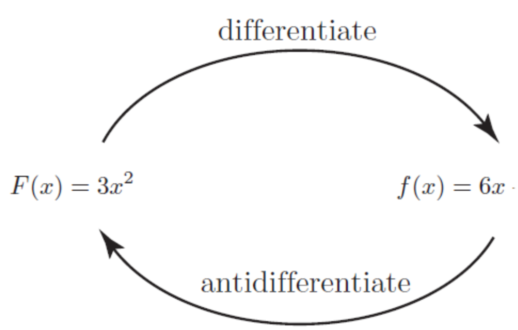
Chapter 12

Indefinite Integral*

Most of the mathematical operations have inverse operations. The inverse operation of differentiation is called integration. For example, describing a process at the given moment knowing the speed of the process at that moment.

12.1 Integration - reverse of differentiation

Consider the function $F(x) = 3x^2$. Suppose we write its derivative as $f(x)$, that is $f(x) = \frac{dF}{dx}$. It is easy to see that $f(x) = 6x$. This process is illustrated in the following figure.



Suppose now that we try to differentiate $F(x) = 3x^2 + 6$ instead. Clearly, the answer to this question is again the function $6x$. We say that $F(x) = 3x^2$ or $F(x) = 3x^2 + 6$ is the **antiderivative** of $f(x) = 6x$. We can see that an antiderivative is not uniquely determined because the antiderivatives of $6x$ can also be $3x^2 + \sqrt{3}$, $3x^2 - \pi$ and any function of the form $3x^2 + C$, where C is an arbitrary constant.
The reason why all of these functions have the same derivative is that the constant term disappeared after differentiation. So, more generally, if $F(x)$ is an antiderivative of $f(x)$, then the antiderivative of $f(x)$ is also every function $F(x) + C$, where C is whatever constant.

Example 12.1.
(a) Let $F(x) = e^{2x} + \cos x$, find $\frac{dF}{dx}$.

*http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-intrevdiff-2009-1.pdf

(b) Write down several antiderivatives of $f(x) = 2e^{2x} - \sin x$.

(a) $\dfrac{dF}{dx} = e^{2x} \cdot 2 - \sin x = 2e^{2x} - \sin x$.

(b) We can deduce from (a) that an antiderivative of $2e^{2x} - \sin x$ is $e^{2x} + \cos x$. All other antiderivatives of $f(x) = 2e^{2x} - \sin x$ will take the form $F(x) + C$ where C is a constant. So, the following can all be the antiderivatives of $f(x)$:

$$e^{2x} + \cos x - 7,\ e^{2x} + \cos x - 15,\ e^{2x} + \cos x,\ e^{2x} + \cos x + 3.$$

▲

From these examples, we have the following definition.

12.2 Indefinite Integral

If the function $F(x)$ is an antiderivative of $f(x)$, then the expression $F(x) + C$, where C is an arbitrary constant, is called the *indefinite integral* of $f(x)$ with respect to x and is denoted by $\int f(x)\ dx$, i.e.,

$$\int f(x)\ dx = F(x) + C.$$

The function $f(x)$ is called the integrand, $\int$ the integral sign, x is called the variable of integration and C the constant of integration.

Example 12.2.

Find $\int 3x^2\, dx$.[†]

Let $y = x^3$, then $\dfrac{dy}{dx} = 3x^2$. Hence, we verify that $\int 3x^2\, dx = x^3$. In general, adding any constant C to x^3 gives the same derivative. Hence, the integral of the function $y = 3x^2$ is given by

$$\int 3x^2\, dx = x^3 + C.$$

▲

Example 12.3.

Find $\int \dfrac{1}{x}\, dx$.

Since

$$\frac{d}{dx} \ln x = \frac{1}{x},$$

Hence

$$\int \frac{dx}{x} = \ln x + C.$$

▲

[†] http://www.maths.manchester.ac.uk/~tv/Teaching/1C2/2013/1-integrals.pdf

Example 12.4.

Let n be an integer not equal to -1. Find $\int x^n \, dx$.

Since

$$\frac{d}{dx}x^{n+1} = (n+1)x^n.$$

Hence

$$\int (n+1)x^n \, dx = x^{n+1} + C_1,$$

dividing $(n+1)$ and adding an arbitrary constant gives

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C$$

. ▲

Example 12.5. Find $\int \cos x \, dx$.

Since

$$\frac{d}{dx}\sin x = \cos x,$$

Hence

$$\int \cos x \, dx = \sin x + C.$$

▲

12.2.1 Basic Properties of Indefinite Integrals

Since integration is the reverse process of differentiation, we can derive some basic properties of indefinite integrals from differentiation.

1. $\int kf(x) \, dx = k \int f(x) \, dx$, where k is a constant.
2. $\int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

Proof. Suppose $F(x)$ is the antiderivative of $f(x)$, i.e., $\frac{d}{dx}F(x) = f(x)$.

$$\begin{aligned} \frac{d}{dx}\left[kF(x)\right] &= k\frac{d}{dx}F(x) \\ &= kf(x) \end{aligned}$$

By the definition of indefinite integral,

$$\int kf(x) \, dx = kF(x) + C_1.$$

On the other hand,

$$\begin{aligned} k\int f(x) \, dx &= k\left[F(x) + C\right] \\ &= kF(x) + C_2. \end{aligned}$$

Since C_1 and C_2 are arbitary constants,

$$\int kf(x) \, dx = k\int f(x) \, dx.$$

■

Proof. Suppose $F(x)$ and $G(x)$ are the antiderivative of $f(x)$ and $g(x)$ respectively. i.e., $\frac{d}{dx}F(x) = f(x)$ and $\frac{d}{dx}G(x) = g(x)$.

$$\begin{aligned} \frac{d}{dx}\left[F(x) \pm G(x)\right] &= \frac{d}{dx}F(x) \pm \frac{d}{dx}G(x) \\ &= f(x) \pm g(x) \end{aligned}$$

By the definition of indefinite integral,

$$\begin{aligned} \int [f(x) \pm g(x)]\ dx &= F(x) \pm G(x) + C \\ &= F(x) + C_1 \pm G(x) + C_2 \\ &= \int f(x)\ dx \pm \int g(x)\ dx. \end{aligned}$$

■

12.2.2 Table of basic Integrals

Similar to our examples 1.2 to 1.5, by reversing our lists of derivatives and rules, we will have a list of elementary integrals results as shown in the following table[‡]:

Some elementary integrals

1.	$\int 1\,dx = x + C$	**2.**	$\int x\,dx = \frac{1}{2}x^2 + C$
3.	$\int x^2\,dx = \frac{1}{3}x^3 + C$	**4.**	$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C$
5.	$\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$	**6.**	$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$
7.	$\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C \quad (r \neq -1)$	**8.**	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
9.	$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$	**10.**	$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$
11.	$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$	**12.**	$\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$
13.	$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax + C$	**14.**	$\int \csc ax\cot ax\,dx = -\frac{1}{a}\csc ax + C$
15.	$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \quad (a > 0)$	**16.**	$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
17.	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$	**18.**	$\int b^{ax}\,dx = \frac{1}{a\ln b}b^{ax} + C$
19.	$\int \cosh ax\,dx = \frac{1}{a}\sinh ax + C$	**20.**	$\int \sinh ax\,dx = \frac{1}{a}\cosh ax + C$

Example 12.6.

Find $\int (2x - \frac{3}{\sqrt{x}})\ dx$.

$$\begin{aligned} \int (2x - \frac{3}{\sqrt{x}})\ dx &= 2\int x dx - 3\int x^{-\frac{1}{2}}\ dx \\ &= 2\left(\frac{x^2}{2}\right) - 3\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right) + C \end{aligned}$$

[‡] http://www.math.hkbu.edu.hk/~felix_kwok/2016/chapter5_notes1.pdf

$$= x^2 - 6x^{\frac{1}{2}} + C$$

▲

Example 12.7.

Find $\int \frac{x^2+6}{x^2}\ dx$.

$$\begin{aligned}\int \frac{x^2+6}{x^2}\ dx &= \int (1+\frac{6}{x^2})\ dx \\ &= \int (1+6x^{-2})\ dx \\ &= x - \frac{6}{x} + C\end{aligned}$$

▲

Example 12.8.

Find $\int \sec x(\cos x + \tan x)\ dx$.

$$\begin{aligned}\int \sec x(\cos x + \tan x)\ dx &= \int (\sec x \cos x + \sec x \tan x)\ dx \\ &= \int (1 + \sec x \tan x)\ dx \\ &= x + \sec x + C\end{aligned}$$

▲

Example 12.9.

