{rank}(A_L) < l$), and $\boldsymbol{b}_L \in \mathbb{C}^l$. Let $l = m + n$ and $\mathrm{rank}(A_L) \leq m < l$. The matrix A_L can be expressed as

$$A_L = \begin{pmatrix} A & AB \\ CA & CAB \end{pmatrix} = C_E A B_E, \tag{2}$$

with $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, and $C \in \mathbb{C}^{n\times m}$. We may need to reorder the rows and columns so that $\mathrm{rank}(A) = \mathrm{rank}(A_L)$. The matrices B_E and C_E are

$$B_E = \begin{pmatrix} E & B \end{pmatrix}, \qquad C_E = \begin{pmatrix} E \\ C \end{pmatrix}, \tag{3}$$

where $E \in \mathbb{C}^{m\times m}$ is an identity matrix. Similarly, $\boldsymbol{x}_L$ and $\boldsymbol{b}_L$ are written

$$\boldsymbol{x}_L = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}_r \end{pmatrix}, \tag{4}$$

$$\boldsymbol{b}_L = \begin{pmatrix} \boldsymbol{b} \\ C\boldsymbol{b} \end{pmatrix} = C_E \boldsymbol{b}, \tag{5}$$

where $\boldsymbol{x} \in \mathbb{C}^m$, $\boldsymbol{x}_r \in \mathbb{C}^n$, and $\boldsymbol{b} \in \mathbb{C}^m$. Equation (5) must be satisfied to ensure the existence of a solution.

Since A_L is singular, the variables in $\boldsymbol{x}_r$ can be regarded as redundant unknowns. In fact, we can eliminate $\boldsymbol{x}_r$ from the system and thereby derive the reduced system of equations

$$A\boldsymbol{y} = \boldsymbol{b}. \tag{6}$$

When a solution of (6) is given, we can easily obtain a solution of the original system (1) by substituting $\boldsymbol{x} = \boldsymbol{y}$ and $\boldsymbol{x}_r = 0$ into (4).

To obtain a solution of (1), we can apply an iterative solver, for example, preconditioned Krylov subspace (PKS) methods, to either (1) or (6). From the viewpoint of the computational cost per iteration and the memory consumption, it appears to be more attractive to solve (6) rather than (1). However, in some practical applications, eliminating $\boldsymbol{x}_r$ causes a significant deterioration in the convergence of PKS solvers. An example of this is applying the incomplete Cholesky preconditioned conjugate gradient (IC-CG) method to eddy-current finite element analyses that lead to singular systems of equations [1]. We present a solution to this problem in the form of a new preconditioning technique which can be employed in a PKS method for (6) [2].

We consider a specific PKS solver for (1). Let the following matrix be the inverse of the preconditioning matrix:

$$M_L = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{7}$$

where $M_{11} \in \mathbb{C}^{m\times m}$, $M_{12} \in \mathbb{C}^{m\times n}$, $M_{21} \in \mathbb{C}^{n\times m}$, and $M_{22} \in \mathbb{C}^{n\times n}$. In general, the approximate solution vector $\boldsymbol{x}_L^{(k)}$ at the k-th step of PKS methods satisfies

$$\boldsymbol{x}_L^{(k)} \in \mathrm{span}\{M_L\boldsymbol{b}_L, M_LA_LM_L\boldsymbol{b}_L, \dots, M_L(A_LM_L)^{(k-1)}\boldsymbol{b}_L\}, \tag{8}$$

where we suppose that $\boldsymbol{x}_L^{(0)} = 0$ for simplicity.

It is worth noting that $\boldsymbol{x}_L^{(k)}$ and another approximate solution, $((B_E\boldsymbol{x}_L^{(k)})T \quad 0)^T$, satisfy the following relation with respect to the residual vector:

$$\boldsymbol{b}_L - A_L\boldsymbol{x}_L^{(k)} = \boldsymbol{b}_L - A_L \begin{pmatrix} B_E\boldsymbol{x}_L^{(k)} \\ 0 \end{pmatrix}. \tag{9}$$

Therefore, it is enough for us to compute $B_E\boldsymbol{x}_L^{(k)}$. Substituting (2) and (5) into (8) and multiplying both sides by B_E, we have

$$\begin{aligned} B_E\boldsymbol{x}_L^{(k)} &\in \mathrm{span}\{B_EM_LC_E\boldsymbol{b}, B_EM_LC_EAB_EM_LC_E\boldsymbol{b}, \dots, B_EM_L(C_EAB_EM_L)^{(k-1)}C_E\boldsymbol{b}\} \\ &\in \mathrm{span}\{M_f\boldsymbol{b}, M_fAM_f\boldsymbol{b}, \dots, M_f(AM_f)^{(k-1)}\boldsymbol{b}\} \end{aligned} \tag{10}$$

where

$$M_f = B_EM_LC_E. \tag{11}$$

This reveals that we are able to derive an algorithm to compute $\boldsymbol{y}^{(k)} = B_E\boldsymbol{x}_L^{(k)}$ in the form of a PKS solver for (6) using M_f as the inverse of the preconditioning matrix. The derived PKS solver has the *same convergence property* as the original PKS solver for (1) in the sense of the equality in (9). In [2], it is proved that the conjugate gradient (CG) solver for (1) with (7) is equivalent to the CG solver for (6) with the corresponding preconditioning (11). See [2] for a more detailed discussion.

We call the new preconditioning technique expressed by (11) the *folded preconditioning.*

Numerical tests: We have examined the performance of iterative solvers using the new preconditioning technique in the full-wave and eddy-current problems in [2] and [4]. We present here numerical results for the magnetostatic finite edge-element analysis, which also leads to a singular linear system of equations. It is known that the elimination of the redundant unknowns, called the tree-zero gauge, results in deterioration of convergence and poor performance of the iterative solvers [3].

When using the tree-zero gauge, we can efficiently implement the multiplication by B_E and C_E in (11), by noting the fact that B_E is identical to the fundamental tie-set matrix.

To compare the performance of conventional solvers and our proposed preconditioning in a sample magnetostatic analysis, we use the IC-CG method for (1) and the CG method with the folded variant of the IC preconditioning (FIC-CG) for (6). For further comparison, we also use the standard IC-CG method for (6), although its performance is very poor.

Table 1 shows that the convergence properties of the IC-CG and FIC-CG are similar, as predicted theoretically. Moreover, the proposed solver outperforms the conventional method in both memory consumption and solution time, due to the reduction in the number of unknowns.

Table 1: Performances of the iterative solvers on a PC (Windows XP x64, Intel Core2 X9770)

Solver	IC-CG for (1)	FIC-CG for (6)	IC-CG for (6)
Number of unknowns	6,390,144	4,210,688	4,210,688
Number of iterations	302	303	7124
Memory consumption [MB]	3016	2293	1462
Solution time [s]	424.0	370.7	4621.2

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