Chapter 4

Radian

4.1 Introduction

Radian is a way to cut and measure an angle, which is more reasonable to use than degree due to the arc length of circle (will be explained below). It is a unit of measurement of angles equal to about 57.3^o, equivalent to the angle subtended at the centre of a circle by an arc equal in length to the radius.

4.2 Why Is a Circle $360°$?

The Sumerians watched the Sun, Moon, and the five visible planets (Mercury, Venus, Mars, Jupiter, and Saturn), primarily for omens. They did not try to understand the motions physically. They did, however, notice the circular track of the Sun's annual path across the sky and knew that it took about 360 days to complete one year's circuit. Consequently, they divided the circular path into 360 degrees to track each day's passage of the Sun's whole journey. This probably happened about 2400 BC.

That's how we got a 360-degree circle. Around 1500 BC, the Egyptians divided the day into 24 hours, though the hours varied with the seasons originally. Greek astronomers made the hours equal. About 300 to 100 BC, the Babylonians subdivided the hour into base-60 fractions: 60 minutes in an hour and 60 seconds in a minute. The base 60 of their number system lives on in our time and angle divisions. *

4.3 Why Radian over Degree?

Students typically learn about degrees before they learn about radians, which brings up the question: Why learn about radians if degrees are good enough for measuring angles? The radian is defined to be the ratio of the length of the arc of a circle s to the length of the radius of the circle r, where each length is measured in the same unit. Therefore, when you divide s by r to get the radian measure of the angle, the units for the two lengths cancel, and you end up with a measure that has no units, i.e.

$$\theta = \frac{s}{r}.$$

*http://math.stackexchange.com/questions/340467/why-is-a-full-circle-360-degrees

When s is the entire circumference of the circle, the corresponding angle is that of the entire circle. Since the circumference of a circle is $s = 2\pi r$, the angle of a full circle is $\frac{2\pi r}{r} = 2\pi$.
Unless there is a good reason, one is free to use any measure whatsoever for angles, such as degrees, radians, or one of the less common ones. There is a good reason to use radian over degree.

If you are working with the derivative of a trigonometric function, then it is preferable to use radian measure for angles, because then derivative formulas (and limit formulas) are easier. For example, using radians, the derivative formulas for sine and cosine are:

$$\frac{d}{d\theta}\sin\theta = \cos\theta$$

and

$$\frac{d}{d\theta}\cos\theta = -\sin\theta.$$

However, if degrees are used, the derivative formulas for sine and cosine are:

$$\frac{d}{d\theta}\sin\theta^\circ = \frac{\pi}{180}\cos\theta^\circ$$

and

$$\frac{d}{d\theta}\cos\theta^\circ = -\frac{\pi}{180}\sin\theta^\circ.$$

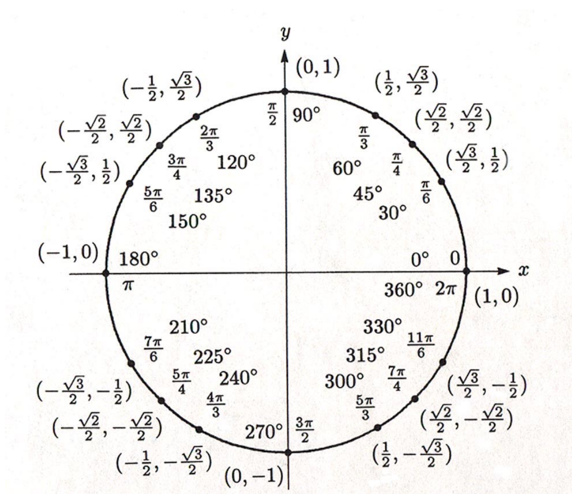
Derivative will be discussed in later chapters. Without knowing what it is, just observe the formulas now. The latter pair of formulas are a pain because of the additional factors, therefore, we use radians whenever we are dealing with derivatives of trigonometric functions.

4.4 Convert Degree to Radian

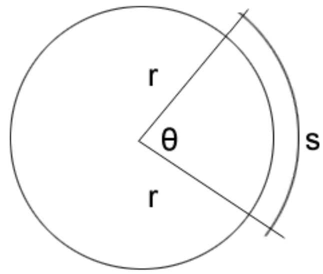
If θ is an angle in degrees while t is an angle in radians, then

$$t = \frac{\pi\theta}{180}.$$

It is obvious to see that radian of a semi-circle (180^o) is equal to π. Then, for easy calculation, we calculate the other special angles based on such a fact, e.g.: radian of a circle (360^o) is equal to two times a semi-circle which is 2π. More can be seen from the following graph.



4.5 Circle and Arc



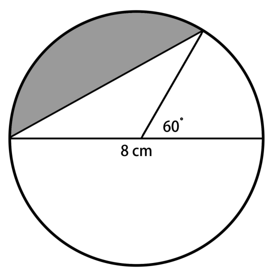
The **arc length** is the distance along the curved line making up the arc of a circle. [†]. And the arc length s of a circle is given by

$$s = r \cdot \theta$$

and the area A of a sector of a circle is given by

$$A = \frac{1}{2} r^2 \theta.$$

Example 4.1. Find the area A of the shaded region.



Area of the shaded part and the triangle

$$= \frac{1}{2} \cdot 4^2 (\pi - \frac{\pi}{3})$$
$$= \frac{16\pi}{3}$$

Area of the triangle

$$= \frac{1}{2} \cdot 4^2 \sin(\frac{\pi}{2})$$
$$= 4\pi \cdot \frac{\sqrt{3}}{2}$$
$$= 2\sqrt{3}\pi$$

Therefore, area of the shaded part

$$= \frac{16\pi}{3} - 2\sqrt{3}\pi.$$

▲

4.6 Exercises

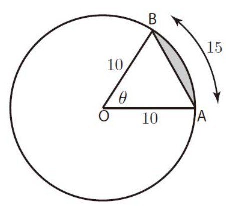
Convert the following radians (degrees) to degrees (radians).

[†] http://www.mathopenref.com/arclength.html

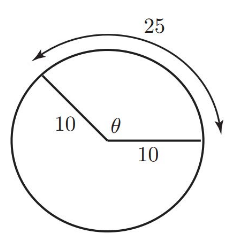
1. $\frac{\pi}{3}$ radians
2. $\frac{3\pi}{4}$ radians
3. $\frac{\pi}{10}$ radians
4. $\frac{7\pi}{4}$ radians
5. $\frac{\pi}{6}$ radians
6. 0.6 radians
7. 120^o
8. 135^o
9. 15^o
10. 45^o
11. 49^o
12. 17^o

Determine the angle (in radians) subtended at the centre of a circle of radius $3cm$ by each of the following arcs:

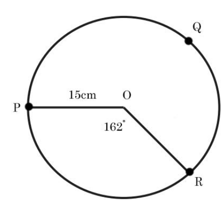
13. arc of length 6 cm
14. arc of length 3π cm
15. arc of length 1.5 cm
16. arc of length 6π cm
17. Suppose we have a circle of radius 10 cm and an arc of length 15 cm. Find the angle θ, the area of the sector OAB and the area of the minor shaded segment.



18. A sector a circle is bounded by two radii $r = 5$ cm and an arc s.
(a) Determine the length of the arc when the angle at the center is $\frac{\pi}{2}$ radians.
(b) Calculate the area of question (a).
(c) A sector of this circle has an area of 50 cm^2. What is the angle (in radians) at the centre of this sector?
19. Consider the circle shown below. Calculate the angle θ in degrees.

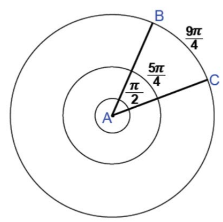


20. Find the arc length of $\widehat{PQR}$.

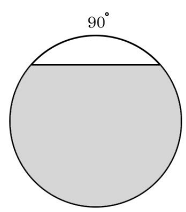


21. $\widehat{AB}$ and $\widehat{CD}$ both have an angle measure of 30^o, but their arc lengths are not the same. $\overline{OB} = 4$ and $\overline{BD} = 2$.
    (a) What are the arc lengths of $\widehat{AB}$ and $\widehat{CD}$?
    (b) What is the ratio of the arc length to the radius for both of these arcs? Explain.
    (c) What are the areas of the sectors AOB and COD?

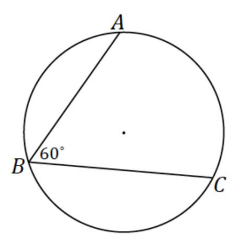
22. The concentric circles all have centre A. The measure of the central common angle $\angle A$ is 45^o. The arc lengths are given as follows.
    (a) Find the radius of each circle.
    (b) Determine the ratio of the arc length to the radius of each circle.



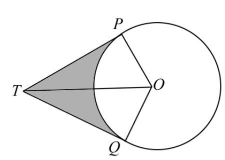
23. Find the area of the shaded part of the circle of radius 2.



24. The radius of the following circle is 36 cm, and the $\angle ABC = 60^o$.
    (a) What is the arc length of $\widehat{AC}$?
    (b) What is the radian measure of the central angle?



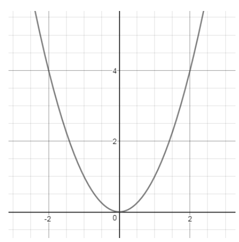
25. In the figure, TP and TQ are tangents to the circle with centre O at P and Q respectively. If $\angle POT = 60^o$ and $TO = 8$ cm, find the area of the shaded region.



Chapter 5

Functions

A function relates an input to an output. And there are different kinds of functions that have different types of input and of output. For example, a human-being can input nearly any kind of food, while dogs can only intake unsweetened dog food. In this chapter, we will go through the varieties of function, examining the characteristics of function by using the example $y = x^2$.



5.1 Domain, Codomain and Range

A function f on a set D into a set S is a rule that assigns a unique element $f(x)$ in S to each element x in D. **Domain**, **co-domain** and **range** are special names for **what can go into**, and **what can come out** of a function:

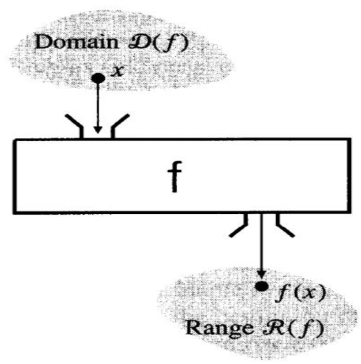
- What can go **into** a function is called the **Domain**
- What **may possibly come out** of a function is called the **Co-domain**
- What **actually comes out** of a function is called the **Range**

$$y = f(x)$$

where the set of all possible input values for x is called the domain of the function, the set of permissible outputs for y is called the co-domain, and the set of all actual outputs for y is called the range of the function. We denote a function f with domain X and codomain Y as

$$f : X \to Y.$$

Note: The co-domain of a function is a set of values that includes the range, but may also include additional values beyond those in the range. Co-domains can be useful when the range is difficult to specify exactly, but a larger set of numbers that includes the entire range can be specified. For example, a co-domain could start out as the set of all real numbers $\mathbb{R}$, and then be gradually reduced in size as sets of values that the function will never produce are figured out. *



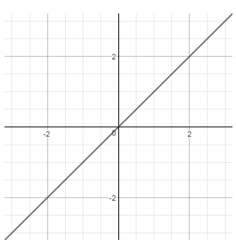
Example 5.1. Find the domain and the range of the function

$$y = f(x) = x^2$$

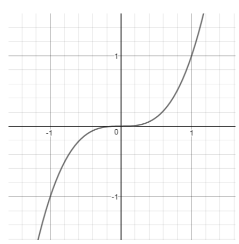
where $x \in [1, 3]$.

The domain of f is given as $D(f) = [1, 3]$. Note that the function is strictly increasing for $x \in [1, 3]$, and so the range of $f(x)$ is $R(f) = [f(1), f(3)] = [1, 9]$. ▲

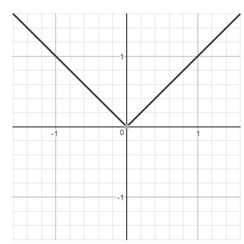
Example 5.2. More examples are shown below.



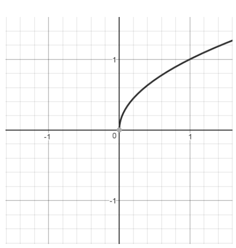
$f(x) = x$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



$f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

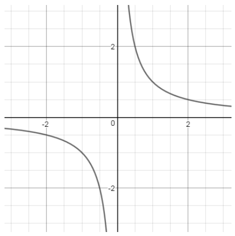


$f(x) = |x|$
Domain: $(-\infty, \infty)$
Range: $(0, \infty)$



$f(x) = \sqrt{x}$
Domain: $(0, \infty)$
Range: $(0, \infty)$

*https://mathmaine.wordpress.com/2013/06/24/domain-and-range/



$$f(x) = \frac{1}{x}$$
Domain: $(-\infty, 0) \bigcup (0, \infty)$
Range: $(-\infty, 0) \bigcup (0, \infty)$

▲

With the basic concept of what a function is, we can further investigate the properties of functions.