Find $\int (6x^2 + \frac{1}{x} - e^x)\ dx$.

$$\begin{aligned}\int (6x^2 + \frac{1}{x} - e^x)\ dx &= 6\left(\frac{x^3}{3}\right) + \ln|x| - e^x + C \\ &= 2x^3 + \ln|x| - e^x + C\end{aligned}$$

▲

Example 12.10.

Find $\int \frac{1}{1-\sin^2 x}\ dx$.

$$\begin{aligned}\int \frac{1}{1-\sin^2 x}\ dx &= \int \frac{1}{\cos^2 x}\ dx \\ &= \int \sec^2 x\ dx \\ &= \tan x + C\end{aligned}$$

▲

Example 12.11.

Find $\displaystyle\int \frac{1+\cos\theta}{1-\cos\theta}\ d\theta$

$$\begin{aligned}
\int \frac{1+\cos\theta}{1-\cos\theta}\ d\theta &= \int \frac{1+\cos\theta}{1-\cos\theta} \times \frac{1+\cos\theta}{1+\cos\theta}\ d\theta \\
&= \int \frac{1+2\cos\theta+\cos^2\theta}{\sin^2\theta}\ d\theta \\
&= \int (\csc^2\theta + 2\cot\theta\csc\theta + \cot^2\theta)\ d\theta \\
&= -\cot\theta - 2\csc\theta + \int (\csc^2\theta - 1)\ d\theta \\
&= -2\cot\theta - 2\csc\theta - \theta + C
\end{aligned}$$

▲

12.3 Integration by Substitution

Some indefinite integrals like $\int \sqrt{x+1}\ dx$ cannot be found directly by using the basic integration table. In this case, we may transform the indefinite integral into another indefinite integral that we know how to integrate. This method is known as integration by substitution. In general, we have:

If $F(u)$ is the antiderivative of $f(u)$, i.e. $F'(u) = f(u)$, and $u = g(x)$ is a differentiable function of x, then we have

$$\int f[g(x)]g'(x)dx = \int f(u)du = F(u) + C.$$

Proof. Since $F(u)$ is an antiderivative of $f(u)$. We have $\int f(u)\ du = F(u) + C$.

By the chain rule,

$$\begin{aligned}
\frac{d}{dx}F(u) &= \frac{dF(u)}{du} \cdot \frac{du}{dx} \\
&= f(u) \cdot g'(x) \\
&= f\,[g(x)]\,g'(x).
\end{aligned}$$

By the definition of indefinite integral,

$$\int f[g(x)]g'(x)\ dx = F(u) + C.$$

Hence,

$$\int f[g(x)]g'(x)dx = \int f(u)du = F(u) + C.$$

■

Example 12.12.

Find $\displaystyle\int x\sqrt{x+1}\ dx.$

Let $u = x + 1$. Then $du = dx$

$$\begin{aligned}
\int x\sqrt{x+1}\ dx &= \int (u-1)(\sqrt{u})\ du \\
&= \int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\ du \\
&= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C \\
&= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C
\end{aligned}$$

▲

Example 12.13.

Find $\int \frac{2x-5}{\sqrt{x^2-5x+3}}\ dx$.

Let $u = x^2 - 5x + 3$. Then $du = (2x-5)dx$

$$\begin{aligned}
\int \frac{2x-5}{\sqrt{x^2-5x+3}}\ dx &= \int \frac{1}{\sqrt{u}}\ du \\
&= 2\sqrt{u} + C \\
&= 2\sqrt{x^2-5x+3} + C
\end{aligned}$$

▲

Example 12.14.

Find $\int e^{-\frac{x}{4}}\ dx$.

Let $u = -\frac{x}{4}$. Then, $du = -\frac{1}{4}dx$ or $-4\ du = dx$

$$\begin{aligned}
\int e^{-\frac{x}{4}}\ dx &= \int e^{u}(-4)\ du \\
&= -4e^{u} + C \\
&= -4e^{-\frac{x}{4}} + C
\end{aligned}$$

▲

Example 12.15.

Find $\int \frac{x^2 - 2\ln x}{x} dx$.

Let $u = \ln x$. Then, $du = \frac{1}{x} dx$

$$\begin{aligned} \int \frac{x^2 - 2\ln x}{x}\ dx &= \int (x - \frac{2\ln x}{x})\ dx \\ &= \frac{x^2}{2} - 2\int u\ du \\ &= \frac{x^2}{2} - 2\left(\frac{u^2}{2}\right) + C \\ &= \frac{x^2}{2} - (\ln x)^2 + C \end{aligned}$$

▲

Example 12.16.

Find $\int x^2 \csc^2(1 - x^3)\ dx$.

Let $u = 1 - x^3$. Then, $du = -3x^2\ dx$ or $-\frac{1}{3} du = x^2\ dx$

$$\begin{aligned} \int x^2 \csc^2(1 - x^3)\ dx &= \int -\frac{1}{3} \csc^2 u\ du \\ &= -\frac{1}{3}(-\cot u) + C \\ &= \frac{1}{3}\cot(1 - x^3) + C \end{aligned}$$

▲

Example 12.17.

Find $\int \frac{\cos\theta}{(\sin\theta + 1)^2}\ d\theta$.

Let $u = \sin\theta + 1$. Then, $du = \cos\theta\ d\theta$

$$\begin{aligned} \int \frac{\cos\theta}{(\sin\theta + 1)^2}\ d\theta &= \int \frac{1}{u^2}\ du \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{\sin\theta + 1} + C \end{aligned}$$

▲

12.3.1 Integration involving $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ and $\sqrt{a^2 + x^2}$

In many cases, the shortest method of integrating such expressions: $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ and $\sqrt{a^2 + x^2}$ where $a > 0$, is to change the variable as follows:

1. For integrands involving $\sqrt{a^2 - x^2}$. We use the substitution
$$x = a\sin\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$
2. For integrands involving $\sqrt{x^2 - a^2}$. We use the substitution
$$x = a\sec\theta, 0 \le \theta \le \pi, \theta \neq \frac{\pi}{2}.$$
3. For integrands involving $\sqrt{a^2 + x^2}$. We use the substitution
$$x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

In each of the above cases, the radical sign in the integrand will be eliminated.

Example 12.18.

Find $\displaystyle\int \frac{dx}{\sqrt{4 - x^2}}$.
Let $x = 2\sin\theta$. Then, $dx = 2\cos\theta\ d\theta$.

$$\begin{aligned} \int \frac{dx}{\sqrt{4 - x^2}} &= \int \frac{2\cos\theta\ d\theta}{2\cos\theta} \\ &= \theta + C \\ &= \sin^{-1}\frac{x}{2} + C \end{aligned}$$

▲

Example 12.19.

Find $\displaystyle\int \frac{\sqrt{x^2 - 9}}{3x}\ dx$.
Let $x = 3\sec\theta$. Then, $dx = 3\sec\theta\tan\theta\ d\theta$.

$$\begin{aligned} \int \frac{\sqrt{x^2 - 9}}{3x}\ dx &= \int \frac{3\tan\theta}{9\sec\theta}(3\sec\theta\tan\theta)\ d\theta \\ &= \int \tan^2\theta\ d\theta \\ &= \int (\sec^2\theta - 1)\ d\theta \\ &= \tan\theta - \theta + C \\ &= \frac{\sqrt{x^2 - 9}}{3} - \cos^{-1}\frac{3}{x} + C \quad \text{(By drawing right-angled triangle and using pythagorean theorem)} \end{aligned}$$

▲

Example 12.20.

Find $\displaystyle\int \frac{x^2}{1 + x^2}\ dx$.
let $x = \tan\theta$. Then, $dx = \sec^2\theta\ d\theta$.

$$\int \frac{x^2}{1 + x^2}\ dx = \int \frac{\tan^2\theta}{\sec^2\theta}(\sec^2\theta)\ d\theta$$

$$\begin{aligned} &= \int \tan^2\theta \; d\theta \\ &= \tan\theta - \theta + C \\ &= x - \tan^{-1} x + C \end{aligned}$$

▲

12.4 Integration by Parts

If $u(x)$ and $v(x)$ are differentiable functions of x, then

$$\int u(x)d(v(x)) = u(x)v(x) - \int v(x)d(u(x)).$$

Proof.

