A Robust And Efficient Proposal For Solving The Linear Systems Arising In Interior-Point Methods For Linear Programming

A. Oliveira
Department of Applied Mathematics, University of Campinas, Brazil
M. González-Lima
Department of Cientific Computation and Statistics, Simón Bolívar University, Venezuela
D. Oliveira
Department of Applied Mathematics, University of Campinas, Brazil

The computational burden of primal-dual interior point methods for linear programming relies on the computation of the search direction by solving one or more linear systems per iterations. The objective of this work is to study an efficient and robust way of solving these systems for large-scale sparse problems. Our proposal combines the use of the stable system and a hybrid iterative method where a conjugate gradient method is preconditioned during the initial interior point iterations by an incomplete Cholesky factorization type.

The linear systems arising at each iteration of an interior point method are indefinite and they can be written in a symmetric form, which is known as augmented system. A common approach in interior point solvers for linear programming reduces the augmented system to a smaller positive definite one, called normal equations system and the search direction is found by solving the smaller system. This latter approach suffers from some numerical drawbacks. The normal equations system matrix can be dense (even if the original constraint matrix A is sparse) when A has dense columns. Furthermore, even A being sparse the Cholesky factors may be dense. These features need to be appropriately handled when solving large scale problems. Besides, the system is ill-conditioned when evaluated at points close to the solution set, specially when working with degenerate problems. However, on the positive side, the ill-conditioning does not seem to have, in general, the detrimental effect that may be expected and approximate solutions for many problems, to the accuracy of 10^{-8}, can be obtained by using the normal equations. However, because the condition number of the normal equations matrix grows to infinity when approaching the solution set [4] many methods for solving these systems become increasingly unstable.

Another approach for finding the search direction is to solve the indefinite augmented system. This system is larger than the normal equations system but it has the advantage that it preserves sparsity, this is, it is not affected by the presence of dense columns in A as the normal equations system. It is also ill-conditioned when the iterates are close to the solution set but this ill-conditioning is easier to handle. However, the efficiency of this approach relies on efficient solvers for sparse symmetric indefinite systems.

Many interior point solvers for linear programming use sparse Cholesky factorization to solve the normal equations system. Sometimes, the use of direct methods becomes prohibitive due to storage and time limitations. In such situations iterative approaches are more interesting.

However, because of the ill-conditioning of the matrix, the success of implementations using iterative methods to solve the normal equations system depends on how to choose an appropriate preconditioner. Several proposals are based on incomplete Cholesky factorization of the positive definite matrix. Typically, such preconditioners are efficient in initial iterations, however it deteriorate as the interior point method converges to a solution.

In [1] an iterative hybrid approach to solve the normal equations system is proposed. The conjugate gradient method is preconditioned during the initial interior points iterations (phase I) using a kind of incomplete factorization called controlled Cholesky factorization proposed in [2] and in the remaining iterations (phase II) using the splitting preconditioner developed in [5]. When the controlled Cholesky preconditioner loses effciency, the system is already highly ill-conditioned and it is a good indicator that the splitting preconditioner will work better.

The class of splitting preconditioners was designed specially for the augmented system. An important feature of this class is the option to reduce the preconditioned indefinite system to the normal equations system, allowing the use of conjugate gradient method. Since this class was developed for the last interior point iterations a diagonal scaling preconditioner was used for the initial iterations. The improvement presented in [1] uses an efficient preconditioner instead of the diagonal one.

Other approach for computing the search direction of an interior point method was proposed in [3] when the constraint matrix of the linear programming problem can be written as (S E) with S well conditioned and easy to factorize. An indefinite system is formed, the so-called stable system, of the same size that the augmented system but with the property that it does not contain the inverse of any linear programming variable. Therefore, this system does not necessarily become highly ill-conditioned since the matrix does not suffer from the presence of large eigenvalues as in other approaches. In fact, in [3] it is shown that the condition number of the matrix is bounded when the primal and dual problems are nondegenerate. Additionally, the linear system is not acted by the presence of dense columns contrary to the normal equations approach.

Even though the approach presented in [3] shows to be robust and outperforms in time the normal equations approach when solving large, sparse, and well-conditioned problems, it is generally slower when solving the degenerate problems from the Netlib test set.

The objective of this work is to propose a method for solving the linear systems that combines the good features and related ideas of the splitting preconditioner implementation and the stable system. We present a way of solving the stable system that it is competitive, in terms of running time, with the traditional normal equations approach for some large, sparse linear programming problems.

Both approaches require the solution of a linear system that involves the inverse of certain original linear programming variables which may prevent the obtain high accurate solutions by using these methods specially when solving very large problems since a certain amount of such variables approach zero close to a solution. Howerver, to get very accurate solutions may be valuable for special applications or needs.

In the case that $A = [SE]$ (maybe after a column reordering) with S an invertible matrix and E a sparse matrix, the stable system for finding the search direction do not contains inverse of variables in the matrix nor the right hand side. The stable system is indefinite but since that it does not contain the inverse of variables it does not necessarily become highly ill-conditioned when close to the solution set since the matrix does not suffer from the presence of large eigenvalues. Then, very accurate solutions of the linear problem can be obtained. However, since the matrix of this system has not a special form, efficient factorizations or iterative methods to solve it are not easy to find.

In order to solve this system in this work we write the stable system, after permutations and change of variables, in a block form such that the block diagonals are diagonal matrices and the off-diagonals are matrices that are close to zero when evaluated in points close to the solution of the linear problem. Therefore, the stable system is transformed in an equivalent system of the same size being symmetric quasidefinite and solved by a low-cost fixed point iterative method. A proof of convergence is also given. The price paid is to find S and to use it to solve the linear system. To find matrix S we use similar ideas to the ones developed in the implementation of the splitting preconditioner presented in [5].

Computational experiments show that many problems can be solved to the accuracy of 10^{-12} or even less by the new approach, while the same problems can not be solved to a tolerance smaller than 10^{-8} by the traditional approaches.

References

[1] S. Bocanegra, F. F. Campos, and A. R. L. Oliveira, *Using a hybrid preconditioner for solving large-scale linear systems arising from interior point methods*, Computational Optimization and Applications, 36 (2007), pp. 149–164.

[2] F. F. Campos and N. R. C. Birkett, *An efficient solver for multi-right hand side linear systems based on the CCCG(η) method with applications to implicit time-dependent partial differential equations*, SIAM J. Sci. Comput., 19 (1998.), pp. 126–138.

[3] M. González-Lima, H. Wei, and H. Wolkowicz, *A stable iterative method for linear programming*, Computational Optimization and Applications – Accepted, (2008).

[4] V. V. Kovacevic-Vujcic and M. D. Asic, *Stabilization of interior-point methods for linear programming*, Computational Optimization and Applications, 14 (1999), pp. 331–346.

[5] A. R. L. Oliveira and D. C. Sorensen, *A new class of preconditioners for large-scale linear systems from interior point methods for linear programming*, Linear Algebra and Its Applications, 394 (2005), pp. 1–24.