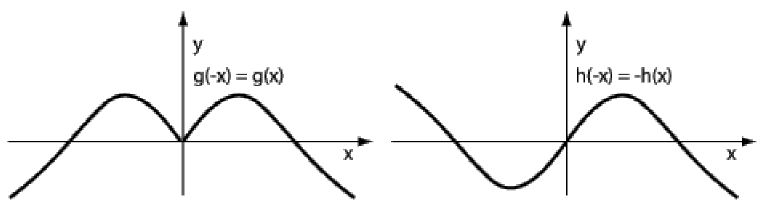
5.2 Odd and Even Function

A function $f : A \to B$ is called an **even function** if

$$f(x) = f(-x) \text{ for all } x \in A.$$

A function $f : A \to B$ is called an **odd function** if

$$-f(x) = f(-x) \text{ for all } x \in A.$$



Even function Odd function

Example 5.3. Again let us consider the function $f(x) = x^2$. For any $x \in R$,

$$f(-x) = (-x)^2 = x^2 = f(x).$$

Therefore $f(x)$ is an even function. ▲

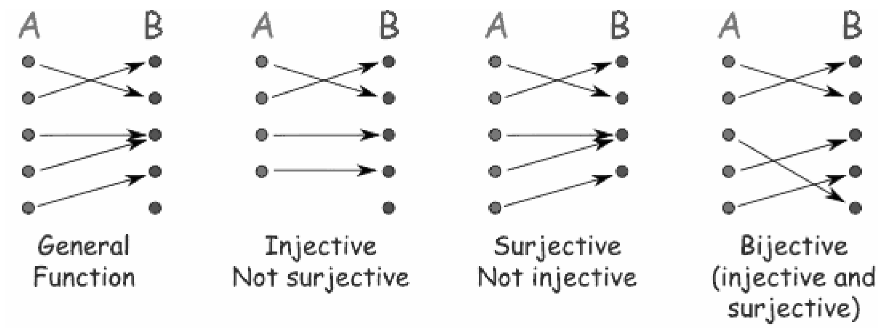
5.3 Injective and Surjective Function

Let $f{:}A \to By = f(x)$ be a function.

1. If for any a, b in A, $f(a) = f(b) \Rightarrow a = b$, f is said to be **injective**.

2. If for any b in B, there exists a in A such that $f(a) = b$, f is said to be **surjective**.

3. If a function is both injective and surjective, it is said to be **bijective**.



Example 5.4. Verify whether the function $f(x) = x^2$ is injective or surjective on real numbers.
To check the injectivity, consider a and $-a$, where a is non-zero,

$$\begin{aligned} f(-a) &= (-a)^2 \\ &= a^2 \\ &= f(a). \end{aligned}$$

Therefore, it is not injective.

To check the surjectivity, it is easily proved not surjective by a counter example.Since $f(x) = x^2 > 0$ for all x, there does not exist a number c such that

$$f(c) = -1$$

. Therefore, it is not surjective. ▲

5.4 Inverse function

A function tells you what y is if you know what x is. The inverse of a function will tell you what x has to be to get that value of y. The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

Example 5.5. Suppose

$$y = 3x - 2.$$

First, we let x be the subject.

$$\frac{y+2}{3} = x$$

Once we have "$x =$", switch x and y, then you will have the inverse function.

$$y = \frac{x+2}{3}$$

To find the domain and range of the inverse, just swap the domain and range from the original function. ▲

Example 5.6. Find the inverse of the function $y = x^2$ on $x \geq 0$.

$$y = x^2$$
$$\pm\sqrt{y} = x$$
$$y = \pm\sqrt{x}$$

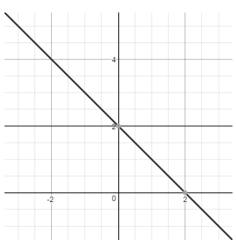
For $x \geq 0$, we choose $y = +\sqrt{x}$ as the inverse function. Please noted that inverse function has to be unique. If the function $y = x^2$ is defined on $x \in \mathbb{R}$, there are two inverse functions $y = \sqrt{x}$ and $y = -\sqrt{x}$ which contradicts the definition of an inverse function. In this case, the inverse function does not exist. ▲

5.4.1 To Prove an Inverse Function

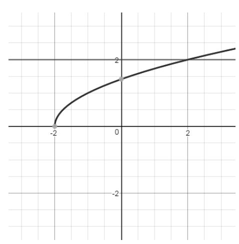
If a function $f : X \to Y$ is bijective, its inverse function $f^{-1} : Y \to X$ exists. We say the function f is invertible and the function f^{-1} is unique.

There is an easy and simple method to verify whether an inverse is a function, called the **horizontal line test**, remember that it is very possible that a function may have an inverse but at the same time not a function.

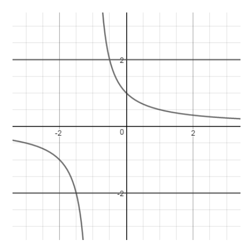
If the horizontal line intersects the graph of a function in all places at **exactly one point**, then the given function should have an inverse that is also a function. We say, this function passes the horizontal line test. Here are some examples of functions that pass the horizontal line test: †



$f(x) = -x + 2$

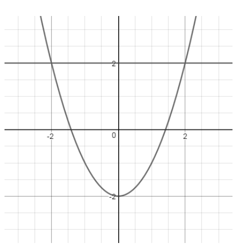


$f(x) = \sqrt{x+2}$

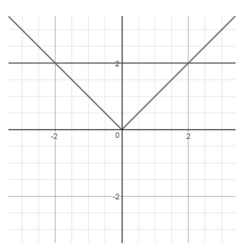


$f(x) = \dfrac{1}{x+1}$

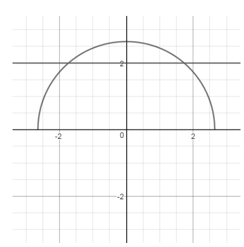
Alternatively, here are some examples that fail the test:



$f(x) = x^2 - 2$



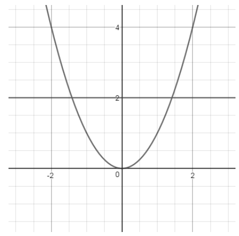
$f(x) = |x|$



$f(x) = \sqrt{7 - x^2}$

Example 5.7. Again let us consider $f(x) = x^2$.

† https://www.chilimath.com/algebra/intermediate/test/horizontal-line-test.html



According to the graph, there exists 2 points $x = \sqrt{2}$ and $x = -\sqrt{2}$ such that $f(x) = 2$.
Therefore, the function fails the horizontal line test and the inverse function does not exist. ▲

5.5 Exercise

1. Find the Domain and Range. Also, state whether each set of ordered pairs is a function or not.
   (a) $\{(-6,-6),(6,1),(-1,-8),(9,4),(-8,2)\}$
   (b) $\{(-7,3),(9,4),(3,-7),(6,-4),(6,-1)\}$
   (c) $\{(-7,9),(8,-2),(-7,-4),(-9,8),(-4,7)\}$
2. Find the domain and the range of function
   (a) $y = \dfrac{x}{x^2+1}$, where $x \in \mathbb{R}$.
   (b) $f : \mathbb{Q} \to \mathbb{Q}$, defined by $f(x) = \dfrac{x}{2}$
   (c) $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = x^2$
   (d) $h : \mathbb{R} \to \mathbb{R}$, defined by $h(x,y) = x + yi$.
   (e) $y = f(x) = \sqrt{9-x^2} + 1$.
3. In each part find the domain and the range of the given function. Then for each of the functions that has an inverse function, state the domain and range of the inverse function.
   (a) $f(x) = x^2 - 6x + 13$
   (b) $g(x) = \ln(x+2)$
   (c) $h(x) = \sqrt{1-x}$
   (d) $k(x) = \dfrac{x+2}{x-3}$
4. Prove the bijection of the following functions.
   (a) $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = 2f(x) + 3$
   (b) $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = x^3$
   (c) $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = x + |x|$
   (d) $g : \mathbb{N} \to \mathbb{N}$, defined by $f(x) = \begin{cases} 2n+1 & \text{if n is even} \\ 2n & \text{if n is odd} \end{cases}$
   (e) $g : \mathbb{N} \to \mathbb{Z}$, defined by $f(n) = (-1)^n 2n + \dfrac{1}{2}[(-1)^n - 1]$

(f) $g : 0, 1, 2, 3 \to \mathbb{N} \bigcup 0$, defined by $g(x) = 2x - x^2$

(g) $g : [0, 1] \to [-4, -1]$, defined by $g(x) = -3x - 1$

(h) $g : \mathbb{R}^{\nvDash} \to \mathbb{R}$, defined by $g(x, y) = x^2 + y^2$

5. Determine, if possible, the inverse function of the followings.

   (a) $f(x) = 5x - 2$

   (b) $f(x) = 2(x - 3)^2 - 5, x \geq 3$

   (c) $f(x) = x^2 - 4x + 6$

   (d) $f(x) = (2x + 8)^3$

   (e) $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = 8x^3 - 2$

   (f) $f : (1, 2) \to (0, 1)$, defined by $f(x) = x - [x]$

6. Let $a, b \in R$ and let f be the function $Z \to Z$ defined by $f(x) = ax + b$. For which values of a and b is f an surjective function? For which values of a and b is f an injective function?

7. Let $A = 1, 2, 3, 4, 5$. The function f is defined on A by

$$f : A \to \mathbb{N} : x \mapsto x^3 - 5x^2 + 3x + 5$$

   (a) By trial-and-error, find the maximum and minimum values of f.

   (b) State whether f is injective. Give a reason for your answer.

Chapter 6

Trigonometric Identities

6.1 Introduction

In geometry, we have learnt some elementary trigonometric identities, which are very useful in tackling problems of many topics. More useful identities will be discussed in this chapter for more advanced problems.

6.2 Imaginary Number

Imaginary number, is a number that when squared gives a negative result. Normally if you square any real number you always obtain a positive or zero result. So how can we square a number and get a negative result? Because we "imagine" that we can, and it turns out that such a number is actually useful and can solve real problems. The unit imaginary numbers (the same as "1" for real numbers) is

$$i = \sqrt{-1}$$

.

6.3 Trigonometric Identities by Euler's Formula

There are so many Trigonometric Identities that it is unwise to memorize them all by spoon-feeding. Instead, we can derive the identities by using Euler's Formula. An Euler's Formula is given by

$$e^{ix} = \cos x + i \sin x,$$

where i is the imaginary number.

Let's try proving $\sin^2 x + \cos^2 x = 1$.

Proof. Suppose that

$$e^{ix} = \cos x + i \sin x$$

and

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x.$$

We multiply both the equations, then

$$\begin{aligned} e^{ix-ix} &= (\cos x + i\sin x)(\cos x - i\sin x) \\ e^0 &= \cos^2 x - i\cos x\sin x + i\sin x\cos x - i^2\sin^2 x \\ 1 &= \cos^2 x + \sin^2 x. \end{aligned}$$

The proof is complete.

■

After experimenting with Euler's Formula, you can get the following results:

- **Double Angle Formulas**
  $\sin(2\theta) = 2\sin\theta\cos\theta$
  $\cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
  $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

- **Sum and Difference Formulas**
  $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
  $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
  $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

- **Product to Sum Formulas**
  $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
  $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
  $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$
  $\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

- **Sum to Product Formulas**
  $\sin\alpha + \sin\beta = 2sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$
  $\sin\alpha - \sin\beta = 2cos(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$
  $\cos\alpha + \cos\beta = 2cos(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$
  $\cos\alpha - \cos\beta = -2sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

6.4 Inverse Trigonometric Functions

The inverse trigonometric functions are the inverse functions of the trigonometric functions, written $\cos^{-1} x$, $\sin^{-1} x$, $\tan^{-1} x$, i.e.

$$\sin(\sin^{-1}(x)) = x.$$

$$\cos(\cos^{-1}(x)) = x.$$

$$\tan(\tan^{-1}(x)) = x.$$

Remark: "-1" in an inverse trigonometric function is NOT an exponent and so,

$$\cos^{-1} x \neq \frac{1}{\cos x}.$$

For different inverse trigonometric functions, they have different domains and ranges.

Function	Domain	Range
$y = sin^{-1}x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = cos^{-1}x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = tan^{-1}x$	$-\infty \leq x \leq \infty$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

6.5 Exercises

1. Prove, by Euler's Formula, that $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

2. Prove, by Euler's Formula, that $\cos(\alpha + \beta) = \cos\alpha + \cos\beta - \sin\alpha\sin\beta$.

3. Find $\sin 3x$ in terms of $\sin x$.

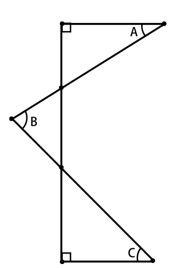
4. Find $\cos 4x$ in terms of $\cos x$.

5. Solve the equation $\sin 2x = \sin x, -\pi \leq x \leq \pi$.

6. Prove the following trigonometric identities.
   (a) $\cos(x + \frac{5\pi}{6}) = -\frac{\sqrt{3}\cos x + \sin x}{2}$.
   (b) $\tan(x - \frac{3\pi}{4}) = \frac{\tan x + 1}{1 - \tan x}$.

7. Prove the following trigonometric identities.
   (a) $\cos(x + \frac{\pi}{6}) - \sin(x + \frac{2\pi}{3}) = 0$.
   (b) $\cos(x + y)\cos(x - y) = (\cos x \cos y)^2 - (\sin x \sin y)^2$.

