Regularized Preconditioner For Ill-Posed Problems By Bidiagonalization Process With Application In Image Restoration

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This talk will address new regularized preconditioner that can be obtained by using a few steps of the Lanczos bidiagonalization and more detailed can be found in [1].

This talk concerns with the computation of a meaningful approximate of least square solution of large scale discrete ill-posed problems of the form

$$Ax = b,$$

by CGLS iterative method, where A is typically ill-conditioned and the right-hand side vector b contaminated by an error such that $b = \hat{b} + e$. Here e and $\hat{b}$ denote the noise and unknown error-free right hand side vectors, respectively. These problems typically arise from discretization of Fredholm integral equations of the first kind in, e.g., geophysics, or image deblurring and are often referred to as linear discrete ill-posed problems.

For large scale ill-posed problems, iterative methods, such as CGLS, are the most popular regularization methods. In fact, CGLS is a semiconvergent method: For some k, in the first k iterations, the method converges to the solution $\hat{x}$, and then suddenly starts to diverge and the noise begins to enter the solution. Hence this is the time that the iteration must be stopped to prevent the noise components to interfere. This shows that CGLS method has regularization property.

However, iterative methods, such as CGLS, have slow rate of convergence for ill-posed problems. So, for speeding up the rate of convergence the use of a suitable preconditioner is needed. But the concept of a preconditioner for discrete ill-posed problems is not the same as that of the standard preconditioners. The standard preconditioners try to speed up the convergence by clustering the whole singular values of preconditioned system around 1, while in the context of ill-posed problems one only needs to take care of the large singular values. So, a preconditioner is needed to be introduced to improve the position of the large magnitude part of the singular value spectrum and leave the remaining singular values unchanged.

In this talk we introduce a new regularized preconditioner that can be obtained by using a few steps of the Lanczos bidiagonalization and it improves the large singular values around 1 and leaves the others unchanged. In contrast to some special structured preconditioners such as circulant and Kronecker product approximations which are proposed for special structured matrices such as Toeplitz and circulant and ... matrices, construction of the new preconditioner is not based on any particular structure of the matrices, and so it presents itself as a general purpose alternative that can be used in special structured matrices as well.

We show that after $k \ll n$ steps of the Lanczos bidiagonalization one can construct a regularized inverse preconditioner such as

$$M = \overline{V}_k (B_k^T B_k)^{-1} \overline{V}_k^T + (I - \overline{V}_k \overline{V}_k^T),$$

which $\overline{V}_k$, $\overline{U}_k$ are orthogonal matrices and $B_k \in R^{(k+1)\times k}$ is bidiagonal matrix that obtaind in k'th step of Lanczos bidiagonalization.

According to characteristic of ill-posed problems the standard preconditioners cannot be used in ill-posed problems because by the first few applications of these preconditioners the signal and noise subspaces will be mixed up and the solution is contaminated by the noise. So in these problems, only the large singular values must be taken into consideration, and we must introduce a preconditioner to improve the large magnitude part of the singular value spectrum and leave the remaining singular values unchanged. For example, consider $A = U\Sigma V^T$ as the singular value decomposition of A, where U and V are orthogonal matrices and $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$. Then $A^T A = V\Sigma^2 V^T$ and for a suitable k

$$P = V \begin{pmatrix} \Sigma_1^{-2} & 0 \\ 0 & I_2 \end{pmatrix} V^T,$$

can be used as inverse regularized preconditioner of $A^T A = A^T b$, where $\Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_k)$, $\Sigma_2 = \text{diag}(\sigma_{k+1}, \ldots, \sigma_n)$ and $I = \begin{pmatrix} I_1 & \\ & I_2 \end{pmatrix}$. In this case the preconditioned system becomes

$$PA^T A = V \begin{pmatrix} I_1 & 0 \\ 0 & \Sigma_2^2 \end{pmatrix} V^T.$$

This shows that P has improved the large singular values of $A^T A$ and has no influence on the small ones. So this preconditioner does not mix the noise and signal subspaces of the problem and has a regularizing effect.

In first view, the construction of this preconditioner seems to be very expensive and difficult to implement, because the computation of all singular vectors in large scale problems is not easily feasible and recommendable. But we now show that by using the orthogonality of the matrix V, a version of this preconditioner can be obtained that its construction is not expensive. In the first step, let $V = [V_1, V_2]$, where $V_1 = [v_1, \ldots, v_k]$, $V_2 = [v_{k+1}, \ldots, v_n]$. Then the closed form of the preconditioner P can be written as

$$P = V_1 \Sigma_1^{-2} V_1^T + V_2 V_2^T.$$

Now from the orthogonality of V one obtains $V_2 V_2^T = I - V_1 V_1^T$, which in turn gives

$$P = V_1 \Sigma_1^{-2} V_1^T + (I - V_1 V_1^T).$$

This shows that for constructing P one needs only to use V_1 and Σ_1. On the other hand, since in ill-posed problems the dimension of signal subspace is small, then its corresponding subspace V_1 has a small dimension (index) k. Incidentally, It can be shown that application of k steps of Lanczos bidiagonalization provides some good approximation of P can be obtained as

$$P \approx \overline{V}_k (B_k^T B_k)^{-1} \overline{V}_k^T + (I - \overline{V}_k \overline{V}_k^T) = M.$$

which $\overline{V}_k$, $\overline{U}_k$ are orthogonal matrices and B_k is the bidiagonal matrix that obtained in the k'th step of Lanczos bidiagonalization. This shows that after k steps of Lanczos bidiagonalization one can construct the regularized inverse preconditioner

$$M = \overline{V}_k (B_k^T B_k)^{-1} \overline{V}_k^T + (I - \overline{V}_k \overline{V}_k^T),$$

in $\mathcal{O}(kn^2)$ or about $\mathcal{O}(n^2)$ operations, since k is very small in comparison with n. By the way at every step of preconditioned iterative method it is needed to compute Mx for some $x \in R^n$. For this preconditioner, Mx can be computed only with $\mathcal{O}(kn) \approx \mathcal{O}(n)$ operations.

We investigated the performance of our new preconditioner on some test problems from first kind fredholm integral equations and also on some image restoration test problems. Also in special structure BTTB matrices the construction and application costs of this preconditioner per iteration is roughly the same or effectively less than commonly used preconditioners such as circulant and Kronecker product approximation.

Reference

[1] M. Rezghi, S. M. Hosseini, Lanczos based Preconditioner for ill-posed problems. *Numerical Algorithm*, submitted.