Since

$$[u(x)v(x)]' = u(x)v'(x) + v(x)u'(x).$$

Hence

$$u(x)v'(x) = [u(x)v(x)]' - v(x)u'(x).$$

Then,

$$\int u(x)v'(x)\; dx = \int [u(x)v(x)]' \; dx - \int v(x)u'(x)\; dx.$$

i.e.,

$$\int u(x)d(v(x)) = u(x)v(x) - \int v(x)d(u(x))$$

■

Some integrals such as $\int \ln x \; dx$, $\int x\cos x \; dx$ and $\int e^x \sin x \; dx$ cannot be found by the techniques we have learnt before. We shall use the above method (integration by parts) to find these types of integrals.

Example 12.21.

Find $\displaystyle\int x^4 \ln x \; dx$.

$$\begin{aligned} \int x^4 \ln x \; dx &= \int \ln x \; d(\frac{x^5}{5}) \\ &= (\ln x)(\frac{x^5}{5}) - \int \frac{x^5}{5} \; d(\ln x) \\ &= \frac{1}{5}x^5 \ln x - \frac{1}{5}\int x^4 \; dx \\ &= \frac{1}{5}x^5 \ln x - \frac{x^5}{25} + C \end{aligned}$$

▲

Example 12.22.

Find $\int x^2 e^x \, dx$.

$$\begin{aligned}
\int x^2 e^x \, dx &= \int x^2 \, d(e^x) \\
&= x^2 e^x - \int e^x \, d(x^2) \\
&= x^2 e^x - 2\int x e^x \, dx \\
&= x^2 e^x - 2\int x \, d(e^x) \\
&= x^2 e^x - 2xe^x + 2\int e^x \, dx \\
&= x^2 e^x - 2xe^x + 2e^x + C
\end{aligned}$$

▲

Example 12.23.

Find $\int x^2 \sin 2x \, dx$.

$$\begin{aligned}
\int x^2 \sin 2x \, dx &= -\frac{1}{2}\int x^2 \, d(\cos 2x) \\
&= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}\int \cos 2x \, d(x^2) \\
&= -\frac{1}{2}x^2 \cos 2x + \int x \cos 2x \, dx \\
&= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}\int x \, d(\sin 2x) \\
&= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x - \frac{1}{2}\int \sin 2x \, dx \\
&= -\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x + C
\end{aligned}$$

▲

Example 12.24.

Find $\int e^{2x} \sin x \, dx$.

$$\begin{aligned}
\int e^{2x} \sin x \, dx &= \int e^{2x} \, d(-\cos x) \\
&= -e^{2x} \cos x + 2\int e^{2x} \cos x \, dx \\
&= -e^{2x} \cos x + 2\int e^{2x} \, d(\sin x) \\
&= -e^{2x} \cos x + 2\left[e^{2x} \sin x - 2\int e^{2x} \sin x \, dx\right]
\end{aligned}$$

$$= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\ dx$$

Hence,

$$\begin{aligned} 5\int e^2x\sin x\ dx &= -e^{2x}\cos x + 2e^{2x}\sin x + C_1 \\ \int e^2x\sin x\ dx &= \frac{1}{5}(-e^{2x}\cos x + 2e^{2x}\sin x) + C \end{aligned}$$

▲

Exercises

1. Find $\int (x-3)(x+2)(1-2x)\ dx$.
2. Find $\int \frac{2e^x - \sqrt[3]{x}}{3}\ dx$.
3. Find $\int (\sec\theta + \csc\theta)(\sec\theta - \csc\theta)\ d\theta$.
4. Find $\int \frac{1-\cos^2 x}{\tan x\cos x}\ dx$.
5. Find $\int (\cos^3 x + \sin^3 x\cot x)\ dx$.
6. Find $\int \frac{1-\csc^4 x}{\cot^2 x}\ dx$.
7. Find $\int \frac{\sqrt{x} - \ln x}{x}\ dx$.
8. Find $\int \cos x\sin^2 x\ dx$.
9. Find $\int x^3 e^{-2x}\ dx$.
10. Find $\int \frac{e^{2x}}{\sqrt[3]{e^x+1}}\ dx$.
11. Find $\int \cos^3 x\sin^4 x\ dx$.
12. Find $\int \tan^2\theta\sec^6\theta\ d\theta$.
13. Find $\int \frac{dx}{(9-x^2)^{\frac{3}{2}}}$.
14. Find $\int \frac{dx}{(x+1)\sqrt{x^2-1}}$.
15. Find $\int \frac{dx}{x^2\sqrt{4x^2-1}}$.
16. Find $\int \frac{x^5}{x^2\sqrt{x^2+1}}\ dx$.

17. Find $\displaystyle\int \frac{dx}{2x^2+2x+1}$.

18. Find $\displaystyle\int x \sin 3x \sin 4x \ dx$.

19. Find $\displaystyle\int \sin(\ln x) \ dx$.

20. Find $\displaystyle\int e^{-2x} \sin 2x \ dx$.

21. (a) Find $\displaystyle\int \frac{dx}{x^2+1}$.

    (b) Using the substitution $u = e^x$ and the result of (a),find $\displaystyle\int \frac{dx}{e^x+e^{-x}}$.

    (c) Using the result of (a), find $\displaystyle\int \frac{dx}{x^2+\sqrt{2}x+\frac{3}{2}}$.

22. (a) Find $\displaystyle\int \tan^6 x dx$.

    (b) Using the result of (a), find $\displaystyle\int \frac{(x^2-1)^{\frac{5}{2}}}{x} dx$.

23. (a) Show that $\displaystyle\frac{d}{dx} \tan \frac{x}{2} = \frac{1}{1+\cos x}$.

    (b) Using (a), or otherwise, find $\displaystyle\int \frac{x+\sin x}{1+\cos x} \ dx$.

24. (a) Prove that $\displaystyle\int \sec x \ dx = \ln|\sec x + \tan x| + C$

    (b) Using the result of (a), evaluate $\displaystyle\int \sec \frac{x}{2} \ dx$ and $\displaystyle\int \sec^3 \frac{x}{2} \ dx$.

Chapter 13

Definite Integral

We have introduced all the skills and procedures of finding *indefinite integral*. Now, we are able to 'anti-differentiate' a wide range of elementary functions. We have to master all the skills of indefinite integral before we move on to this chapter.

In this chapter, we deal with *definite integral*. The main difference between definite and indefinite integral is the existence of upper and lower limits. Usually, we denote the upper limit by b and lower limit by a. The notation of definite integral is $\int_a^b f(x)\ dx$.

13.1 Definition of Definite Integrals

For a function f defined on $[a, b]$, a partition P of $[a, b]$ into a collection of n subintervals

$$[x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n],$$

and for each $k = 1, 2, ..., n$, a point ξ_k in $[x_{k-1}, x_k]$, the sum under graph is given by

$$f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + ... + f(\xi_n)(x_n - x_{n-1})$$

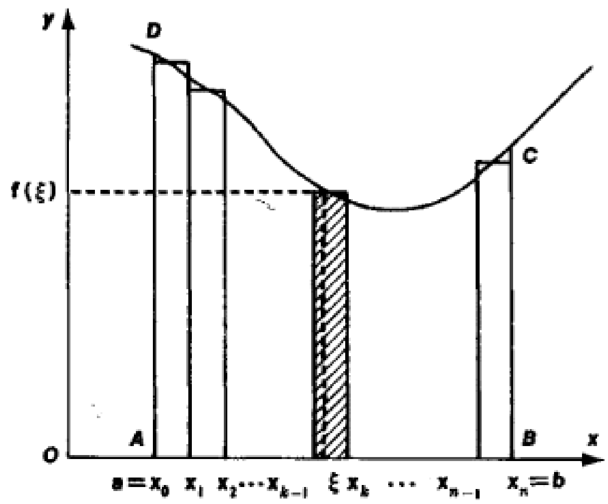
or in summation notation

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta x.$$

Usually, we will take P to be n regular partition (i.e. $\Delta x = \frac{b-a}{n}$). This is called **Riemann sum** of the function f. Notice that as n tends to infinity, the sum would be a more and more accurate representation of the area under graph.

The **definite integral** of f from a to b is defined to be

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)(\frac{b-a}{n}).$$



Example 13.1.

Compute the integral $\int_0^4 x^3\ dx$ by computing Riemann sums for a regular partition.