8. Complete both sections with the picture below.



(a) Show that $\tan B = \dfrac{\tan A + \tan C}{1 - \tan A \tan C}$.
(b) If $A = 32^o, B = 89^o$, what is the value of C?

9. Evaluate, if possible
(a) $\cot(\sin^{-1} 2)$ and $\sin(\tan^{-1} 2)$.
(b) $\sin(\cot^{-1}(-\frac{1}{2}))$ and $\cos(\cot^{-1}(-\frac{1}{2}))$.

10. Solve the following equations for $0 \leq x \leq \pi$.

    (a) $\tan 2x - \tan x = 0$
    (b) $4\sin^2 x - 2\cos^2 x + \cos 2x = 0$
    (c) $\cos 3x = \cos x$
    (d) $4\sin\left(\dfrac{\pi}{4} + x\right)\cos\left(\dfrac{3\pi}{4} - x\right) = 1$

11. (a) Prove that $\dfrac{\sec 2A + \tan 2A + 1}{\sec 2A + \sec 2A - 1} = \cot A$.
    (b) Hence, solve the equation $\dfrac{\sec 2A + \tan 2A + 1}{\sec 2A + \tan 2A - 1} = 1$ for $0 \leq A \leq \pi$.
    (c) i. Show that $\sin 2\theta = \sin(90^o - 3\theta)$ when $\theta = 18^o$.
        ii. Using the identity $\cos 3A = 4\cos^3 A - 3\cos A$, express $\sin 18^o$ in surd form.

12. (a) Express $2\cos\left(\dfrac{5\pi}{8} + \theta\right)$ in terms of $\cos 2\theta$.
    (b) Hence, solve the equation $2\cos\left(x + \dfrac{3\pi}{8}\right)\cos\left(x + \dfrac{5\pi}{8}\right) = 0$ for $0 \leq x \leq \pi$.

13. (a) Prove that $\cos 3x = 4\cos^3 x - 3\cos x$.
    (b) Hence, solve the equation $\cos 4x + 6\cos^2 x - 8\cos^4 x = 1$ for $0 \leq x \leq \pi$.

14. (a) Express $\sin\left(x - \dfrac{\pi}{3}\right)$ in terms of $\sin x$ and $\cos x$.
    (b) Hence, solve the equation $\sin x - \sqrt{3}\cos x = 1$ for $0 \leq x \leq \pi$.

15. (a) Prove that $\tan^2\theta + \cot^2\theta = \dfrac{4}{\sin^2\theta} - 2$, where $0 \leq \theta \leq \dfrac{\pi}{2}$.
    (b) Hence. find the least value of $\tan^2\theta + \cot^2\theta$.

16. A and B are two unequal acute angles.
    (a) Prove that $\dfrac{\sin 2A - \sin 2B}{\cos 2A - \cos 2B} = -\cot(A + B)$.
    (b) If $\sqrt{3}\sin A\cos A - \sin^2 A = \sqrt{3}\sin B\cos B - \sin^2 B$, find the value of $\cot(2A + 2B)$ in surd form.

17. Prove by mathematical induction that

$$\cos A + \cos 3A + \cos 5A + \cdots + \cos(2n-1)A = \frac{\sin 2nA}{2\sin A}$$

for all positive integers n, where $\sin A \neq 0$.

18. **HKCEE 1998 Additional Mathematics Paper 2 Q7**
    Show that $\sin\left(3x+\frac{\pi}{4}\right)\cos\left(3x-\frac{\pi}{4}\right)=\frac{1+\sin 6x}{2}$.

19. **HKCEE 2002 Additional Mathematics Q8**
    Given $0<x<\frac{\pi}{2}$. Show that $\frac{\tan x-\sin^2 x}{\tan x+\sin^2 x}=\frac{4}{2+\sin 2x}-1$.
    Hence, or otherwise, find the least value of $\frac{\tan x-\sin^2 x}{\tan x+\sin^2 x}$.

20. **HKCEE 2003 Additional Mathematics Q10**
    Given two acute angles α and β.
    (a) Show that $\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=\tan\left(\frac{\alpha+\beta}{2}\right)$.
    (b) If $3\sin\alpha-4\cos\alpha=4\cos\beta-3\sin\beta$, find the value of $\tan(\alpha+\beta)$.

Chapter 7

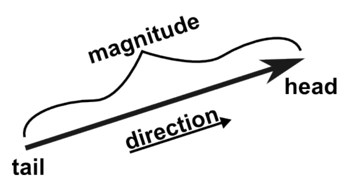
Vector

In everyday life, we always encounter different varieties of quantities, and there are mainly two different kind of quantities: scalars and vectors. **Scalar** is the measurement of a medium strictly in magnitude (size), like time, weight, height, the number of teeth etc., while **vector** is a measurement that refers to both the magnitude of the medium as well as the direction of the movement the medium has taken, like velocity, force etc.

7.1 Introduction

A vector, $\overrightarrow{AB}$ is represented by a directed line segment from an initial point A to a terminal point B. The magnitude (or length) of $\overrightarrow{AB}$ is denoted by $use\|\overrightarrow{AB}\|$.

In the figure, we call it a vector from the tail A to the head B.



For scalar, we say two numbers are the same if their magnitudes are the same, like 4 and 4, 3.14159 and π. For vector, the two vectors have to be the same in magnitude and direction in order to be the same. i.e. if $AB = CD$ and $AB//CD$, then $\overrightarrow{AB} = \overrightarrow{CD}$.

7.2 Definition

7.2.1 Zero Vector

A vector with zero magnitude is called a **zero vector**. It is usually denoted by 0. E.g. $\overrightarrow{AA}$ is a zero vector.

7.2.2 Unit Vector

A vector $\mathbf{v}$ with magnitude of 1 unit is called a unit vector. It is usually denoted by $\hat{\text{v}}$. Unit vector is usually used to indicate direction. We shall have further discussion about it later.

7.2.3 Negative Vector

Given a vector $\mathbf{v}$, the negative vector of $\mathbf{v}$, denoted by $-\mathbf{v}$, is a vector with the same magnified as $\mathbf{v}$ but in the opposite direction of $\mathbf{v}$. E.g. $\overrightarrow{BA}$ is the negative vector of $\overrightarrow{AB}$, i.e. $\overrightarrow{BA} = -\overrightarrow{AB}$.

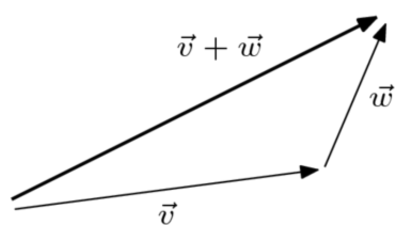
7.3 Operation

Like normal scalar, we can also do addition, subtraction and multiplication on vector, except that we cannot do division on vectors.

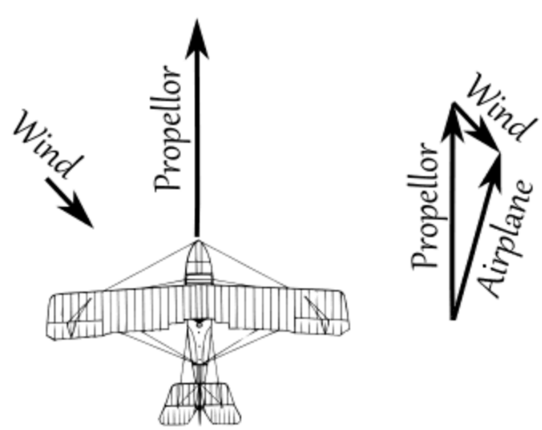
7.3.1 Addition

Triangle Law of Addition

The length of the line shows its magnitude and the arrowhead points in the direction. We can add two vectors by joining them head-to-tail.



Example 7.1. A plane is flying along, pointing North, but there is a wind coming from the North-West as shown below.

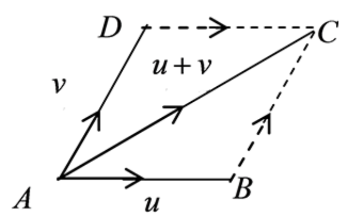


The two vectors (the velocity caused by the propeller, and the velocity of the wine) result in a slightly slower ground speed heading a little East of North.

If you watched the plane from the ground, you will see that the plane is not actually going straight, but slipping sideway a little.* ▲

Parallelogram Law of Addition

Draw the vectors so that their initial points coincide. Then draw lines to form a complete parallelogram. The diagonal from the initial point to the opposite vertex of the parallelogram is the resultant. †
In the figure, $\mathbf{v}=\overrightarrow{AD}$ and $\mathbf{u}=\overrightarrow{AB}$.

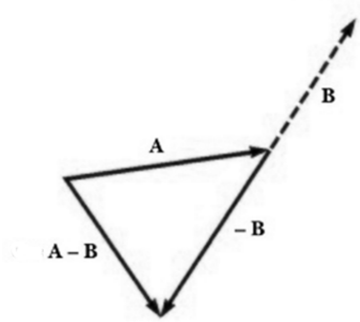


Consider the parallelogram $ABCD$. Because $\overrightarrow{AB}=\overrightarrow{DC}$, therefore

$$\begin{aligned}\mathbf{v}+\mathbf{u} &= \overrightarrow{AD}+\overrightarrow{AB}\\ &= \overrightarrow{AD}+\overrightarrow{DC}\\ &= \overrightarrow{AC}\end{aligned}$$

7.3.2 Subtraction

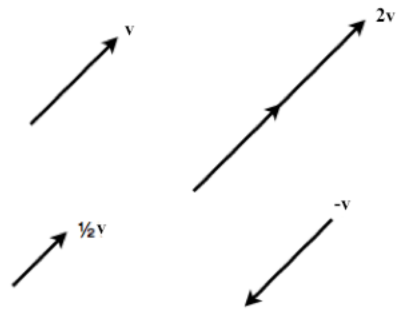
The subtraction of a vector $\mathbf{b}$ from another vector $\mathbf{a}$, denoted by $\mathbf{a}-\mathbf{b}$, is defined as $\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$, where $-\mathbf{b}$ is the negative vector of $\mathbf{b}$. Then process the two vectors with the head-to-tail method and treat it like an addition, there you go.



7.3.3 Scalar multiplication

Scalar multiplication refers to the multiplication of a vector by a constant λ, producing a vector in the same (for $\lambda > 0$) or opposite (for $\lambda < 0$) direction but of different length. If the constant λ is zero, the vector becomes zero as well. E.g. $\mathbf{2v}$ has the same direction but double magnitude of $\mathbf{v}$.

*https://www.mathsisfun.com/algebra/vectors.html
†`http://hotmath.com/hotmath_help/topics/adding-and-subtracting-vectors.html`



7.4 Rule of Vectors

We have just studied the basic operations on vectors. Next, let us investigate the properties of vectors operations. With the help of these properties, we can manipulate the vectors operations more easily.

For any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and scalars λ, μ, λ_1 and μ_1, the following properties hold.

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + 0 = \mathbf{a}$
4. $0\mathbf{a} = 0$
5. $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$
6. $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$
7. $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$
8. $\mathbf{a} = k\mathbf{b}$ if and only if $\mathbf{a} \,//\, \mathbf{b}$
9. Suppose $\mathbf{a}$ and $\mathbf{b}$ are non-zero and are not parallel to each other.

   (a) If $\lambda\mathbf{a} + \lambda_1\mathbf{b} = 0$, then $\lambda = \lambda_1 = 0$

   (b) If $\lambda\mathbf{a} + \mu\mathbf{b} = \lambda_1\mathbf{a} + \mu_1\mathbf{b}$,, then $\lambda = \lambda_1$ and $\mu = \mu_1$.

   (c) If $(\lambda\mathbf{a} + \mu\mathbf{b})//(\lambda_1\mathbf{a} + \mu_1\mathbf{b})$, then $\dfrac{\lambda}{\lambda_1} = \dfrac{\mu}{\mu_1}$.

Proof. To prove rule $9(a)$, suppose $\mathbf{a}$ and $\mathbf{b}$ are non-zero and are not parallel to each other.

$$\begin{aligned} \lambda\mathbf{a} + \mu\mathbf{b} &= \lambda_1\mathbf{a} + \mu_1\mathbf{b} \\ \lambda\mathbf{a} - \lambda_1\mathbf{a} &= \mu_1\mathbf{b} - \mu\mathbf{b} \\ (\lambda - \lambda_1)\mathbf{a} &= -(\mu - \mu_1)\mathbf{b} \end{aligned}$$

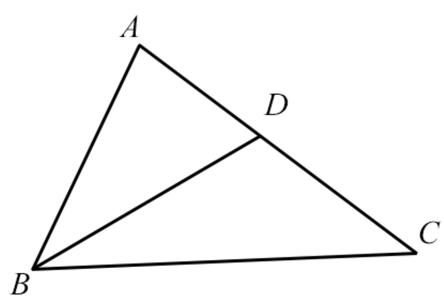
Without loss of generality, assume that $\lambda - \lambda_1 \neq 0$. Then $\mathbf{a} = -\dfrac{\mu - \mu_1}{\lambda - \lambda_1}$
From the equation, we can see $\mathbf{a}$ and $\mathbf{b}$ are parallel which contradicts to the given condition that $\mathbf{a}$ and $\mathbf{b}$ are not parallel to each other.
The assumptions that $\lambda - \lambda_1 \neq 0$ or $\mu - \mu_1 \neq 0$ are wrong.

