Preconditioned Saddle Point Problems In Finite Precision Arithmetic

Miro Rozložník
Institute of Computer Science, Czech Academy of Sciences, CZ-18207 Prague, Czech Republic
miro@cs.cas.cz

Symmetric indefinite saddle-point problems arise in many application areas such as computational fluid dynamics, electromagnetism, optimization and nonlinear programming. Particular attention has been paid to their iterative solution. In this contribution we analyze several theoretical issues and practical aspects related to the application of preconditioners in Krylov subspace methods. Two main representatives of the segregated solution approach are analyzed: the Schur complement reduction method, based on an (iterative) elimination of primary variables and the null-space projection method which relies on a basis for the null-space for the constraints. We illustrate our theory mainly on the constraint (null-space projection) pre-conditioner, but several results hold and can be extended for other classes of methods and preconditioners.

Several structure-dependent schemes have been proposed and analyzed. Indeed, the nature of these systems enables to take into account not only simple preconditioning strategies and scalings, but also preconditioners with a particular block structure. It is well-known that the application of positive definite block-diagonal preconditioner still leads to preconditioned system with a symmetric structure similar to the original saddle point system. On the other hand, the application of symmetric indefinite or nonsymmetric block-triangular preconditioner leads to nonsymmetric triangular preconditioned systems and therefore general nonsymmetric iterative solvers should be considered. It can be shown from a particular structure of the preconditioned matrix (as it is e.g. in the case of the constraint preconditioner) that its eigenvalues are all real and positive. A careful analysis shows that in general this system is not diagonalizable since there are 2×2 nontrivial blocks in its Jordan decomposition. On the other hand, experiments indicate that Krylov subspace methods perform surprisingly well on practical problems even those which should theoretically work only for symmetric systems.

The research in case of the indefinite constraint preconditioner has focused on the use of the conjugate gradient method (PCG). The convergence of PCG for a typical choice of right-hand side has been analyzed and it was shown that solving the preconditioned system by means of PCG is mathematically equivalent to using the CG method applied to the projected system onto the kernel of the constraint operator. Consequently, the primary variables in the PCG approximate solution always converge to the exact solution, while the dual variables may not converge or they can even diverge. The (non)convergence of the dual variables is then reflected onto the (non)convergence of the total residual vector. These considerations naturally lead to the development of safeguard strategies and stopping criteria to avoid possible misconvergence of dual variables. It can be often observed in practical problems that even simple scaling of the leading diagonal block by diagonal entries may easily recover the convergence of dual iterates. An alternative strategy consists in changing the conjugate gradient direction vector when computing the dual iterates into a minimum residual direction vector.

The necessity of scaling the system is even more profound if the method is applied in .nite precision arithmetic. It can be shown that rounding errors may considerably in.uence the numerical behavior of the scheme. More precisely, bad scaling, and thus nonconvergence of dual iterates, affects significantly the maximum attainable accuracy of the computed primary iterates. Therefore, applying a safeguard or pre-scaling technique, not only ensures the convergence of the method, but it also leads to a high maximum attainable accuracy of (all) iterates computed in finite precision arithmetic.

Reference

[1] M. Rozložník and V. Simoncini, *Krylov subspace methods for saddle point problems with indefinite preconditioning*, SIAM J. Matrix Anal. Appl., 24 (2002), pp. 368–391.