Notice that $a = 0$, $b = 4$ and $f(x) = x^3$. For regular partition and for each positive integer n, we have

$$\begin{aligned} \Delta x &= \frac{4-0}{n} \\ &= \frac{4}{n}; \\ x_k &= a + k\Delta x \\ &= \frac{4k}{n}, \quad \text{for } k = 1, 2, \cdots, n; \\ f(\xi_k) &= \left(\frac{4k}{n}\right)^3 \\ &= \frac{64k^3}{n^3}, \quad \text{for } k = 1, 2, \cdots, n. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{k=1}^{n} f(\xi_k)\Delta x &= \sum_{k=1}^{n} \left(\frac{64k^3}{n^3}\right)\left(\frac{4}{n}\right) \\ &= \sum_{k=1}^{n} \frac{256k^3}{n^4} \\ &= \frac{256}{n^4}\sum_{k=1}^{n} k^3 \\ &= \frac{256n^2(n+1)^2}{4n^4}. \end{aligned}$$

(Notice that the last step of the simplification follows from summation formula.)

Then, by the definition of definite integral,

$$\begin{aligned} \int_0^4 x^3\ dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x \\ &= \lim_{n\to\infty} \left(\frac{64k^3}{n^3}\right)\left(\frac{4}{n}\right) \end{aligned}$$

$$\begin{aligned} &= \lim_{n\to\infty} \frac{256n^2(n+1)^2}{4n^4} \\ &= 64 \end{aligned}$$

▲

13.2 Basic properties of Definite Integrals

In the last section, we have defined definite integral of $f(x)$ on the closed interval $[a, b]$. This is to imply that $a < b$. We shall also define integrals in which the upper limit is less than the lower limit.

Definition 13.1
Let $f(x)$ be integrable on $[a, b]$. Then, the definite integral of $f(x)$ from b to a is defined by

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx.$$

Directly from this definition, we have the following properties:

1. $\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$, where k is a constant.
2. $\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$
3. $\int_a^a f(x)\ dx = 0$
4. $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$, where $a < c < b$.

We will prove property 3. The proofs of the other properties are easy and trivial. They are left to our students as an exercise.

Proof.

Since

$$\int_a^a f(x)\ dx = -\int_a^a f(x)\ dx.$$

Then,

$$\begin{aligned} 2\int_a^a f(x)\ dx &= 0 \\ \int_a^a f(x)\ dx &= 0. \end{aligned}$$

■

13.3 Fundamental Theorem of Calculus

As you will see in section 12.1 that it is quite difficult to evaluate a definite integral directly from the definition. This problem can be easily solved by means of the following famous theorem:

If $f(x)$ is a continuous function on $[a, b]$ and $F(x)$ is the antiderivative of $f(x)$ (i.e. $F'(x) = f(x)$).
We have,

$$\int_a^b f(x)\ dx = [F(x)]_a^b = F(b) - F(a).$$

The Fundamental Theorem of Calculus enables us to evaluate definite integrals by finding the indefinite integrals. To evaluate the definite integral $\int_a^b f(x)\ dx$, we first find the indefinite integral $\int f(x)\ dx$. Then, we substitute the upper and lower limits into the resulting function to evaluate the value of $\int_a^b f(x)\ dx$.

In other words, we will repeat all the skills of the last chapter again and do the upper and lower limits substitution to the final function without the integration constant C.
Note: For Integration by Substitutions, we need to change the upper and lower limits to the new upper and lower limits for the new variable after substitution.

Example 13.2.

Evaluate $\int_{-2}^{0} (3x - 4x^3)\ dx$.

$$\begin{aligned}\int_{-2}^{0} (3x - 4x^3)\ dx &= \left[\frac{3}{2}x^2 - x^4\right]_{-2}^{0} \\ &= \left[\frac{3}{2}(0)^2 - 0^4\right] - \left[\frac{3}{2}(-2)^2 - (-2)^4\right] \\ &= 10\end{aligned}$$

▲

Example 13.3.

Evaluate $\int_0^{\frac{\pi}{3}} \sec x \tan x\ dx$.

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sec x \tan x\ dx &= \Big[\sec x\Big]_0^{\frac{\pi}{3}} \\ &= \sec\left(\frac{\pi}{3}\right) - \sec 0 \\ &= 1\end{aligned}$$

▲

Example 13.4.

Evaluate $\int_0^2 (x^2+4x)\ dx - \int_3^2 (x^2+4x)\ dx$.

$$\begin{aligned}
\int_0^2 (x^2+4x)\ dx - \int_3^2 (x^2+4x)\ dx &= \int_0^2 (x^2+4x)\ dx + \int_2^3 (x^2+4x)\ dx \\
&= \int_0^3 (x^2+4x)\ dx \\
&= \left[\frac{x^3}{3} + 2x^2\right]_0^3 \\
&= \frac{3^3}{3} + 2(3)^2 \\
&= 27
\end{aligned}$$

▲

Example 13.5.

Evaluate $\int_0^4 6x\sqrt{x^2+9}\ dx$.

Let $u = x^2+9$. Then, $du = 2x\ dx$. When $x=0, u=9$ and $x=4, u=25$.

$$\begin{aligned}
\int_0^4 6x\sqrt{x^2+9}\ dx &= \int_9^{25} 3\sqrt{u}\ du \\
&= 2\left[u^{\frac{3}{2}}\right]_9^{25} \\
&= 2[125-27] \\
&= 196
\end{aligned}$$

▲

Example 13.6.

Evaluate $\int_e^{e^2} \frac{\ln x+1}{x}\ dx$.

Let $u = lnx+1$. Then, $du = \frac{1}{x}\ dx$. When $x=e, u=2$ and $x=e^2, u=3$.

$$\begin{aligned}
\int_e^{e^2} \frac{\ln x+1}{x}\ dx &= \int_2^3 u\ du \\
&= \left[\frac{u^2}{2}\right]_2^3 \\
&= \frac{3^2-2^2}{2} \\
&= \frac{5}{2}
\end{aligned}$$

▲

Example 13.7.

Evaluate $\int_0^{\frac{\pi}{4}} \tan^3 x \sec x \; dx$.

Let $u = \sec x$. Then $du = \sec x \tan x \; dx$. When $x = 0, u = 1$ and $x = \frac{\pi}{4}, u = \sqrt{2}$.

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \tan^3 x \sec x \; dx &= \int_1^{\sqrt{2}} (u^2 - 1) \; du \\
&= \left[\frac{u^3}{3} - u \right]_1^{\sqrt{2}} \\
&= \left[\frac{(\sqrt{2})^3}{3} - \sqrt{2} \right] - \left[\frac{(1)^3}{3} - 1 \right] \\
&= \frac{2\sqrt{2}}{3} - \sqrt{2} - \frac{1}{3} + 1 \\
&= \frac{2 - \sqrt{2}}{3}
\end{aligned}$$

▲

Example 13.8.

Evaluate $\int_0^3 \frac{x^3}{\sqrt{9 - x^2}} \; dx$.

Let $x = 3 \sin \theta$. Then, $dx = 3 \cos \theta \; d\theta$. When $x = 0, u = 0$ and $x = 3, u = \frac{\pi}{2}$.

$$\begin{aligned}
\int_0^3 \frac{x^3}{\sqrt{9 - x^2}} \; dx &= \int_0^{\frac{\pi}{2}} \frac{(3 \sin \theta)^3}{3 \cos \theta} 3 \cos \theta \; d\theta \\
&= 27 \int_0^{\frac{\pi}{2}} \sin^3 \theta \; d\theta \\
&= -27 \int_0^{\frac{\pi}{2}} (1 - \cos^2 \theta) \; d(\cos \theta) \\
&= -27 \left[\cos \theta - \frac{\cos^3 \theta}{3} \right]_0^{\frac{\pi}{2}} \\
&= -27 \left[\cos \frac{\pi}{2} - \frac{\cos^3 \frac{\pi}{2}}{3} \right] + 27 \left[\cos 0 - \frac{\cos^3 0}{3} \right] \\
&= 27 \left[1 - \frac{1}{3} \right] \\
&= 18
\end{aligned}$$

▲

Example 13.9.

Evaluate $\int_0^1 x e^x \; dx$.

$$\int_0^1 x e^x \; dx = \int_0^1 x \; d(e^x)$$

$$\begin{aligned} &= [xe^x]_0^1 - \int_0^1 e^x \ dx \\ &= \left[(1)e^1 - 0(e^0)\right] - [e^x]_0^1 \\ &= e - \left[e^1 - e^0\right] \\ &= 1 \end{aligned}$$

▲

Example 13.10.