Hence, $\lambda = \lambda_1$ and $\mu = \mu_1$.
The proof is thus complete. The rest are left for student exercises.

■

Example 7.2. In the figure, ABC is a triangle and D is the mid-point of AC. Prove that

$$\overrightarrow{BD} = \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{BC}).$$

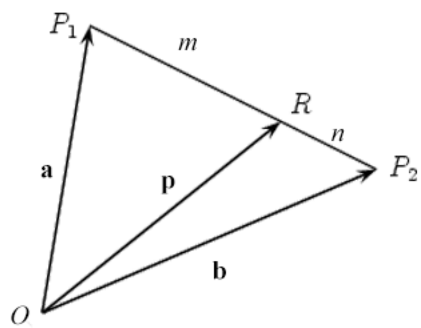


Since D is the mid-point of AC, $\overrightarrow{AD} = \frac{1}{2}\overrightarrow{AC}$.

$$\begin{aligned}
L.H.S. &= \overrightarrow{BD} \\
&= \overrightarrow{BA} + \overrightarrow{AD} \\
&= \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC} \\
&= \overrightarrow{BA} + \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) \\
&= \overrightarrow{BA} + \frac{1}{2}(-\overrightarrow{BA} + \overrightarrow{BC}) \\
&= \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{BC} \\
&= \frac{1}{2}(\overrightarrow{BA} + \overrightarrow{BC})
\end{aligned}$$

▲

7.5 Point of Division



In the figure, let the position vectors of P_1, P_2 and R with respect to a reference point O be $\mathbf{a}, \mathbf{b}$ and $\mathbf{p}$ respectively. If $\mathbf{p}$ divides the line segment P_1P_2 in the ratio $m : n$, we can express $\mathbf{p}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ as

$$\mathbf{p} = \frac{n\mathbf{a} + m\mathbf{b}}{m+n},$$

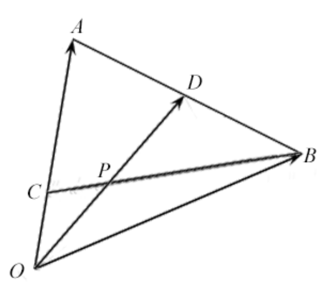
Let us prove the above result.

Proof. Since $\overrightarrow{P_1R}$ and $\overrightarrow{RP_2}$ are in the same direction, and $P_1R : RP_2 = m : n$,

$$\begin{aligned} n\overrightarrow{P_1R} &= m\overrightarrow{RP_2} \\ n(\mathbf{p} - \mathbf{a}) &= m(\mathbf{b} - \mathbf{p}) \\ (m+n)\mathbf{p} &= n\mathbf{a} + m\mathbf{b} \\ \mathbf{p} &= \frac{n\mathbf{a} + m\mathbf{b}}{m+n} \end{aligned}$$

■

Example 7.3. In the figure, P is the mid-point of OD and it divides CB in the ratio $1 : 3$. Let the position vectors of A and B with respect to O be $\mathbf{a}$ and $\mathbf{b}$ respectively. D divides AB in the ratio $1 : m$ and $\overrightarrow{OC} = n\overrightarrow{OA}$.



(a) Express $\overrightarrow{OD}$ in terms of $\mathbf{a}, \mathbf{b}$ and m.

(b) Express $\overrightarrow{OP}$ in terms of $\mathbf{a}, \mathbf{b}$ and n.

(c) Hence, find $AD : DB$.

(a) $\overrightarrow{OD} = \dfrac{m\overrightarrow{OA} + \overrightarrow{OB}}{1+m} = \dfrac{m\mathbf{a} + \mathbf{b}}{1+m}$

(b) $\overrightarrow{OP} = \dfrac{3\overrightarrow{OC} + \overrightarrow{OB}}{1+3} = \dfrac{3n\overrightarrow{OA} + \overrightarrow{OB}}{4} = \dfrac{3n\mathbf{a} + \mathbf{b}}{4}$

(c) Since P is the mid-point of OD,

$$\begin{aligned} \overrightarrow{OP} &= \frac{1}{2}\overrightarrow{OD} \\ \frac{3n\mathbf{a} + \mathbf{b}}{4} &= \frac{m\mathbf{a} + \mathbf{b}}{2(1+m)}. \end{aligned}$$

Compare the components in the direction of $\mathbf{b}$.

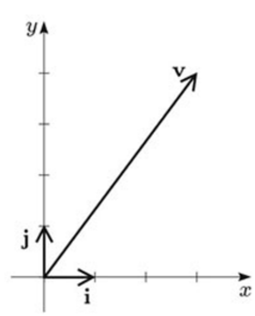
$$\begin{aligned} \frac{1}{4} &= \frac{1}{2(1+m)} \\ m &= 1 \end{aligned}$$

$\therefore AD : DB = 1 : 1$

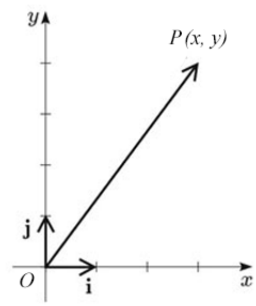
▲

7.6 Representation of Vectors in Cartesian Form

For any vector $\mathbf{v}$, if $\mathbf{v} = \mathbf{a} + \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are not parallel to each other, then $\mathbf{a}$ and $\mathbf{b}$ are called the components of $\mathbf{v}$.
In a rectangular coordinate plane, the unit vectors along the positive x-axis and the positive y-axis are denoted by $\mathbf{i}$ and $\mathbf{j}$ respectively.



Every point $P(x, y)$ on a cartesian plane can be represented by a vector $\overrightarrow{OP}$ with respect to the origin O in terms of $\mathbf{i}$ and $\mathbf{j}$. i.e.



$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j}$$

By Pythagoras' theorem, the magnitude of $\overrightarrow{OP}$ can be expressed as

$$\|\overrightarrow{OP}\| = \sqrt{x^2 + y^2}$$

The direction of $\overrightarrow{OP}$ is specified by angle θ which is measure anti-clockwise from the positive x-axis to $\overrightarrow{OP}$. Then

$$\tan\theta = \frac{y}{x}, \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}, \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

Example 7.4. It is given that $\overrightarrow{AB} = 2\mathbf{i} - \mathbf{j}$ and $\overrightarrow{CD} = -\overrightarrow{AB}$. (a)If the coordinates of A are $(1, 4)$, find the coordinates of B; (b)if the coordinates of D are $(3, 1)$, find the coordinates of C.

(a)

$$\begin{aligned}
\overrightarrow{OA} &= \mathbf{i} + 4\mathbf{j} \\
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
\overrightarrow{OB} &= \overrightarrow{OA} + \overrightarrow{AB} \\
&= (\mathbf{i} + 4\mathbf{j}) + (2\mathbf{i} - \mathbf{j}) \\
&= (1 + 2)\mathbf{i} + (4 - 1)\mathbf{j} \\
&= 3\mathbf{i} + 3\mathbf{j}
\end{aligned}$$

Therefore, the coordinates of B are $(3, 3)$.

(b)

$$\begin{aligned}
\overrightarrow{OD} &= 3\mathbf{i} + \mathbf{j} \\
\overrightarrow{CD} &= \overrightarrow{OD} - \overrightarrow{OC} \\
-\overrightarrow{AB} &= \overrightarrow{OD} - \overrightarrow{OC} \\
\overrightarrow{OC} &= \overrightarrow{OD} + \overrightarrow{AB} \\
&= (3\mathbf{i} + \mathbf{j}) + (2\mathbf{i} - \mathbf{j}) \\
&= 5\mathbf{i}
\end{aligned}$$

Therefore, the coordinates of C are $(5, 0)$. ▲

7.7 Scalar Product

7.7.1 Definition

The **scalar product** of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \cdot \mathbf{b}$, is defined as

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ with $0 \leq \theta \leq \pi$. Recall that $\|\mathbf{a}\|$ represents the magnitude of $\mathbf{a}$.

Note:

1. $\mathbf{a} \cdot \mathbf{b}$ is read as "a dot b". The symbol "·" cannot be omitted or replaced by "×".
2. The scalar product is also called the **dot product**.
3. The result of a scalar product is a scalar, not a vector.
4. If one of the vectors $\mathbf{a}$ and $\mathbf{b}$ is a zero vector, then $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 7.5. If k is any positive number, what is the size of the angle between the vectors $\mathbf{a} = k\mathbf{i} + k\mathbf{j}$ and $\mathbf{b} = -3\mathbf{i} + 4\mathbf{j}$?

By scalar product, $\mathbf{ab} = (-3) \times k + 4 \times k = k$
$\|\mathbf{a}\| = \sqrt{k^2 + k^2} = \sqrt{2}k$, $\|\mathbf{b}\| = \sqrt{(-3)^2 + 4^2} = 5$.
Then by the formula,

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta \\
k &= \sqrt{2}k \cdot 5 \cdot \cos\theta \\
\Rightarrow \cos\theta &= \frac{1}{5\sqrt{2}} \\
\Rightarrow \theta &= \cos^{-1}\frac{1}{5\sqrt{2}} = 0.455\pi.
\end{aligned}$$

▲

7.7.2 Properties of Scalar Product

For any vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, and scalar λ, the following are true.

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. $\mathbf{a} \cdot (\lambda\mathbf{b}) = (\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2 \geq 0$
5. $\mathbf{a} \cdot \mathbf{a} = 0$ if and only if $\mathbf{a} = 0$
6. $\|\mathbf{a}\|\|\mathbf{b}\| \geq \|\mathbf{a} \cdot \mathbf{b}\|$
7. $\|\mathbf{a\text{-}b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2(\mathbf{a} \cdot \mathbf{b})$

7.7.3 Scalar Product in the Rectangular Coordinate System

In two-dimensional space, we denote $\mathbf{i}$ as the vertical (y-axis) unit vector and $\mathbf{j}$ as the horizontal (x-axis) unit vector. They are perpendicular to each other. Therefore,
$\mathbf{i}\cdot\mathbf{i} = \|\mathbf{i}\|\|\mathbf{i}\|\cos 0 = 1$ $\quad$ $\mathbf{j}\cdot\mathbf{j} = \|\mathbf{j}\|\|\mathbf{j}\|\cos 0 = 1$ $\quad$ $\mathbf{i}\cdot\mathbf{j} = \|\mathbf{i}\|\|\mathbf{j}\|\cos\dfrac{\pi}{2} = 0.$

Consider two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ in two-dimensional space. Then

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j})\cdot(b_1\mathbf{i} + b_2\mathbf{j}) \\
&= (a_1b_1)(\mathbf{i}\cdot\mathbf{i}) + (a_1b_2 + a_2b_1)(\mathbf{i}\cdot\mathbf{j}) + (a_2b_2)(\mathbf{j}\cdot\mathbf{j}) \\
&= a_1b_1 + a_2b_2.
\end{aligned}$$

Example 7.6. Determine whether or not the vectors $2\mathbf{i} + 4\mathbf{j}$ and $-\mathbf{i} + \dfrac{1}{2}\mathbf{j}$ are perpendicular.

$$\begin{aligned}
(2\mathbf{i} + 4\mathbf{j})\cdot(-\mathbf{i} + \frac{1}{2}\mathbf{j}) &= (2)(-1) + (4)(\frac{1}{2}) \\
&= -2 + 2 \\
&= 0
\end{aligned}$$

Since they are non-zero vectors, and their scalar product is zero, we deduce that they must be perpendicular.

▲

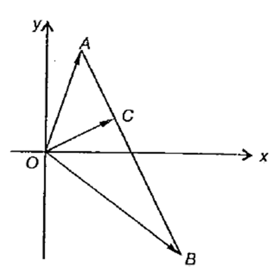
Exercises‡

1. Given $\overrightarrow{OA} = 3\mathbf{i} - 2\mathbf{j}$, $\overrightarrow{OB} = -\mathbf{i} + \mathbf{j}$

   (a) Find the unit vector in the direction of $\overrightarrow{AB}$.

   (b) If P is a point such that $\overrightarrow{AP} = m\overrightarrow{AB}$, express $\overrightarrow{OP}$ in terms of m.

2.



   In the figure, $\overrightarrow{OA} = \mathbf{i} + 3\mathbf{j}, \overrightarrow{OB} = 4\mathbf{i} - 3\mathbf{j}$. C is a point on AB such that $\dfrac{AC}{CB} = r$.

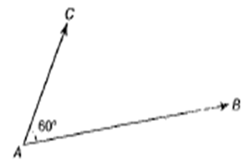
   (a) Express $\overrightarrow{OC}$ in terms of r.

‡HKCEE 1980-2001

(b) Find the value of r if OC is perpendicular to AB. Hence, find the coordinates of C.

3. The angle between the two vectors $\mathbf{i}+\mathbf{j}$ and $(c+4)\mathbf{i}+(c-4)\mathbf{j}$ is θ, where $\cos\theta = -\frac{3}{5}$. Find the value of the constant c.

4.



In the figure, $\|\overrightarrow{AB}\| = 3$, $\|\overrightarrow{AC}\| = 1$ and $\angle CAB = \frac{\pi}{3}$. Find

(a) $\overrightarrow{AB}\cdot\overrightarrow{AC}$,

(b) $\|\overrightarrow{AB}+2\overrightarrow{AC}\|$.

