Modified Hermitian And Skew-Hermitian Splitting Preconditioner For Saddle Point Problems

Jun-Feng Yin
Department of Mathematics, Tongji University, 1239 Siping Road, Shanghai 200092, P. R. China
yinjf@tongji.edu.cn

Consider saddle point problem of the form

$$\begin{bmatrix} A & B^* \\ -B & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \qquad \text{or} \quad \mathcal{A}x = b$$

where A is nonsymmetric, a modified Hermitian/skew-Hermitian splitting (MHSS) preconditioner,

$$\mathcal{P} = \frac{1}{\alpha+\beta}(\alpha I + \mathcal{H})(\beta I + \mathcal{S}), \qquad \mathcal{H} = \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} S & B^* \\ -B & 0 \end{pmatrix}$$

is presented for the Krylov subspace solution method where H and S is Hermitian and skew-Hermitian part of matrix A; see [1, 2] for details.

Let η and $x = [u, v]^T \neq 0$ be eigenvalue of $\mathcal{P}^{-1}\mathcal{A}$ and the corresponding eigenvector respectively, they satisfy

$$(\mathcal{H} + \mathcal{S})x = \frac{\eta}{\alpha+\beta}(\alpha I + \mathcal{H})(\beta I + \mathcal{S})x.$$

Denote $\Re(\eta)$ and $\Im(\eta)$ be the real and imaginary part of η. If $u \neq 0$, $v = 0$, theoretical analysis shows that,

$$g(\lambda_{\min}(H)) \leq \Re(\eta) \leq g(\lambda_{\max}(H))$$
$$g(\lambda_{\min}(S/i)) \leq \Im(\eta) \leq g(\lambda_{\min}(S/i))$$

where

$$g(x) = \frac{(\alpha+\beta)}{\alpha} - \frac{(\alpha+\beta)\beta}{\alpha(\beta + \alpha x/(\alpha+x))}.$$

If $u \neq 0$, $v \neq 0$, we show that $|\frac{\eta\alpha\beta}{\alpha+\beta-\eta\alpha}| < 1$ if and only if

$$\frac{a^2}{\frac{1+c}{1-c}} + \frac{b^2}{\frac{1-c}{1+c}} < 1$$

where

$$a = \frac{u^*K^*HKu}{u^*K^*Lu}, \qquad b = \frac{u^*K^*(S/i)Ku}{u^*K^*Lu}, \qquad c = \frac{u^*K^*B^*Bu}{u^*K^*Lu}.$$

In Tables 1 and 2, the optimal parameters, number of iteration and CPU time are listed for preconditioned BiCGSTAB and GMRES method for discrete Navier-Stokes equation of the form

$$\begin{aligned} -\nu\nabla u + \mu\nabla u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= f \quad \text{on } \Omega. \end{aligned}$$

Numerical results further verify that our preconditioner is efficient and better than standard HSS preconditioner.

Table 1: Numerical results of preconditioned BiCGSTAB.

μ	HSS			MHSS			
	α^*	IT	CPU	α^*	β^*	IT	CPU
10	5	13	8.99	5	170	8	5.68
50	17	22.5	15.74	5	70	9.5	6.74
100	30	28	19.51	5	40	12.5	8.85
500	52	25	17.51	5	30	10.5	7.59

Table 2: Numerical results of preconditioned GMRES.

μ	HSS			MHSS			
	α^*	IT	CPU	α^*	β^*	IT	CPU
10	3	15	6.06	5	20	12	5.06
50	10	28	10.46	5	40	14	5.81
100	12	33	12.46	5	40	15	6.22
500	40	28	10.70	20	50	24	9.13

This work was supported by Program for Young Excellent Talents in Tongji University, and the National Natural Science Foundation of China (Grant No. 10801106).

References

[1] Z.-Z. Bai, G. H. Golub and M. K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM J. Matrix Anal. Appl., 24(2003), pp. 603–626.

[2] M. Benzi, A. J. Wathen, Some Preconditioning Techniques for Saddle Point Problems, Model Order Reduction: Theory, Research Aspects and Appl., 13(2008), pp. 195–211.