Evaluate $\int_0^\pi x^2 \cos x \ dx$.

$$\begin{aligned} \int_0^\pi x^2 \cos x \ dx &= \int_0^\pi x^2 \ d(\sin x) \\ &= \left[x^2 \sin x\right]_0^\pi - 2\int_0^\pi x \sin x \ dx \\ &= 2\int_0^\pi x \ d(\cos x) \\ &= 2\left[x \cos x\right]_0^\pi - 2\int_0^\pi \cos x \ dx \\ &= 2\left[\pi \cos \pi\right] - 2\left[\sin x\right]_0^\pi \\ &= -2\pi - 2\left[0 - 0\right] \\ &= -2\pi \end{aligned}$$

▲

13.4 Definite Integrals of Even, Odd and Periodic Functions

13.4.1 Even and Odd functions

A function $f(x)$ is an even function if $f(-x) = f(x)$ for all values of x.

Example 13.11.
$f(x) = x^2$ is an even function since $f(-x) = (-x)^2 = x^2 = f(x)$. ▲

A function $f(x)$ is an odd function if $f(-x) = -f(x)$ for all values of x.

Example 13.12.
$f(x) = x^3$ is an odd function since $f(-x) = (-x)^3 = -x^3 = -f(x)$. ▲

When $f(x)$ is an even function or an odd function, we have the following result.

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx \text{ , if } f(x) \text{ is even.} \\ 0 \text{ , if } f(x) \text{ is odd.} \end{cases}$$

Proof.

$$\int_{-a}^{a} f(x)\ dx = \int_{-a}^{0} f(x)\ dx + \int_{0}^{a} f(x)\ dx$$

Consider $\int_{-a}^{0} f(x)\ dx$.

Let $u = -x$. Then, $-du = dx$. When $x = -a, u = a$ and $x = 0, u = 0$.

$$\begin{aligned} \int_{-a}^{0} f(x)\ dx &= \int_{a}^{0} f(-u)\ (-du) \\ &= \int_{0}^{a} f(-u)\ du \\ &= \int_{0}^{a} f(-x)\ dx \end{aligned}$$

Hence,

$$\int_{-a}^{a} f(x)\ dx = \int_{0}^{a} f(-x)\ dx + \int_{0}^{a} f(x)\ dx$$

When $f(x)$ is even function, $f(-x) = f(x)$.

$$\int_{-a}^{a} f(x)\ dx = 2\int_{0}^{a} f(x)\ dx$$

When $f(x)$ is even function, $f(-x) = -f(x)$.

$$\int_{-a}^{a} f(x)\ dx = 0$$

■

Example 13.13.

Evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x\ dx$.

Let $f(x) = \cos x$. Notice that $f(-x) = \cos(-x) = \cos x = f(x)$. Hence, $\cos x$ is an even function.

$$\begin{aligned} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x\ dx &= 2\int_{0}^{\frac{\pi}{4}} \cos x\ dx \\ &= 2\left[\sin x\right]_{0}^{\frac{\pi}{4}} \\ &= 2\left[\sin\frac{\pi}{4} - \sin 0\right] \\ &= \sqrt{2} \end{aligned}$$

▲

Example 13.14.

Evaluate $\int_{-1}^{1} \frac{x}{x^2+2}\ dx$.

Let $f(x) = \frac{x}{x^2+2}$. Notice that $f(-x) = \frac{-x}{(-x)^2+2} = -f(x)$. Hence, $\frac{x}{x^2+2}$ is an odd function.

$$\int_{-1}^{1} \frac{x}{x^2+2}\ dx = 0$$

▲

13.4.2 Periodic Functions

If a function $f(x)$ is a periodic function with period T, then $f(x+T) = f(x)$ for all values of x.

Example 13.15. $f(x) = \sin x$ is a periodic function with period 2π. ▲

Note: If $f(x+T) = f(x)$, then $f(x+nT) = f(x)$, where n is an integer.

When $f(x)$ is a periodic function with period T, we have the following result.

$$\int_0^{nT} f(x)\ dx = n\int_0^T f(x)\ dx, \quad \text{where } n \text{ is an integer.}$$

Proof.

Case I: n is a positive integer.

$$\int_0^{nT} f(x)\ dx = \int_0^T f(x)\ dx + \int_T^{2T} f(x)\ dx + \cdots + \int_{(n-1)T}^{nT} f(x)\ dx$$

Consider $\int_T^{2T} f(x)\ dx$. Let $u = x - T$. Then $du = dx$. When $x = T, u = 0$ and $x = 2T, u = T$.

$$\begin{aligned}\int_T^{2T} f(x)\ dx &= \int_0^T f(u+T)\ du \\ &= \int_0^T f(u)\ du \\ &= \int_0^T f(x)\ dx\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{2T}^{3T} f(x)\ dx &= \int_0^T f(x)\ dx \\ &\vdots \\ \int_{(n-1)T}^{nT} f(x)\ dx &= \int_0^T f(x)\ dx\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^{nT} f(x)\ dx &= \int_0^T f(x)\ dx + \int_T^{2T} f(x)\ dx + \cdots \int_{(n-1)T}^{nT} f(x)\ dx \\ &= n\int_0^T f(x)dx\end{aligned}$$

Case II: $n = 0$. The statement is trivial.

Case III: n is a negative integer.

Let $u = x - nT$ and $m = -n$. Then, $du = dx$. When $x = 0, u = -nT = mT$ and $x = nT, u = 0$

$$\begin{aligned}\int_0^{nT} f(x)\ dx &= \int_{mT}^0 f(u)\ du \\ &= -\int_0^{mT} f(u)\ du\end{aligned}$$

$$\begin{aligned} &= -m\int_0^T f(u)\ du \quad \text{(by the result of case I)} \\ &= n\int_0^T f(x)\ dx \end{aligned}$$

To conclude,

$$\int_0^{nT} f(x)\ dx = n\int_0^T f(x)\ dx, \quad \text{where } n \text{ is an integer.}$$

■

Example 13.16.

It is given that $f(x)$ is a periodic function with period 2π, $\int_0^{\frac{\pi}{3}} f(x)\ dx = \frac{\sqrt{3}}{4}$ and $\int_0^{2\pi} f(x)\ dx = 0$.

(a) Show that $\int_{a+2n\pi}^{b+2n\pi} f(x)\ dx = \int_a^b f(x)\ dx$, where a, b are real numbers and n is an integer.

(b) Hence, evaluate $\int_0^{\frac{13\pi}{3}} f(x)\ dx$.

(a)

Let $u = x - 2n\pi$. Then $du = dx$. When $x = a + 2n\pi, u = a$ and $x = b + 2n\pi, u = b$.

$$\begin{aligned} \int_{a+2n\pi}^{b+2n\pi} f(x)\ dx &= \int_a^b f(u+2n\pi)\ du \\ &= \int_a^b f(u)\ du \\ &= \int_a^b f(x)\ dx \end{aligned}$$

(b)

$$\begin{aligned} \int_0^{\frac{13\pi}{3}} f(x)\ dx &= \int_0^{4\pi+\frac{\pi}{3}} f(x)\ dx \\ &= \int_0^{4\pi} f(x)\ dx + \int_{4\pi}^{4\pi+\frac{\pi}{3}} f(x)\ dx \\ &= \int_0^{2(2\pi)} f(x)\ dx + \int_0^{\frac{\pi}{3}} f(x)\ dx \\ &= 2\int_0^{2\pi} f(x)\ dx + \frac{\sqrt{3}}{4} \\ &= 2(0) + \frac{\sqrt{3}}{4} \\ &= \frac{\sqrt{3}}{4} \end{aligned}$$