5. Let $a = 3\mathbf{i}+4\mathbf{j}$, $b = 8\mathbf{i}+6\mathbf{j}$ and $c = 2\mathbf{i}+k\mathbf{j}$ where k is a constant. Find $\mathbf{a}\cdot\mathbf{c}$ and $\mathbf{b}\cdot\mathbf{c}$ in terms of k. If $\mathbf{c}$ makes equal angles with $\mathbf{a}$ and $\mathbf{b}$, evaluate k.

6. Let $\overrightarrow{OA} = \mathbf{i}+3\mathbf{j}$, $\overrightarrow{OB} = 4\mathbf{i}-\mathbf{j}$ and C be a point dividing AB internally in the ratio $k:1$.

(a) Express $\overrightarrow{OC}$ in terms of $k, \mathbf{i}$ and $\mathbf{j}$.

(b) If OC is perpendicular to AB, find the value of k.

7. Given $\overrightarrow{OA} = 5\mathbf{j}, \overrightarrow{OB} = -\mathbf{i}+7\mathbf{j}$. P is a point such that $\overrightarrow{AP} = t\overrightarrow{AB}$.

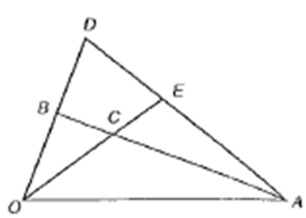
(a) Express $\overrightarrow{OP}$ in terms of t.

(b) If OP is perpendicular to AB, find

i. the value of t.

ii. $\overrightarrow{OP}$.

8.



In the figure, OAD is a triangle and B is the mid-point of OD. The line OE cuts the line AB at C such that $AC : CB = 3:1$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) i. Let $OC : CE = k : 1$. Express $\overrightarrow{OE}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$.

ii. Let $AE : ED = m : 1$. Express $\overrightarrow{OE}$ in terms of m, $\mathbf{a}$ and $\mathbf{b}$. Hence, find k and m.

9. Given $\overrightarrow{OA} = 5\mathbf{i} - \mathbf{j}$, $\overrightarrow{OB} = -3\mathbf{i} + 5\mathbf{j}$ and APB is a straight line.

(a) Find $\overrightarrow{AB}$ and $\|\overrightarrow{AB}\|$

(b) If $\|\overrightarrow{AP}\| = 4$, find $\overrightarrow{AP}$.

10. Given $\overrightarrow{OA} = 3\mathbf{i} - 2\mathbf{j}, \overrightarrow{OB} = \mathbf{i} + \mathbf{j}$. C is a point such that $\angle ABC$ is a right angle.

(a) Find $\overrightarrow{AB}$.

(b) Find $\overrightarrow{AB} \cdot \overrightarrow{AB}$ and $\overrightarrow{AB} \cdot \overrightarrow{BC}$. Hence, find $\overrightarrow{AB} \cdot \overrightarrow{AC}$.

11. P, Q and R are points on a plane such that $\overrightarrow{OP} = \mathbf{i} + 2\mathbf{j}$, $\overrightarrow{OQ} = 3\mathbf{i} + \mathbf{j}$ and $\overrightarrow{PR} = -3\mathbf{i} - 2\mathbf{j}$, where O is the origin.

(a) Find $\overrightarrow{PQ}$ and $\|\overrightarrow{PQ}\|$.

(b) Find the value of $cos\angle QPR$.

12. Let $\overrightarrow{OP} = 2\mathbf{i} + 3\mathbf{j}$ and $\overrightarrow{OQ} = -6\mathbf{i} + 4\mathbf{j}$. Let R be a point on PQ such that $PR : RQ = k : 1$, where $k > 0$.

(a) Express $\overrightarrow{OR}$ in terms of k, $\mathbf{i}$ and $\mathbf{j}$

(b) Express $\overrightarrow{OP} \cdot \overrightarrow{OR}$ and $\overrightarrow{OQ} \cdot \overrightarrow{OR}$ in terms of k.

(c) Find the value of k such that OR bisects $\angle POQ$.

13. Given $\overrightarrow{OA} = 4\mathbf{i} + 3\mathbf{j}$ and C is a point on OA such that $\|\overrightarrow{OC}\| = \dfrac{16}{5}$.

(a) Find the unit vector in the direction of $\overrightarrow{OA}$. Hence, find $\overrightarrow{OC}$.

(b) If $\overrightarrow{OB} = \mathbf{i} + 4\mathbf{j}$, show that BC is perpendicular to OA.

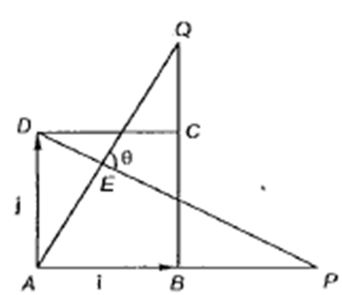
14. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors such that $\mathbf{a} = 2\mathbf{i} + 4\mathbf{j}$, $\|\mathbf{b}\| = \sqrt{5}$ and $cos\theta = \dfrac{4}{5}$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

(a) Find $\|\mathbf{a}\|$.

(b) Find $\mathbf{a} \cdot \mathbf{b}$.

(c) If $\mathbf{b} = m\mathbf{i} + n\mathbf{j}$, find the values of m and n.

15.



In the figure, $ABCD$ is a square with $\overrightarrow{AB} = \mathbf{i}$ and $\overrightarrow{AD} = \mathbf{j}$. P and Q are respectively points on AB and BC produced with $BP = k$ and $CQ = m$. AQ and DP intersect at E and $\angle QEP = \theta$.

(a) By calculating $\overrightarrow{AQ} \cdot \overrightarrow{DP}$, find $cos\theta$ in terms of m and k.

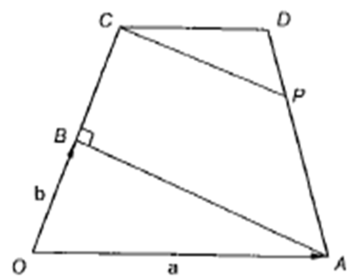
(b) Given that $\dfrac{DE}{EP} = \dfrac{1}{4}$.

  i. Express $\overrightarrow{AE}$ in terms of k.

  ii. Let $\dfrac{AE}{AQ} = r$. Express $\overrightarrow{AE}$ in terms of r and m.

  iii. If $\theta = \dfrac{\pi}{2}$, use the above results to find the values of k, m and r.

16.



In the figure, $\triangle OBA$ is a right-angled at B. OB is produced to C such that $OB = BC$. CD is drawn in the direction of OA such that $CD = kOA$. P is a point on AD such that $CP//BA$. Let $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{DP} = \lambda\overrightarrow{DA}$.

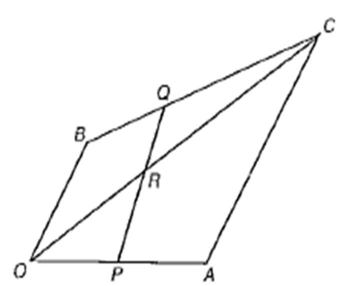
(a) i. Express $\overrightarrow{OD}$ and $\overrightarrow{DA}$ in terms of $\mathbf{a}, \mathbf{b}$ and k.

  ii. Find $\overrightarrow{BA}$ in terms of $\mathbf{a}, \mathbf{b}$ and express $\overrightarrow{CP}$ in terms of $\mathbf{a}, \mathbf{b}, \lambda$ and k. Hence, find λ in terms of k.

(b) i. Show that $\mathbf{a} \cdot \mathbf{b} = OB^2$

  ii. If $OB = \dfrac{1}{4}OA$, show that $\overrightarrow{OD} \cdot \overrightarrow{DA} = (-16k^2 + 12k - 2)OB^2$. Hence, find the values of k and λ if $OD \perp DA$.

17.



In the figure, $OACB$ is a trapezium with $OB//AC$ and $AC = 2OB$. P and Q are points on OA and BC respectively such that $OP = \dfrac{1}{2}OA$ and $BQ = \dfrac{1}{3}BC$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OA}, \overrightarrow{BC}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) OC intersects PQ at the point R. Let $PR : RQ = h : 1 - h$.

i. Express $\overrightarrow{OR}$ in terms of $\mathbf{a}, \mathbf{b}$ and h.

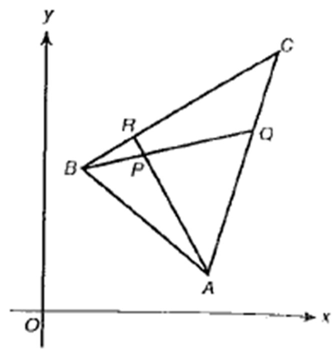
ii. If $\overrightarrow{OR} = k\overrightarrow{OC}$, find h and k.

(c) OB and PQ are produced to meet T and $\overrightarrow{OT} = \lambda\mathbf{b}$.

i. Express $\overrightarrow{PQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and express $\overrightarrow{PT}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and λ.

ii. Hence, or otherwise, find the value of λ.

18.



In the figure, R is a point on BC such that $BR : RC = m : 1$. Q is a point on AC. BQ intersects AR at P. $\overrightarrow{OA} = 4\mathbf{i} + \mathbf{j}$, $\overrightarrow{OB} = \mathbf{i} + 4\mathbf{j}$, $\overrightarrow{OC} = 7\mathbf{i} + 7\mathbf{j}$ and $\overrightarrow{BQ} = 5\mathbf{i} + \mathbf{j}$.

(a) i. Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

ii. Express $\overrightarrow{AR}$ in terms of m, $\mathbf{i}$ and $\mathbf{j}$.

(b) Suppose AR is perpendicular to BC.

i. Show that $m = \dfrac{1}{4}$.

ii. Find $\angle QPR$.

iii. If $\overrightarrow{BQ} = \lambda\overrightarrow{BA} + \mu\overrightarrow{BC}$, find the values of λ and μ.

iv. If $AP : PR = n : 1$, express $\overrightarrow{BP}$ in terms of n, $\mathbf{i}$ and $\mathbf{j}$. Hence, find the value of n.

19. A, B and C are three points on a plane such that $\overrightarrow{OA} = \mathbf{i}+2\mathbf{j}$, $\overrightarrow{OB} = -6\mathbf{i}-2\mathbf{j}$ and $\overrightarrow{AB}-\overrightarrow{BC} = -12\mathbf{i}+6\mathbf{j}$, where O is the origin.

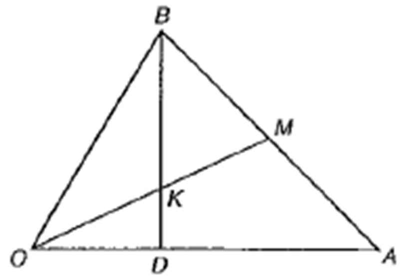
(a) Find $\overrightarrow{AB}$ and $\overrightarrow{OC}$.

(b) X is a point on the plane such that $\overrightarrow{AX} = k\overrightarrow{OX}$.

i. Express $\overrightarrow{OX}$ in terms of k, $\mathbf{i}$ and $\mathbf{j}$.

ii. If $OX \perp BX$, find the value of k and hence find $\overrightarrow{AX} + \overrightarrow{BX} + \overrightarrow{CX}$. Furthermore, if M is the mid-point of BC, find $\overrightarrow{AM}$ and hence show that X lies on AM.

20.



In the figure, M is the mid-point of AB and D is a point on OA such that $OD : DA = 1 : 2$, OM intersects BD at K. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) i. Express $\overrightarrow{OM}$ and $\overrightarrow{OD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

ii. Suppose $BK : KD = \lambda : 1 - \lambda$. Express $\overrightarrow{OK}$ in terms of $\mathbf{a}, \mathbf{b}$ and λ. Let $\overrightarrow{OK} = \mu\overrightarrow{OM}$. Find the values of λ and μ.

(b) Suppose $\mathbf{a} = 12\mathbf{i}$ and $\mathbf{b} = 2\mathbf{i} + 8\mathbf{j}$.

i. Find $\overrightarrow{OM}$ and $\overrightarrow{DB}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.

ii. Evaluate $\overrightarrow{OM} \cdot \overrightarrow{DB}$ and hence find $\angle BKM$.

iii. Suppose P is the point on OB such that $OP : PB = 1 : 2$. Find $\overrightarrow{AP}$ and $\overrightarrow{AK}$, and hence show that A,K,P are collinear.

21. A, B and C are three points on a plane such that $\overrightarrow{OA} = 3\mathbf{i} - \mathbf{j}$,$\overrightarrow{BC} = 7\mathbf{i} + \mathbf{j}$ and $\overrightarrow{OC} = x\mathbf{i} + y\mathbf{j}$, where O is the origin.

(a) Find $\overrightarrow{CA}$, $\overrightarrow{OB}$ and $\overrightarrow{AB}$ in terms of x, y, $\mathbf{i}$ and $\mathbf{j}$.

(b) Given $\overrightarrow{AB} \cdot \overrightarrow{BC} = 4\overrightarrow{BC} \cdot \overrightarrow{CA}$

i. Show that $y = 30 - 7x$.

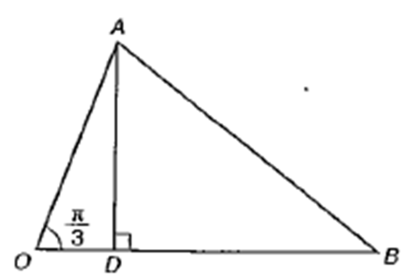
ii. If $\|\overrightarrow{BC}\| = \sqrt{5}\|\overrightarrow{CA}\|$ and x, y are positive,

A. find x and y,

B. show that CA is perpendicular to AB,

C. show that O lies on AB.

22.



In the figure, OA=2, $OB = 3$ and $\angle AOB = \dfrac{\pi}{3}$. D is a point on OB such that AD is perpendicular to OB. Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$.

(a) Find $\mathbf{a} \cdot \mathbf{a}$ and $\mathbf{b} \cdot \mathbf{b}$.

(b) Find the length of OD. Hence, express $\overrightarrow{OD}$ in terms of $\mathbf{b}$.

(c) Let H be a point on AD such that $AH : HD = 1 : k$ and $\overrightarrow{OH}$ is perpendicular to $\overrightarrow{AB}$.

  i. Express $\overrightarrow{OH}$ in terms of k, $\mathbf{a}$ and $\mathbf{b}$. Hence, find the value of k.

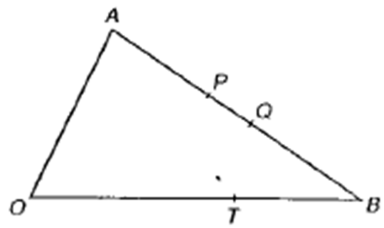
  ii. OH produced meets AB at a point C. Let $AC : CB = 1 : m$ and $OH : HC = 1 : n$.

    A. Express $\overrightarrow{OC}$ in terms of m, $\mathbf{a}$ and $\mathbf{b}$.

    B. Express $\overrightarrow{OC}$ in terms of n, $\mathbf{a}$ and $\mathbf{b}$.

    C. Hence, find m and n.

23.



In the figure, OAB is a triangle, P, Q are two points on AB such that $AP : PB = PQ : QB = r : 1$, where $r > 0$. T is a point on OB such that $OT : TB = 1 : r$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OA} = \mathbf{b}$.

(a) Express $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of r, $\mathbf{a}$ and $\mathbf{b}$.

(b) Express $\overrightarrow{OT}$ in terms of r and $\mathbf{b}$. Hence, show that $\overrightarrow{TQ} = \dfrac{\mathbf{a} + (r^2 + r - 1)\mathbf{b}}{(r+1)^2}$.

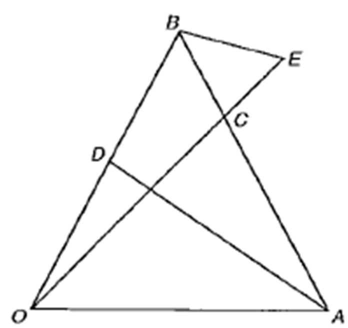
(c) Find the value(s) of r such that $\overrightarrow{OA}$ is parallel to $\overrightarrow{TQ}$.

(d) Suppose $OA = 2$, $OB = 16$ and $\angle AOB = \dfrac{\pi}{3}$.

i. Find $\mathbf{a} \cdot \mathbf{a}$ and $\mathbf{a} \cdot \mathbf{b}$.

ii. Find the value(s) of r such that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{TQ}$.

24.



In the figure, D is the mid-point of OB and C is a point on AB such that $AC : CB = 2 : 1$. OC is produced to a point E such that $OC : CE = 1 : k$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Express $\overrightarrow{OC}$ and $\overrightarrow{DA}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Show that $\overrightarrow{BE} = \dfrac{k+1}{3}\mathbf{a} + \dfrac{2k-1}{3}\mathbf{b}$.

(c) Find the value of k such that $\overrightarrow{BE}$ is parallel to $\overrightarrow{DA}$.

(d) Given $\|\mathbf{a}\| = 1, \|\mathbf{b}\| = 2$ and $\angle BOA = \dfrac{\pi}{3}$.

i. Find $\mathbf{a} \cdot \mathbf{b}$.

ii. Find the value of k such that $\overrightarrow{BE}$ is perpendicular to $\overrightarrow{OE}$. Hence, find the distance of B from OC.

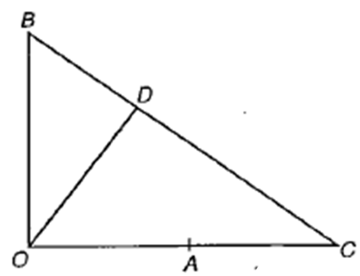
25. Let $\mathbf{a}, \mathbf{b}$ be two vectors such that $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\|\mathbf{b}\| = 4$. The angle between $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{\pi}{3}$.

(a) Find $\|\mathbf{a}\|$.

(b) Find $\mathbf{a} \cdot \mathbf{b}$.

(c) If the vector $(m\mathbf{a} + \mathbf{b})$ is perpendicular to $\mathbf{b}$, find the value of m.

26.



In the figure, $\overrightarrow{OA} = \mathbf{i}$, $\overrightarrow{OB} = \mathbf{j}$. C is a point on OA produced such that $AC = k$, where $k > 0$. D is a point on BC such that $BD : DC = 1 : 2$.

(a) Show that $\overrightarrow{OD} = \dfrac{1+k}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j}$.

(b) If $\overrightarrow{OD}$ is a unit vector, find

i. k,

ii. $\angle BOD$, giving your answer correct to the nearest degree.

Chapter 8

Determinant and Matrix

8.1 Matrices

A matrix is a rectangular table of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

A matrix is said to be of dimension $m \times n$ when it has m rows and n columns. This method of describing the size of a matrix is necessary in order to avoid all confusion between two matrices containing the same amount of entries. For example, a matrix of dimension 3×4 has 3 rows and 4 columns. It would be distinct from a matrix 4×3, that has 4 rows and 3 columns, even if it also has 12 entries. A matrix is said to be **square** when it has the same number of rows and columns.
The elements are matrix entries a_{ij}, that are identified by their position. The element a_{32} would be the entry located on the third row and the second column of matrix A. This notation is essential in order to distinguish the elements of the matrix. The element a_{23}, distinct from a_{32}, is situated on the second row and the third column of the matrix A.*

Example 8.1. Consider the matrix

$$A = \begin{pmatrix} 1 & 9 & -13 \\ 20 & 5 & -6 \end{pmatrix},$$

the dimension of matrix A is 2×3 with $a_{11} = 1, a_{12} = 9, a_{13} = -13, a_{21} = 20, a_{22} = 5$ and $a_{23} = -6$. ▲

Matrices which have a single row are called row vectors, and those which have a single column are called column vectors. A matrix with an infinite number of rows or columns (or both) is called an infinite matrix†.

8.1.1 Special matrices

We will introduce some matrices with special patterns in the array of numbers first.

*http://www.hec.ca/en/cam/help/topics/Matrix_determinants.pdf
†https://en.wikipedia.org/wiki/Matrix_(mathematics)

1. Zero matrix. All elements are 0, for example, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

2. Diagonal matrix. A square matrix whose elements not on the main diagonal are all 0, for example, $\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$.

3. Identity matrix. A diagonal matrix whose elements on the main diagonal are all 1. Usually, it is denoted by I, for example, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

4. Scalar matrix. A diagonal matrix whose elements on the main diagonal are the same, for example, $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{pmatrix}$.

8.2 Basic operations of matrix

There are a number of basic operations that can be applied to modify matrices, called matrix addition, scalar multiplication, transposition, matrix multiplication.

8.2.1 Matrix addition[‡]

Matrix addition is the operation of adding two matrices by adding the corresponding entries together. Two matrices must have an equal number of rows and columns to be added. The sum of two matrices A and B will be a matrix which has the same number of rows and columns as do A and B. The sum of A and B, denoted by $A + B$, is computed by adding corresponding elements of A and B.
Similarly, we can also subtract one matrix from another, as long as they have the same dimensions. $A - B$ is computed by subtracting corresponding elements of A and B, and has the same dimensions as A and B.

$$A \pm B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$$

Example 8.2.

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 7 & 5 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 8 & 5 \\ 3 & 3 \end{pmatrix}.$$

▲

[‡] https://en.wikipedia.org/wiki/Matrix_addition

Example 8.3.

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 7 & 5 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -6 & -5 \\ -1 & 1 \end{pmatrix}.$$

▲

8.2.2 Scalar multiplication

The product cA of a scalar number c and a matrix A is computed by multiplying every entry of A by c. This operation is called scalar multiplication.

$$c \cdot A = c \cdot \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}$$

Example 8.4.

$$2 \cdot \begin{pmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{pmatrix}$$

▲

8.2.3 Transposition

The transpose of a matrix A gives another matrix A^T by performing any one of the following equivalent actions:

1. Reflect A over its main diagonal (which runs from top-left to bottom-right) to obtain A^T.
2. Write the rows of A as the columns of A^T.
3. Write the columns of A as the rows of A^T.

Formally, the ith row, jth column element of A is the jth row, ith column element of A^T. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.

Example 8.5.

$$\begin{pmatrix} 1 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

▲

Example 8.6.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

▲

Example 8.7.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

▲

8.2.4 Matrix multiplication

Multiplication of two matrices is defined as the number of columns of the left matrix is the same as the number of rows of the right matrix. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then their matrix product AB is the $m \times p$ matrix whose entries are given by the dot product of the corresponding row of A and the corresponding column of B, i.e.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{pmatrix}$$

Then,

$$AB = \begin{pmatrix} (AB)_{11} & (AB)_{12} & \cdots & (AB)_{1p} \\ (AB)_{21} & (AB)_{22} & \cdots & (AB)_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ (AB)_{m1} & (AB)_{m2} & \cdots & (AB)_{mp} \end{pmatrix} \qquad \text{where} \qquad (AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Example 8.8.

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1000 \\ 1 & 100 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 2 \cdot 0 + 3 \cdot 1 + 4 \cdot 0 & 2 \cdot 1000 + 3 \cdot 100 + 4 \cdot 10 \\ 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 & 1 \cdot 1000 + 0 \cdot 100 + 0 \cdot 10 \end{pmatrix} = \begin{pmatrix} 3 & 2340 \\ 0 & 1000 \end{pmatrix}$$

▲

Example 8.9.

$$A = \begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix}, B = \begin{pmatrix} d & e & f \\ h & i & j \\ x & y & z \end{pmatrix}$$

$$AB = \begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix} \begin{pmatrix} d & e & f \\ h & i & j \\ x & y & z \end{pmatrix} = \begin{pmatrix} ad + bh + cx & ae + bi + cy & af + bj + cz \\ pd + qh + rx & pe + qi + ry & pf + qj + rz \\ ud + vh + wx & ue + vi + wy & uf + vj + wz \end{pmatrix}$$

▲

Matrix multiplication satisfies the rules $(AB)C = A(BC)$ (associativity), and $(A+B)C = AC+BC$ as well as $C(A+B)$=$CA+CB$ (left and right distributivity). The product AB may be defined without BA being defined, namely if A and B are $m \times n$ and $n \times k$ matrices respectively and $m \neq k$. Even if both products are defined, they need not to be equal. That is, in general, matrix multiplication is not commutative. i.e.,

$$AB \neq BA,$$

Let us look at the following counter example and verify this result.

Example 8.10.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix},$$

whereas

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix}.$$

▲

8.3 Inverse of matrices

Consider the following matrix equation:

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

The above matrix representation is equivalent to

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}.$$

Now, we denote

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Then, we have

$$Ax = B.$$

If there exists matrix Q such that $QA = I$, where I is the identity matrix, then

$$\begin{aligned} Ax &= B \\ QAx &= QB \\ Ix &= QB \\ x &= QB \end{aligned}$$

If Q can be found, x (the matrix of the unknowns) can be solved easily.

Definition 1.1 Let A be an $n \times n$ square matrix. If there exists an $n \times n$ matrix Q such that $AQ = QA = I$ where I is the $n \times n$ identity matrix, then Q is called the inverse matrix of A and is written as $Q = A^{-1}$. In this case, A is said to be an invertible matrix or a non-singular matrix.

Before we move on to the methods of finding A^{-1}, we need to introduce determinant first.

8.3.1 Determinants and their expansion

Definition 1.2 The determinant of order 2 of the square matrix A is defined by the relation

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

The result is obtained by multiplying opposite elements and by calculating the difference between these two products.

Example 8.11. Given the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}.$$

The determinant of A is given by

$$det(A) = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = 2(-2) - 1(3) = -7.$$

▲

Definition 1.3 The determinant of order 3 of the square matrix A is defined by the relation

$$\begin{aligned} det(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \end{aligned}$$

This is referred to the expansion of the determinant by the first row. In expanding determinants by any row (or column), each element a_{ij} of the row (or column) is multiplied by a determinant formed by excluding the row and column of the element concerned together with an appropriate sign $(-1)^{i+j}$. These reduced determinants together with the sign are called the cofactor of a_{ij} and are denoted by A_{ij}, i.e.,

$$A_{ij} = (-1)^{i+j} \begin{vmatrix} a_{1,1} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,j-1} & a_{m,j+1} & \cdots & a_{m,n} \end{vmatrix}.$$

The expansion of a 3×3 determinant by the first row can then be defined by:

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13},$$

where $A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, $A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ and $A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

Note that some books may present a method of expansion for the 3×3 determinant which is known as the Sarrus' rule. However, that rule is only applicable to 3×3 determinant and sometimes students may wrongly generalize that rule to higher order determinant. In this review, we leave out this definition in order to avoid confusion.