▲

Exercises

1. Evaluate the following definite integrals from the definition of definite integrals.
   (a) $\displaystyle\int_1^3 (4x+3)\ dx.$
   (b) $\displaystyle\int_0^2 (7+x^2)\ dx.$
2. Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \left(\frac{1}{1-\sin x}+\frac{1}{1+\sin x}\right)\ dx.$
3. Evaluate $\displaystyle\int_1^4 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\ dx.$
4. Evaluate $\displaystyle\int_{-3}^3 \frac{e^x}{e^x+2e^{-x}}\ dx - 2\int_3^{-3} \frac{e^{-x}}{e^x+2e^{-x}}\ dx.$
5. Evaluate $\displaystyle\int_1^{e^5} \frac{dx}{x\ln x + x}.$
6. Evaluate $\displaystyle\int_0^3 (x+1)e^{x^2+2x}\ dx.$
7. Evaluate $\displaystyle\int_{3\sqrt{2}}^{3\sqrt{3}} \frac{1}{x^2\sqrt{36-x^2}}\ dx.$
8. Evaluate $\displaystyle\int_{-1}^3 \frac{1}{\sqrt{(x+1)^2+16}}\ dx.$
9. (a) Evaluate $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}.$
   (b) Using the substitution $u=\dfrac{1}{x}$ and the result of (a), evaluate $\displaystyle\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{(x^3+1)(x^2+1)}.$
10. Evaluate $\displaystyle\int_1^4 \sqrt{x}\ln x\ dx.$
11. Evaluate $\displaystyle\int_{-\pi}^0 e^{2x}(\sin x+\cos x)^2\ dx.$
12. Evaluate $\displaystyle\int_0^{\sqrt{3}} x^3 e^{x^2}\ dx.$
13. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x\cos^2 x\ dx.$
14. (a) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} x\sec^2 x\ dx.$
    (b) Using the result of (a), evaluate $\displaystyle\int_0^{\frac{\pi}{3}} x\tan^2 x\ dx.$
15. Evaluate $\displaystyle\int_{-\pi}^{\pi} (e^x+e^{-x})\sin x\ dx.$

16. Suppose $f(x)$ is a periodic function with period T.

    (a) Using the substitution $u = x + T$, show that $\int_a^b f(x)\ dx = \int_{a+T}^{b+T} f(x)\ dx$, where a and b are real numbers.

    (b) Using the result of (a), show that $\int_0^T f(x)\ dx = \int_a^{a+T} f(x)\ dx$, where a is a real number.

17. Evaluate $\int_{\sqrt{2}}^{2} \frac{dx}{x^2\sqrt{x^2-1}}$.

18. Evaluate $\int_{-1}^{0} x^2 e^{-3x}\ dx$.

19. (a) Using the substitution $u = a - x$, show that $\int_0^a \frac{x^3}{x^3+(x-a)^3}\ dx = \int_0^a \frac{(x-a)^3}{x^3+(x-a)^3}\ dx$.

    (b) Using the result of (a), evaluate $\int_0^4 \frac{x^3}{x^3+(x-4)^3}\ dx$.

20. (a) Show that $\int_0^{\frac{\pi}{2}} f(x)\ dx = \int_0^{\frac{\pi}{2}} f(\frac{\pi}{2} - x)\ dx$.

    (b) Using the result of (a), evaluate the following definite integrals.

    i. $\int_0^{\frac{\pi}{2}} \frac{\cos x - 3\sin x}{\sin x + \cos x}\ dx$.

    ii. $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x + \cos x}\ dx$. (Hint: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$)

Chapter 14

Application of Integration

In this chapter, we will see how definite and indefinite integrals be applied to real life situation. The first application of indefinite integral would be on straight line motion and the next application would be on graphs and functions.

14.1 Straight Line Motion

Consider the motion of an object along a straight line. If the displacement of the object from a reference point at any time t is s, then its velocity v at time t is given by $v = \frac{ds}{dt}$ and its acceleration a at time t is given by $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Conversely, the displacement and the velocity of the object can be determined from the given velocity and acceleration respectively. For any given acceleration a at time t, the velocity v and displacement s can be found by

$$s = \int v \ dt$$

$$v = \int a \ dt.$$

Example 14.1.

An object moves along a straight line through a fixed point O. The acceleration $a \ ms^{-2}$ of the object t seconds after the object has passed through O is given by $a = 10t - 2$. The object moves with a velocity of $18 \ ms^{-1}$ when $t = 2$.

(a) Find the velocity of the object when $t = 6$.

(b) Find the displacement of the object from O when $t = 6$.

(a) Let $v \ ms^{-1}$ be the velocity of the object.

$$\begin{aligned} v &= \int a \ dt \\ &= \int (10t - 2) \ dt \end{aligned}$$

$$= 5t^2 - 2t + C_1$$

When $t = 2$, $v = 18$. Hence, $C_1 = 18 - 5(2)^2 + 2(2) = 2$.

i.e. $v = 5t^2 - 2t + 2$.

When $t = 6$, $v = 5(6)^2 - 2(6) + 2 = 170$.

(b) Let s m be the displacement of the object from O.

$$\begin{aligned} s &= \int v\ dt \\ &= \int (5t^2 - 2t + 2)\ dt \\ &= \frac{5}{3}t^3 - t^2 + 2t + C_2 \end{aligned}$$

When $t = 0$, $s = 0$. Hence, $C_2 = 0$.

i.e. $s = \dfrac{5}{3}t^3 - t^2 + 2t$.

When $t = 6$, $s = \dfrac{5}{3}(6)^3 - (6)^2 + 2(6) = 336$.

▲

Example 14.2.
A ball is thrown vertically upwards from the ground. Its velocity v ms^{-1} after t seconds is given by $v = 20 - 10t$. Let s m be the height of the ball above the ground.

(a) Express s in terms of t.

(b) Find the time taken for the ball to reach the highest point and the maximum height reached.

(c) Find the total distance travelled by the ball from $t = 1$ to $t = 3$.

(a)

$$\begin{aligned} s &= \int v\ dt \\ &= \int (20 - 10t)\ dt \\ &= 20t - 5t^2 + C \end{aligned}$$

When $t = 0$, $s = 0$. Hence, $C = 0$.

i.e. $s = 20t - 5t^2$.

(b) When the ball reaches the highest point, its velocity would be zero. i.e. $v = 0$.

Hence, $20 - 10t = 0$ and $t = 2$.

The time taken for the ball to reach the highest point is 2 s.

When $t = 2$, $s = 20(2) - 5(2)^2 = 20$.

Hence, the maximum height reached is 20 m.

(c) When $t = 1$, $s = 20(1) - 5(1)^2 = 15$.

When $t = 3$, $s = 20(3) - 5(3)^2 = 15$.

From (b), the ball will reach its highest position and start to fall after 2 seconds.

Hence, the total distance travelled by the ball from $t = 1$ to $t = 3$

$$\begin{aligned} &= [(20-15)+(20-15)] \\ &= 10\ m \end{aligned}$$

▲

14.2 Geometrical Application

We have learnt that the slope of a curve $y = f(x)$ at any point (x, y) is given by $f'(x)$ or $\frac{dy}{dx}$. In particular, $f'(x_1)$ is the slope of the curve at (x_1, y_1).
If the slope of a curve at any point is given, we can use integration to find the equation of the curve.

Example 14.3.
The slope of a curve at any point (x, y) is given by $\frac{dy}{dx} = 2x - 4$. If the curve passes through the point $(1, -2)$, find the equation of the curve.

$$\begin{aligned} \frac{dy}{dx} &= 2x - 4 \\ y &= \int (2x-4)\ dx \\ &= x^2 - 4x + C \end{aligned}$$

Since the curve passes through $(1, -2)$,

$$\begin{aligned} -2 &= (1)^2 - 4(1) + C \\ C &= 1 \end{aligned}$$

Hence, the equation of the curve is $y = x^2 - 4x + 1$.

▲

Example 14.4.
At any point (x, y) of a curve, $\frac{d^2y}{dx^2} = 12x + 6$. If $(0, -7)$ and $(2, 11)$ lie on the curve, find the equation of the curve.

$$\begin{aligned} \frac{d^2y}{dx^2} &= 12x + 6 \\ \frac{dy}{dx} &= \int (12x+6)\ dx \\ \frac{dy}{dx} &= 6x^2 + 6x + C_1 \end{aligned}$$

Then,

$$y = \int (6x^2 + 6x + C_1)\ dx.$$

Hence,

$$y = 2x^3 + 3x^2 + C_1x + C_2.$$

Since $(0, -7)$ and $(2, 11)$ lies on the curve, we have

$$-7 = 2(0)^3 + 3(0)^2 + C_1(0) + C_2$$

and

$$11 = 2(2)^3 + 3(2)^2 + C_1(2) + C_2.$$

On solving, $C_1 = -5$ and $C_2 = -7$.

i.e. The equation of the curve is $y = 2x^3 + 3x^2 - 5x - 7$.

▲

Example 14.5.

At any point (x, y) of a curve, $\dfrac{d^2y}{dx^2} = 24x^2 - 10$. If the slope of the curve at $(-1, -8)$ is 9, find the equation of the curve.

$$\begin{aligned}
\frac{d^2y}{dx^2} &= 24x^2 - 10 \\
\frac{dy}{dx} &= \int (24x^2 - 10)\ dx \\
&= 8x^3 - 10x + C_1
\end{aligned}$$

Since the slope of the curve at $(-1, -8)$ is 9

$$\begin{aligned}
9 &= 8(-1)^3 - 10(-1) + C_1 \\
C_1 &= 7 \\
\frac{dy}{dx} &= 8x^3 - 10x + 7 \\
y &= \int (8x^3 - 10x + 7)\ dx \\
&= 2x^4 - 5x^2 + 7x + C_2
\end{aligned}$$

Since $(-1, -8)$ is a point on the curve

$$\begin{aligned}
-8 &= 2(-1)^4 - 5(-1)^2 + 7(-1) + C_2 \\
C_2 &= 2
\end{aligned}$$

i.e. The equation of the curve is $y = 2x^4 - 5x^2 + 7x + 2$.

▲

14.3 Area of a Plane Figure

From the definition of integrals, we know that

$$\int_a^b f(x)\ dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)(\frac{b-a}{n}).$$

Hence, the area of a region bounded by a curve $f(x)$, the x-axis and the line $x = a$ and $x = b$ can be evaluated by the definite integrals:

$$\int_a^b f(x)\ dx.$$

Similarly, the area of a region bounded by a curve $f(y)$, the y-axis and the line $y = a$ and $y = b$ can be evaluated by the definite integrals:

$$\int_a^b f(y)\ dy.$$

Example 14.6.
Find the area of the region bounded by the line $y = x + 1$, the x-axis and the lines $x = 1$ and $x = 2$.

The required area:

$$\begin{aligned}\int_1^2 (x+1)\ dx &= \left[\frac{x^2}{2} + x\right]_1^2 \\ &= \left[4 - \frac{3}{2}\right] \\ &= \frac{5}{2}\end{aligned}$$

▲

Example 14.7.
Find the area of the region bounded by the curve $y = -\frac{1}{x^2}$, the x-axis and the lines $x = 1$ and $x = 2$.

Notice that the region is below the x-axis when x lies between 1 and 2.

Hence, the required area:

$$\begin{aligned}-\int_1^2 \left(-\frac{1}{x^2}\right)\ dx &= \left[-\frac{1}{x}\right]_1^2 \\ &= \frac{1}{2}\end{aligned}$$

▲

Example 14.8.
Find the area of the region bounded by the curve $y = x(x-1)(x-2)$ and the x-axis.

When $y = 0$, $x = 0, 1$ or 2. Hence, the x-intercepts of the curve are $0, 1$ and 2. Notice that from $x = 0$ to $x = 1$, the area region is above the x-axis while from $x = 1$ to $x = 2$, the area region is below the x-axis.

Hence, the required area:

$$\int_0^1 x(x-1)(x-2)\ dx - \int_1^2 x(x-1)(x-2)\ dx \quad = \quad \int_0^1 (x^3 - 3x^2 + 2x)\ dx - \int_1^2 (x^3 - 3x^2 + 2x)\ dx$$

$$\begin{aligned} &= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 \\ &= \frac{1}{2} \end{aligned}$$

▲

Example 14.9.
Find the area of the region below the x-axis bounded by the curve $x^2 + y^2 = 36$ and the line $x = -3$.

Notice that $y = \pm\sqrt{36 - x^2}$ and the required area is below the x-axis. Hence, we will take $y = -\sqrt{36 - x^2}$.

The required area:

$$-\int_{-3}^{6} -\sqrt{36 - x^2}\ dx = \int_{-3}^{6} \sqrt{36 - x^2}\ dx$$

Let $x = 6\sin\theta$. Then $dx = 6\cos\theta\ d\theta$. When $x = -3, \theta = -\frac{\pi}{6}$ and $x = 6, \theta = \frac{\pi}{2}$.

$$\begin{aligned} \int_{-3}^{6} \sqrt{36 - x^2}\ dx &= \int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} 36\cos^2\theta\ d\theta \\ &= 18\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \cos 2\theta)\ d\theta \\ &= 18\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= 12\pi + \frac{9\sqrt{3}}{2} \end{aligned}$$

▲

Example 14.10.
Find the area of the region bounded by the curve $x = y(y-2)(y+1)$ and the y-axis.

Notice that we are now considering the area with respect to the *y-axis* instead of the *x-axis*. When $x = 0$, $y = -1, 0$ or 2. Hence, the y-intercepts of the curve are $-1, 0$ and 2. Moreover, the required area is leftward of the y-axis from $y = 0$ to $y = 2$ and rightward of the y-axis from $y = -1$ to $y = 0$.

The required area:

$$\begin{aligned} \int_{-1}^{0} y(y-2)(y+1)\ dy - \int_{0}^{2} y(y-1)(y-2)\ dy &= \int_{-1}^{0} (y^3 - y^2 - 2y)\ dy - \int_{0}^{2} (y^3 - y^2 - 2y)\ dy \\ &= \left[\frac{y^4}{4} - \frac{y^3}{3} - y^2\right]_{-1}^{0} - \left[\frac{y^4}{4} - \frac{y^3}{3} - y^2\right]_{0}^{2} \\ &= \frac{37}{12} \end{aligned}$$

▲

Example 14.11.
Find the area of the region bounded by the curve $y = x^2$ and the line $y = 2x$.

By solving $y = x^2$ and $y = 2x$ simultaneously, the x-coordinate of their points of intersection are $x = 0$ and $x = 2$. Notice that when $0 \le x \le 2$, $2x \ge x^2$.

Hence, the required area:

$$\begin{aligned}\int_0^2 (2x - x^2)\ dx &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= \frac{4}{3}\end{aligned}$$

▲

Example 14.12.
Find the area of the region bounded by the curve $y = 2\sin x + 2$ and $y = \sin x + 2$ between $x = 0$ and $x = 2\pi$.

By solving $y = 2\sin x + 2$ and $y = \sin x + 2$ simultaneously, the x-coordinate of their points of intersection are $x = 0$, $x = \pi$ and $x = 2\pi$. From $x = 0$ to $x = \pi$, $2\sin x + 2 \geq \sin x + 2$ while from $x = \pi$ to $x = 2\pi$, $\sin x + 2 \geq 2\sin x + 2$.

Hence, the required area:

$$\begin{aligned}\int_0^{\pi} (2\sin x + 2 - \sin x - 2)\ dx + \int_{\pi}^{2\pi} (\sin x + 2 - 2\sin x - 2)\ dx &= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} \\ &= 4\end{aligned}$$

▲

Example 14.13.
Find the area of the region bounded by the curve $x = y^2$ and the line $x = -y + 6$.

By solving $x = y^2$ and $x = -y + 6$ simultaneously, the y-coordinate of their points of intersection are 2 and -3. When $-3 \leq y \leq 2$, $-y + 6 \geq y^2$.

Hence, the required area:

$$\begin{aligned}\int_{-3}^{2} (-y + 6 - y^2)\ dy &= \left[-\frac{y^2}{2} + 6y - \frac{y^3}{3}\right]_{-3}^{2} \\ &= \frac{125}{6}\end{aligned}$$

▲

Example 14.14.
Find the area of the region bounded by the curve $y = x^2$ and the lines $y = 5x$ and $y = 4 - 3x$.

By solving $y = x^2$ and $y = 5x$ simultaneously, the x-coordinate of their points of intersection is $\frac{1}{2}$. Similarly, by solving $y = x^2$ and $y = -4 - 3x$ simultaneously, the x-coordinate of their points of intersection is 1.

Hence, the required area:

$$\begin{aligned}\int_0^{\frac{1}{2}} (5x - x^2)\ dx + \int_{\frac{1}{2}}^{1} (4 - 3x - x^2)\ dx &= \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^{\frac{1}{2}} + \left[4x - \frac{3x^2}{2} - \frac{x^3}{3}\right]_{\frac{1}{2}}^{1} \\ &= \frac{7}{6}\end{aligned}$$

▲

14.4 Volume of a Solid of Revolution

The volume of a solid obtained by revolving the region bounded by a curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ about the axis is given by

$$\pi \int_a^b [f(x)]^2 \; dx.$$

Similarly, the volume of a solid obtained by revolving the region bounded by a curve $x = g(y)$, the y-axis and the lines $y = a$ and $y = b$ is given by

$$\pi \int_a^b [f(y)]^2 \; dy.$$

Example 14.15.
Find the volume of the solid of revolution generated by revolving the region bounded by the x-axis and the lines $y = x + 1$, $x = 1$ and $x = 5$ about the axis.