Example 8.12. Evaluate

$$\begin{vmatrix} 6 & 7 \\ -5 & 8 \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} 1 & 3 & -2 \\ 2 & -3 & 2 \\ 1 & 4 & 0 \end{vmatrix}.$$

$$\begin{aligned}
\begin{vmatrix} 6 & 7 \\ -5 & 8 \end{vmatrix} &= 6 \times 8 - 7 \times (-5) \\
&= 83 \\
\begin{vmatrix} 1 & 3 & -2 \\ 2 & -3 & 2 \\ 1 & 4 & 0 \end{vmatrix} &= 1 \begin{vmatrix} -3 & 2 \\ 4 & 0 \end{vmatrix} - 3 \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} + (-2) \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} \\
&= (-3 \times 0 - 2 \times 4) - 3(2 \times 0 - 2 \times 1) - 2\,[2 \times 4 - (-3) \times 1] \\
&= -24
\end{aligned}$$

▲

Example 8.13. Find the value(s) of x if $\begin{vmatrix} x & -4 & 0 \\ 1 & 2 & x \\ -1 & 4 & 0 \end{vmatrix} = 0$. By expanding along the third column, we have

$$\begin{aligned}
0 \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} - x \begin{vmatrix} x & -4 \\ -1 & 4 \end{vmatrix} + 0 \begin{vmatrix} x & -4 \\ 1 & 2 \end{vmatrix} &= 0 \\
-x(4x - 4) &= 0 \\
x &= 0 \qquad \text{or} \qquad -1.
\end{aligned}$$

▲

8.3.2 Properties of determinants

Sometimes, we may need much effort to evaluate some complicated determinants. However, there are a number of useful properties of determinants that can be used to simplify the process of expanding determinants of order 3 or higher.

1. Interchanging rows and columns does not change the value of the determinant.

$$\begin{vmatrix} a_1 & b_1 & c_3 \\ a_2 & b_2 & c_3 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

2. Interchanging any two columns changes the sign of the value of the determinant.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$$

3. If each element of any column is multiplied by a constant k, the value of the determinant is multiplied by k.
$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
In particular, if each element of a column is 0 (i.e. $k = 0$), the value of the determinant is 0.
$$\begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix} = 0$$

4. If the corresponding elements of two columns are in proportion, the value of the determinant is 0.
$$\begin{vmatrix} a_1 & b_1 & kb_1 \\ a_2 & b_2 & kb_2 \\ a_3 & b_3 & kb_3 \end{vmatrix} = 0$$
In particular, if the corresponding elements of two columns are equal (i.e. $k = 1$), the value of the determinant is 0.

5. If each element of a column can be expressed as the sum of two numbers, the determinant can be expressed as the sum of two determinants.
$$\begin{vmatrix} a_1+p & b_1 & c_1 \\ a_2+q & b_2 & c_2 \\ a_3+r & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{vmatrix}$$

6. If a multiple of any column is added to another column, the value of the determinant does not change.
$$\begin{vmatrix} a_1+kb_1 & b_1 & c_1 \\ a_2+kb_2 & b_2 & c_2 \\ a_3+kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Properties 2 to 6 still hold when the operations on columns are changed to operations on rows. We can prove the above properties by expanding the determinants. We will prove property 2 and property 5. The proofs of the other properties are left to our students as an exercise.

Proof.

$$\begin{aligned} L.H.S. &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 \\ R.H.S. &= -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&= -(-a_1b_2c_3 - a_3b_1c_2 - a_2b_3c_1 + a_3b_2c_1 + a_1b_3c_2 + a_2b_1c_3) \\
&= a_1b_2c_3 + a_2b_3c_1 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2
\end{aligned}
$$

$$
\text{i.e.} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}
$$

■

Proof.

$$
\begin{aligned}
\begin{vmatrix} a_1+p & b_1 & c_1 \\ a_2+q & b_2 & c_2 \\ a_3+r & b_3 & c_3 \end{vmatrix} &= (a_1+p)b_2c_3 + (a_2+q)b_3c_1 + (a_3+r)b_1c_2 - (a_3+r)b_2c_1 - (a_2+q)b_1c_3 - (a_1+p)b_3c_2 \\
&= (a_1b_2c_3 + a_2b_3c_1 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2) \\
&\quad + (pb_2c_3 + qb_3c_1 + rb_1c_2 - rb_2c_1 - qb_1c_3 - pb_3c_2) \\
&= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{vmatrix}
\end{aligned}
$$

$$
\text{i.e.} \begin{vmatrix} a_1+p & b_1 & c_1 \\ a_2+q & b_2 & c_2 \\ a_3+r & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p & b_1 & c_1 \\ q & b_2 & c_2 \\ r & b_3 & c_3 \end{vmatrix}
$$

■

Example 8.14. Evaluate $\begin{vmatrix} 3 & -7 & 1 \\ 2 & -7 & 1 \\ 4 & -13 & 4 \end{vmatrix}$.

$$
\begin{aligned}
\begin{vmatrix} 3 & -7 & 1 \\ 2 & -7 & 1 \\ 4 & -13 & 4 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ 2 & -7 & 1 \\ 4 & -13 & 4 \end{vmatrix} \\
&= 1\begin{vmatrix} -7 & 1 \\ 13 & 4 \end{vmatrix} \\
&= -28 + 13 \\
&= -15
\end{aligned}
$$

In this example, we use property 6 to simplify our calculation. The 1st and the 2nd rows have two equal elements -7 and -1. We replace the 1st row by the sum of it and the 2nd row so that only one of the elements of the new 1st row is non-zero. We can use the symbols $R_1 + (-1)R_2 \to R_1$ to represent the above process, where R_i denotes the ith row. ▲

Example 8.15. Evaluate $\begin{vmatrix} 13 & 4 & 5 \\ 6 & 1 & 2 \\ 12 & -3 & 3 \end{vmatrix}$.

$$\begin{aligned}
\begin{vmatrix} 13 & 4 & 5 \\ 6 & 1 & 2 \\ 12 & -3 & 3 \end{vmatrix} &= 3\begin{vmatrix} 13 & 4 & 5 \\ 6 & 1 & 2 \\ 4 & -1 & 1 \end{vmatrix} \\
&= 3\begin{vmatrix} 29 & 4 & 5 \\ 10 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} \\
&= 3\begin{vmatrix} 29 & 4 & 9 \\ 10 & 1 & 3 \\ 0 & -1 & 0 \end{vmatrix} \\
&= 3 \times [-(-1)]\begin{vmatrix} 29 & 9 \\ 10 & 3 \end{vmatrix} \\
&= 3(87-90) \\
&= -9
\end{aligned}$$

In our first step, we take out the common factor 3 of the elements of the 3rd row. In our second step, we have $C_1 + 4C_2 \to C_1$ where C_i denotes the ith column. In our third step, we have $C_3 + C_2 \to C_3$. ▲

Example 8.16. Let $\Delta = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$. Show that $\Delta = \begin{vmatrix} b^2-a^2 & b^3-a^3 \\ c^2-a^2 & c^3-a^3 \end{vmatrix}$, where a, b and c are non-zero real numbers.

$$\begin{aligned}
abc\Delta &= \begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix} \\
&= abc\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \\
\Delta &= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \\
&= \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix} \\
&= \begin{vmatrix} b^2-a^2 & b^3-a^3 \\ c^2-a^2 & c^3-a^3 \end{vmatrix}
\end{aligned}$$

In this example, we first multiply the 1st, 2nd and 3rd rows of the determinant by a, b and c respectively to obtain a column with identical element (abc). Then, we reduce two elements of this column to zero by $R_2 - R_1 \to R_2$ and $R_3 - R_1 \to R_3$. Finally, we expand the resulting first column of the determinant. ▲

Example 8.17. Factorize $\begin{vmatrix} a & a & x \\ a & x & a \\ x & a & a \end{vmatrix}$.

$$\begin{aligned}
\begin{vmatrix} a & a & x \\ a & x & a \\ x & a & a \end{vmatrix} &= \begin{vmatrix} 2a+x & 2a+x & 2a+x \\ a & x & a \\ x & a & a \end{vmatrix} \\
&= (2a+x)\begin{vmatrix} 1 & 1 & 1 \\ a & x & a \\ x & a & a \end{vmatrix} \\
&= (2a+x)\begin{vmatrix} 1 & 0 & 0 \\ a & x-a & 0 \\ x & a-x & a-x \end{vmatrix} \\
&= (2a+x)\begin{vmatrix} x-a & 0 \\ a-x & a-x \end{vmatrix} \\
&= -(2a+x)(x-a)^2
\end{aligned}$$

In this example, we first perform $R_1 + R_2 + R_3 \to R_1$. Then, we take out the common factor $(2a + x)$ and perform $C_2 - C_1 \to C_2, C_3 - C_1 \to C_3$ in the third step. Finally, we expand the determinant by the 1st row to obtain the result. ▲

We can now move on to discuss the methods of finding inverse matrix.

1. Using Direct Method

   Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an inverse matrix of $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   By expanding the matrices, we have:

   $$\begin{cases} pa + rb = 1 \\ qa + sb = 0 \\ pc + rd = 0 \\ qc + sd = 1 \end{cases}.$$

   Then, we can solve the above equation to find a, b, c and d.

   Example 8.18. Let $A = \begin{pmatrix} 1 & 5 \\ 2 & 2 \end{pmatrix}$. Find A^{-1}.

Let A^{-1} be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 5 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{cases} a + 2b = 1 \\ 5a + 2b = 0 \\ c + 2d = 0 \\ 5c + 2d = 1 \end{cases}.$$

On solving the above equations, we have $a = -\frac{1}{4}, b = \frac{5}{8}, c = \frac{1}{4}, d = -\frac{1}{8}$. Hence, we have

$$A^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{5}{8} \\ \frac{1}{4} & -\frac{1}{8} \end{pmatrix}.$$

▲

This method works well for a 2×2 matrix because it is simple. In finding the inverse matrix of a 3×3 matrix, we have to set up 9 equations. This may be tedious. In this case, the inverse matrix of a 3×3 matrix can be found by the following method which involves determinants.

2. Using Determinants

   Let A be an $n \times n$ matrix. Given that the cofactor of a_{ij} is A_{ij}, the transpose of the matrix $(A_{ij})_{n\times n}$ is called the adjoint matrix of A and is denoted by $adj(A)$. Consider a 3×3 matrix A.

$$adj(A) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

Example 8.19. Let $A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$. Then, $adj(A) = \begin{pmatrix} 4 & -(-2) \\ -3 & -1 \end{pmatrix}^T = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$ ▲

The inverse matrix of a 3×3 matrix can be found by the following relation:

$$\text{If } det(A) \neq 0, \text{then } A^{-1} = \frac{1}{det(A)} adj(A)$$

Let us prove this theorem by considering a 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where $det(A) \neq 0$.

Proof. Let $C = (c_{ij})_{3\times 3}$ and $C = A \cdot adj(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$.

$$\begin{aligned} c_{11} &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} &&= det(A) \\ c_{22} &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} &&= det(A) \end{aligned}$$

$$
\begin{aligned}
c_{33} &= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \quad = \quad det(A) \\
c_{12} &= a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} \\
&= -a_{11}\begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} + a_{12}\begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} - a_{13}\begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} \\
&= -a_{11}a_{12}a_{33} + a_{11}a_{13}a_{32} + a_{12}a_{11}a_{33} - a_{12}a_{13}a_{31} - a_{13}a_{11}a_{32} + a_{13}a_{12}a_{31} \\
&= 0
\end{aligned}
$$

Similarly, $c_{13}, c_{21}, c_{23}, c_{31}, c_{32}$ can be shown to be all 0.

$$
\begin{aligned}
A \cdot adj(A) &= \begin{pmatrix} det(A) & 0 & 0 \\ 0 & det(A) & 0 \\ 0 & 0 & det(A) \end{pmatrix} \\
&= det(A)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
A \cdot \frac{1}{det(A)} adj(A) &= I
\end{aligned}
$$

Similarly, we can show that

$$
\begin{aligned}
\frac{1}{det(A)} adj(A) \cdot A &= I \\
\text{i.e.} \qquad A^{-1} &= \frac{1}{det(A)} adj(A).
\end{aligned}
$$

For square matrix of order $n \times n$, where n is an integer greater than 1, the same result can be obtained. ■

In general, we usually use the determinant method rather than the direct method to find the inverse of an matrix due to the complexity of direct method in solving system of linear equations.

Example 8.20. Find the inverse matrix of $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$.

$$
\begin{aligned}
det(A) &= 3 \times 2 - 5 \times 1 \\
&= 1 \\
adj(A) &= \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}^T \\
&= \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \\
A^{-1} &= \frac{1}{1}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \\
det(B) &= 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= 2(4-3)-(6-1)+3(9-2) \\
&= 18 \\
adj(B) &= \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{pmatrix}^T \\
&= \begin{pmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{pmatrix}^T \\
&= \begin{pmatrix} 1 & 7 & -5 \\ -5 & 1 & 7 \\ 7 & -5 & 1 \end{pmatrix} \\
B^{-1} &= \frac{1}{18} \begin{pmatrix} 1 & 7 & -5 \\ -5 & 1 & 7 \\ 7 & -5 & 1 \end{pmatrix}
\end{aligned}
$$

▲

Example 8.21. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 8 & -3 & 2 \\ 20 & -10 & 6 \end{pmatrix}$.