The required volume:

$$\begin{aligned} \pi \int_1^5 (x+1)^2 \; dx &= \pi \left[\frac{(x+1)^3}{3} \right]_1^5 \\ &= \frac{208\pi}{3} \end{aligned}$$

▲

Example 14.16.
Find the volume of the solid of revolution generated by revolving the region bounded by the y-axis and the lines $y = x + 1$ and $y = 7$ about the y-axis.

The required volume:

$$\begin{aligned} \pi \int_1^7 (y-1)^2 \; dy &= \pi \left[\frac{(y-1)^3}{3} \right]_1^7 \\ &= 72\pi \end{aligned}$$

▲

Example 14.17.
Find the volume of the solid generated by revolving the region bounded by

(a) the curve $y = x^2$ and the lines $y = 1$, $x = 1$ and $x = 3$ about the line $y = 1$,

(b) the curve $y = x^2$ and the lines $x = 1$, $y = 1$ and $y = 4$ about the line $x = 1$.

(a) The required volume:

$$\begin{aligned} \pi \int_1^3 (x^2-1)^2 \; dx &= \pi \int_1^3 (x^4 - 2x^2 + 1) \; dx \\ &= \pi \left[\frac{x^5}{5} - \frac{2x^3}{3} + x \right]_1^3 \\ &= \frac{496\pi}{15} \end{aligned}$$

(b) Since $x = y^{\frac{1}{2}}$, the required volume:

$$\begin{aligned} \pi \int_1^4 (y^{\frac{1}{2}} - 1)^2 \; dy &= \pi \int_1^4 (y - 2y^{\frac{1}{2}} + 1) \; dy \\ &= \pi \left[\frac{y^2}{2} - \frac{4y^{\frac{3}{2}}}{3} + y \right]_1^4 \\ &= \frac{7\pi}{6} \end{aligned}$$

▲

Example 14.18.

Find the volume of the solid generated by revolving the region bounded by the curve $x = \sqrt{y}$ and the line $x = 1$, $y = 1$ and $y = 4$ about the y-axis.

The required volume:

$$\begin{aligned} \pi \int_1^4 \left[(\sqrt{y})^2 - 1^2 \right] \; dy &= \pi \int_1^4 (y - 1) \; dy \\ &= \pi \left[\frac{y^2}{2} - y \right]_1^4 \\ &= \frac{9\pi}{2} \end{aligned}$$

▲

Example 14.19.

Find the volume of the solid generated by revolving the region bounded by the curve $y = \dfrac{1}{3}x^2 + 1$ and the line $y = x$, $x = 0$ and $x = 3$ about the x-axis.

The required volume:

$$\begin{aligned} \pi \int_0^3 \left[\left(\frac{1}{3}x^2 + 1 \right)^2 - x^2 \right] \; dx &= \pi \int_0^3 \left(\frac{x^4}{9} + \frac{2x^2}{3} + 1 - x^2 \right) \; dx \\ &= \pi \left[\frac{x^5}{45} - \frac{x^3}{9} + x \right]_0^3 \\ &= \frac{27\pi}{5} \end{aligned}$$

▲

Exercises

1. A toy car initially at rest moves along a straight line from a point O. Its velocity $v \; ms^{-1}$ after t seconds is given by $v = -t^2 + 25$. Let s m be the displacement of the toy car from O.
   (a) Express s in terms of t.
   (b) Find the displacement of the toy car from O after 2 seconds.

2. There is an apple on a tree. The apple is 4 m above the ground. Polly throws a ball vertically upwards towards the apple from the ground. The velocity $v\ ms^{-1}$ of the ball after t seconds is given by $v = 7 - 10t$. Can the ball hit the apple? Explain your answer briefly.

3. The acceleration $a\ ms^{-2}$ of a moving object at time t s is given by $a = \cos t$. The velocity $v\ ms^{-1}$ of the object is 0.5 ms^{-1} when $t = 30$. (a) Express v in terms of t. (b) Find the maximum velocity of the object.

4. At any point (x, y) of a curve, $\frac{d^2y}{dx^2} = 24(1 - x^2)$. If the slope of the curve at $(3, -7)$ is 4, find the equation of the curve.

5. At any point (x, y) of a curve, $\frac{d^2y}{dx^2} = 48x^2$. If $(1, 12)$ and $(-1, 18)$ lie on the curve, find the equation of the curve.

6. At any point (x, y) of a curve, $\frac{d^2y}{dx^2} = kx$, where k is a constant. The curve attains its maximum at $(-1, 5)$ and its y-intercept is 1.
   (a) Find the value of k.
   (b) Find the equation of the curve.
   (c) Find the minimum point of the curve.
   (d) Sketch the curve for $-2 \le x \le 2$.

7. HKCEE 2005 Additional Mathematics Q10
   (a) Show that $\frac{d}{dx}[x(x + 1)^n] = (x + 1)^{n-1}[(n + 1)x + 1]$, where n is a rational number.
   (b) The slope of any point (x, y) of a curve C is given by $\frac{dy}{dx} = (x + 1)^{2004}(2006x + 1)$. If C passes through the point $(-1, 1)$, find the equation of C.

8. HKCEE 2006 Additional Mathematics Q10
   The slope at any point (x, y) of a curve is given by $\frac{dy}{dx} = 3 + 2\cos 2x$. If the curve passes through the point $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$, find its equation.

9. Two curves $y = f(x)$ and $y = g(x)$ intersect at a point A, where $f(x) = x^2+2x+1$ and $g(x) = f(x-4)$.
   (a) Find the coordinates of A.
   (b) Find the area of the region bounded by the two curves and the x-axis.

10. The curves $y = -x^2 + 4$ and $y = \frac{x^2}{2} - x$ both pass through a point A on the x-axis.
    (a) Find the coordinates of A.
    (b) Find the area of the region bounded by the two curves and y-axis.

11. The curve $y^2 = 2x$ and the straight line $x - 2y + 2 = 0$ intersect at a point A.
    (a) Find the coordinates of A.
    (b) Find the area of the region bounded by the curve, the straight line and the y-axis.

12. The curve $y = \frac{1}{x}$ cuts the straight lines $y = x$ and $x = e$ at P and Q respectively.
    (a) Find the coordinates of P and Q.
    (b) Find the area of the region bounded by the curve $y = \frac{1}{x}$, the x-axis and the lines $y = x$ and $x = e$.

13. Find the area in the first quadrant bounded by the curves $y = e^x, y = e^{-x}$ and $x = 2$.

14. Find the area bounded by the curve $2y^2 = 3 - x$ and the straight line $x + 2y + 1 = 0$.

15. Let $f(x) = \dfrac{p - x}{x + q}$ for all $x \neq -q$ and $g(x) = \dfrac{x + 2q}{4 - x}$ for all $x \neq 4$, where p and q are positive constants. The y-intercept of the curve $y = f(x)$ is 5 and the x-intercept of the curve $y = g(x)$ is -2.

    (a) Find the values of p and q.

    (b) Find the point of intersection of two curves.

    (c) If the area of the shaded region bounded by the two curves, the y-axis and the line $x = k$ is $6 \ln \dfrac{3}{2}$, where $0 < k < \dfrac{3}{2}$, find the value of k.

16. (a) Evaluate $\displaystyle\int \sqrt{b^2 - y^2}\ dy$.

    (b) Find the area bounded by the lower half of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and the straight line $y = -(b-h)$ where $0 < h < b$.

17. Find the volume of the solid generated by revolving the region bounded by the curve $x = \sqrt{4 - y^2}$ and the line $x = -y + 2$ about the y-axis.

18. Find the volume of the solid generated by revolving the region bounded by the curves $y^2 = x$ and $8y = x^2$ about the x-axis.

19. Show, by using integration, that the volume of a sphere with radius r is $\dfrac{4}{3}\pi r^3$.

20. Find the volume of the solid of revolution generated by revolving the region bounded by the x-axis and the curves $y = \sqrt{x}$ and $y = \sqrt{6 - x}$ about the x-axis.

21. Find the volume of the solid of revolution generated by revolving the region bounded by the x-axis, $x = 0$, $x = \dfrac{\pi}{2}$, $y = \sin x$ and $y = \cos x$ about the x-axis.

22. Given two curves $y = f(x)$ and $y = g(x)$, where $f(x) = -x^2 - 6x + 2$ and $g(x) = -f(x) + 4$.

    (a) Find the points of intersection of the two curves.

    (b) Find the area of the shaded region bounded by the two curves.

    (c) Find the volume of the solid of revolution generated by revolving the shaded region about the y-axis.