Show that $A^2 = 3A - 2I$, where I is the 3×3 identity matrix. Hence, find A^{-1}.

$$
\begin{aligned}
A^2 &= \begin{pmatrix} 1 & 0 & 0 \\ 8 & -3 & 2 \\ 20 & -10 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 8 & -3 & 2 \\ 20 & -10 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 24 & -11 & 6 \\ 60 & -30 & 16 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 0 & 0 \\ 24 & -9 & 6 \\ 60 & -30 & 18 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= 3 \begin{pmatrix} 1 & 0 & 0 \\ 8 & -3 & 2 \\ 20 & -10 & 6 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= 3A - 2I \\
A^2 &= 3A - 2I \\
A^{-1}A^2 &= A^{-1}(3A - 2I) \\
A &= 3I - 2A^{-1}
\end{aligned}
$$

$$\begin{aligned} A^{-1} &= \frac{1}{2}(3I - A) \\ &= \frac{1}{2}\left(\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 8 & -3 & 2 \\ 20 & -10 & 6 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -4 & 3 & -1 \\ -10 & 5 & -\frac{3}{2} \end{pmatrix} \end{aligned}$$

▲

Example 8.22. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$. Find B^{-1} and $B^{-1}AB$. Hence, find A^n.

$$\begin{aligned} det(B) &= 1 \\ adj(B) &= \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}^T \\ &= \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \\ B^{-1} &= \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \\ B^{-1}AB &= \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} -4 & -2 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \\ \text{Now,}\quad B^{-1}AB &= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \\ (B^{-1}AB)^n &= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^n \\ \underbrace{(B^{-1}AB)(B^{-1}AB)\cdots(B^{-1}AB)}_{n \text{ factors}} &= \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \\ B^{-1}A^nB &= \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \\ A^n &= B\begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}B^{-1} \\ &= \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2^n & 3^n \\ 2^n & -2\cdot 3^n \end{pmatrix}\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 2^{n+1} - 3^n & 2^n - 3^n \\ -2^{n+1} + 2 \cdot 3^n & -2^n + 2 \cdot 3^n \end{pmatrix}$$

▲

8.3.3 Properties of inverse matrices

After we have learnt how to find inverse matrices. Let us discuss some properties of inverse matrices. Suppose the inverse matrices A and B are A^{-1} and B^{-1} respectively, and k is a non-zero scalar. For the examples in the following properties, let $A = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$. Then $A^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$.

1. $$\left(A^{-1}\right)^{-1} = A$$

   For example,
   $$\left(A^{-1}\right)^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = A.$$

2. $$\left(A^n\right)^{-1} = \left(A^{-1}\right)^n$$

   For example, suppose $n = 2$,
   $$\left(A^2\right)^{-1} = \left(\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\right)^{-1} = \begin{pmatrix} 14 & 25 \\ 5 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} 9 & -25 \\ -5 & 14 \end{pmatrix} \quad \text{and} \quad \left(A^{-1}\right)^2 = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}^2 = \begin{pmatrix} 9 & -25 \\ -5 & 14 \end{pmatrix}.$$

3. $$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

   For example,
   $$\left(A^T\right)^{-1} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \quad \text{and} \quad \left(A^{-1}\right)^T = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}.$$

4. $$\left(kA^{-1}\right) = k^{-1}A^{-1}$$

   For example, suppose $k = 3$,
   $$\left(3A^{-1}\right) = \begin{pmatrix} 9 & 15 \\ 3 & 6 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad 3^{-1}A^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}.$$

5. $$(AB)^{-1} = B^{-1}A^{-1}$$

For example,

$$(AB)^{-1} = \left(\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\right)^{-1} = \begin{pmatrix} 30 & 19 \\ 11 & 7 \end{pmatrix}^{-1} = \begin{pmatrix} 7 & -19 \\ -11 & 30 \end{pmatrix}$$

and

$$B^{-1}A^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & -19 \\ -11 & 30 \end{pmatrix}.$$

6.

$$\left|A^{-1}\right| = |A|^{-1}$$

For example,

$$\left|A^{-1}\right| = \begin{vmatrix} 2 & -5 \\ -1 & 3 \end{vmatrix} = 1 \quad \text{and} \quad |A|^{-1} = \frac{1}{\begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix}} = 1.$$

The above properties can be proved by using the definition of inverse matrices. Let us prove property 5 and the proofs of the other properties are left to students as an exercise.

Proof.

$$\begin{aligned}
(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\
&= AIA^{-1} \\
&= AA^{-1} \\
&= I \\
(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\
&= B^{-1}IB \\
&= B^{-1}B \\
&= I \\
\text{i.e.} \quad (AB)^{-1} &= B^{-1}A^{-1}
\end{aligned}$$

■

Example 8.23. Let $A = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}$.

(a) Find A^2 and A^{-1}.

(b) Find A^n, where n is a positive integer.

(c) Hence, find $\left(A^{-1}\right)^{2008}$ and $\left(A^{-1}\right)^{2009}$.

(a)

$$A^2 = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \\
det(A) &= -4 \\
adj(A) &= \begin{pmatrix} -2 & 0 \\ -1 & 2 \end{pmatrix} \\
\text{i.e.} \quad A^{-1} &= \frac{1}{-4}\begin{pmatrix} -2 & 0 \\ -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}
\end{aligned}$$

(b)

$$\begin{aligned}
A^2 &= 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= 2^2 I
\end{aligned}$$

When n is even, $n = 2k$ for some positive integer k.

$$\begin{aligned}
A^n &= A^{2k} \\
&= \left(A^2\right)^k \\
&= (2^2 I)^k \\
&= 2^{2k} I^k \\
&= 2^n I, \qquad \text{where } I \text{ is the } 2 \times 2 \text{ identity matrix.}
\end{aligned}$$

When n is odd, $n = 2k + 1$ for some positive integer k.

$$\begin{aligned}
A^n &= A^{2k+1} \\
&= A^{2k} \cdot A \\
&= 2^{2k} I \cdot A \\
&= 2^{n-1} A
\end{aligned}$$

(c)

$$\begin{aligned}
\left(A^{-1}\right)^{2008} &= \left(A^{2008}\right)^{-1} \\
&= \left(2^{2008} I\right)^{-1} \\
&= 2^{-2008} I^{-1} \\
&= \begin{pmatrix} 2^{-2008} & 0 \\ 0 & 2^{-2008} \end{pmatrix} \\
\left(A^{-1}\right)^{2009} &= \left(A^{2009}\right)^{-1} \\
&= \left(2^{2008} A\right)^{-1} \\
&= 2^{-2008} A^{-1}
\end{aligned}$$

$$= 2^{-2008}\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2^{-2009} & 0 \\ 2^{-2010} & -2^{-2009} \end{pmatrix}$$

▲

Exercises

1. Evaluate $\begin{vmatrix} 2 & 10 & 4 \\ 4 & 5 & 3 \\ 7 & 10 & 3 \end{vmatrix}$.

2. Evaluate $\begin{vmatrix} -2 & 3 & -1 \\ 4 & 6 & 7 \\ 8 & -9 & 5 \end{vmatrix}$.

3. Evaluate $\begin{vmatrix} -2 & -2 & 4 \\ -2 & 8 & -2 \\ 4 & -2 & -2 \end{vmatrix}$.

4. Evaluate $\begin{vmatrix} 8 & 2 & 4 \\ 5 & 7 & 3 \\ 1 & \frac{1}{4} & \frac{1}{2} \end{vmatrix}$.

5. $\begin{vmatrix} 1 & 1 & 1 \\ x & 5 & 2 \\ x^2 & 25 & 4 \end{vmatrix} = 6$, find the value(s) of x.

6. $\begin{vmatrix} -2 & 7 & 49 \\ 14 & x & x^2 \\ -7 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 49 & 1 & 4 \\ 0 & -8 & -20 \end{vmatrix}$, find the value(s) of x.

7. Prove that $\begin{vmatrix} a^2 & b^2 & (a+b)^2 \\ a^2 & (a+c)^2 & c^2 \\ (b+c)^2 & b^2 & c^2 \end{vmatrix} = -2abc(a+b+c)^3$.

8. (a) Prove that $\begin{vmatrix} a & c & b \\ b & a & c \\ b+c & a+b & a+c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2-ab-ac-bc)$.

   (b) Hence. evaluate $\begin{vmatrix} -4 & 8 & 10 \\ 5 & -2 & 4 \\ 9 & 3 & 2 \end{vmatrix}$.

9. If $\begin{pmatrix} -5 & 1 & -2 \\ -3 & 4 & -1 \end{pmatrix} \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}$, find the values of x and y.

10. Let $P = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}$. Show that $P^3 + 3P = 0$.

11. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$. Prove that $A - A^3 + 2A^2$ is a scalar matrix.

12. Let $P = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \\ -1 & 2 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} -2 & 0 & 2 \\ -7 & 4 & 5 \\ 6 & -4 & -6 \end{pmatrix}$

    (a) Prove that PQ is a scalar matrix.

    (b) It is given that $det(P) = -4$. Using the result of (a), find $det(Q)$ without expansion.

13. Let $\begin{pmatrix} 8 & 6 \\ -3 & -3 \end{pmatrix}$.

    (a) Prove that $det(I + A) = 0$.

    (b) Hence, show that $det(I - A^4) = 0$.

14. Let $X = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

    (a) Show that $(X - 2I)(4I - X) = I$, where I is the 3×3 identity matrix.

    (b) Hence, show that $-X^2 + 6X - 9I = 0$.

15. Let $A = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & -2 \end{pmatrix}$.

    (a) Show that $A^2 + 4A + 4I = 0$, where I is the 3×3 identity matrix.

    (b) Show that $(-I - A)(A + 3I) = I$.

16. Let $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $X = \begin{pmatrix} a \\ b \end{pmatrix}$.

    (a) Find BAX.

    (b) Let $V = BAX$. If $V^T \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} V = 0$, express b in terms of a.

17. Let $A = \begin{vmatrix} 15 & 6 & -2 \\ -10 & -3 & 2 \\ -10 & -8 & 3 \end{vmatrix}$ and $B = \begin{vmatrix} 18 & 10 & -2 \\ -15 & -9 & 1 \\ -5 & 1 & -1 \end{vmatrix}$.

   (a) Show that $A^2 - B^2 - I = 0$.

   (b) Show that $AB = BA$.

   (c) Hence, find the inverse matrix of $\begin{pmatrix} 33 & 16 & -4 \\ -25 & -12 & 3 \\ -15 & -7 & 2 \end{pmatrix}$.

18. Let $P^3 - P^2 + P - I = 0$, where P is a 2×2 matrix and I is the 2×2 identity matrix.

   (a) Prove that P has an inverse, and find P^{-1} in terms of P.

   (b) Show that

      i. $P^4 = I$,

      ii. $(P^{-1})^3 - (P^{-1})^2 + P^{-1} - I = 0$.

19. Let $A = \begin{pmatrix} -2 & 2 \\ -1 & 2 \end{pmatrix}$.

   (a) Find A^2 and A^{-1}.

   (b) Find A^n, where n is a positive integer.

   (c) Hence, find $(A^{-1})^{1000}$.

20. Let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are real numbers. Suppose $det(P) = 0$. Prove, by mathematical induction, that $P^{n+1} = (a+d)^n P$ for any positive integer n.

21. (a) If A is an invertible matrix and $A^{-1} = A^T$, prove that $|A| = \pm 1$.

   (b) Let $B = \begin{pmatrix} 1 & -6 & 82 \\ 6 & 1 & 59 \\ -82 & -59 & 1 \end{pmatrix}$.

      i. Using the result of (a), show that $det(I - B) = 0$.

      ii. Hence, find the value of $det(I - B^4)$.

22. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ -1 & 1 & 2 \end{pmatrix}$.

   (a) Show that $A^3 - 3A^2 + 7A = 6I$.

   (b) Show that $A^2 - 3A + 7I$ is a multiple of A^{-1}.

   (c) Using the result of (b), find A^{-1}.

23. Let $P = \begin{pmatrix} 2 & 1 \\ -\frac{4}{3} & -\frac{1}{3} \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$. Find

(a) Q^{-1},

(b) $Q^{-1}PQ$,

(c) P^n, where n is a positive integer.

24. (a) If $\begin{vmatrix} m+1 & -1 \\ -2 & m \end{vmatrix} = 0$, find the values of m.

(b) Let m_1 and m_2 be the values obtained in (a) where $m_1 < m_2$. If $\begin{pmatrix} m_1+1 & -1 \\ -2 & m_1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} m_2+1 & -1 \\ -2 & m_2 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, find the values of a and b.

(c) Let $A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ b & a \end{pmatrix}$ where a and b are the values obtained in (b). Evaluate A^n, where n is a positive integer.

(d) Prove that $A^{-1} \begin{pmatrix} -7 & 2 \\ 4 & 0 \end{pmatrix} A$ is a matrix of the form $\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$, where k_1 and k_2 are real numbers.

(e) Evaluate $\begin{pmatrix} -7 & 2 \\ 4 & 0 \end{pmatrix}^{100}$.