Math3806 Lecture Note 4 Appendix

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P2. The selection of significant level α is relatively subjective. Accepting the null hypothesis cannot say that the null hypothesis is right, and only can say there is no strong evidence to reject it.

P3. Two degree of freedoms of Hotelling's T^2 or F distribution: p is from the sample mean, $n - p$ is the degree of freedom of the sample covariance matrix.

P5. Example 4.1.

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{6+10+8}{3} \\ \frac{9+6+3}{3} \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$s_{11} = \frac{(6-8)^2 + (10-8)^2 + (8-8)^2}{3-1} = 4,$$

$$s_{22} = \frac{(9-6)^2 + (6-6)^2 + (3-6)^2}{3-1} = 9,$$

$$s_{11} = \frac{(6-8)(9-6) + (10-8)(6-6) + (8-8)(3-6}{3-1} = -3$$

- Example 4.1. (continuous) Hence

$$\mathbf{S} = \begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix}, \text{and } \mathbf{S}^{-1} = \frac{1}{4 \cdot 9 - (-3)(-3)} \begin{bmatrix} 9 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{bmatrix}$$

Hence

$$T^2 = 3[8-9, 6-5] \begin{bmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{4}{27} \end{bmatrix} \begin{bmatrix} 8-9 \\ 6-5 \end{bmatrix} = 3[-1, 1] \begin{bmatrix} -\frac{2}{9} \\ \frac{1}{27} \end{bmatrix} = \frac{7}{9}$$

and Since $n = 3$ and $p = 2$, T^2 has the distribution of a

$$\frac{(3-1)2}{3-2} F_{2,3-2} = 4F_{2,1}$$

random variable.

P5. Example 4.2. $n = 20, p = 3,$

$$\bar{\mathbf{x}} = \begin{bmatrix} 4.640 \\ 45.400 \\ 9.965 \end{bmatrix}, \ \mathbf{S} = \begin{bmatrix} 2.879 & 10.010 & -1.810 \\ 10.010 & 199.788 & -5.640 \\ -1.810 & -5.640 & 3.628 \end{bmatrix}$$

and

$$\mathbf{S}^{-1} = \begin{bmatrix} .586 & -.022 & .258 \\ -.022 & .006 & -.002 \\ .258 & -.002 & .402 \end{bmatrix}.$$

$$T^2 = 20[4.640 - 4, 45.400 - 50, 9.965 - 10] \times \begin{bmatrix} .586 & -.022 & .258 \\ -.022 & .006 & -.002 \\ .258 & -.002 & .402 \end{bmatrix} \begin{bmatrix} 4.640 - 4 \\ 45.400 - 50 \\ 9.965 - 10 \end{bmatrix} = 9.74.$$

The critical value is

$$\frac{(n-1)p}{n-p} F_{p,n-p}(.10) = \frac{19 \cdot 3}{17} F_{3,17}(.10) = 3.353 \cdot 2.44 = 8.18$$

But $T^2 = 9.74 > 8.18$, and hence consequently, we reject H_0 at the 10% level of significance.

P9. Example 4.3. $n = 42, p = 2$.

$$\bar{\mathbf{x}} = \begin{bmatrix} .563 \\ .603 \end{bmatrix}, \mathbf{S} = \begin{bmatrix} .0144 & .0117 \\ .0117 & .0146 \end{bmatrix}, \mathbf{S}^{-1} = \begin{bmatrix} 203.018 & -163.391 \\ -163.391 & 200.228 \end{bmatrix}$$

The eigenvalue and eigenvector pairs for $\mathbf{S}$ are

$$\lambda_1 = .026, \mathbf{e}_1^T = [.704, .710], \text{and } \lambda_2 = .002, \mathbf{e}_2^T = [-.710, .704].$$

The 95% confidence ellipse for $\boldsymbol{\mu}$ consists of all values (μ_1, μ_2) satisfying

$$42[.564 - \mu_1, .603 - \mu_2] \begin{bmatrix} 203.018 & -163.391 \\ -163.391 & 200.228 \end{bmatrix} \begin{bmatrix} .563 - \mu_1 \\ .603 - \mu_2 \end{bmatrix}$$
$$\leq \frac{2 \cdot 41}{40} F_{2,40}(.05)$$

Since $F_{2,40}(.05) = 3.23$,

$$42(203.018)(.564 - \mu_1)^2 + 42(200.228)(.603 - \mu_2)^2$$
$$- 84(163.391)(.564 - \mu_1)(.603 - \mu_2) \leq 6.62$$

- Example 4.3. continuous. If $\boldsymbol{\mu}_0 = [.562, .589]^T$, then

$$\begin{aligned} 42(203.018)(.564-.562)^2 &+ 42(200.228)(.603-.589)^2 \\ &- 84(163.391)(.564-.562)(.603-.589) \\ &= 1.30 \leq 6.62 \end{aligned}$$

  Hence $\boldsymbol{\mu}_0 = [.562, .589]^T$ is in the confidence region. Equivalently, for a test of $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ vs $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$ at $\alpha = 0.05$ level of significance, we will not reject the null hypothesis H_0.
- The center of the joint confidence ellipsoid is at $\bar{\mathbf{x}}^T = [.564, .603]$, the axes lie along $\mathbf{e}_1$ and $\mathbf{e}_2$ with the half-lengths of the major and minor axes are given by

$$\sqrt{\lambda_1}\sqrt{\frac{p(n-1)}{n(n-p)}F_{p,n-p}(\alpha)} = \sqrt{0.026}\sqrt{\frac{2(41)}{42(40)}(3.23)} = 0.64$$

$$\sqrt{\lambda_2}\sqrt{\frac{p(n-1)}{n(n-p)}F_{p,n-p}(\alpha)} = \sqrt{0.002}\sqrt{\frac{2(41)}{42(40)}(3.23)} = 0.018$$

P14. Example 4.4. The 95% simultaneous T^2 intervals for the two component means are

$$
\begin{aligned}
& \left(\bar{x}_1 - \sqrt{\frac{p(n-1)}{(n-p)}F_{p,n-p}(.05)}\sqrt{\frac{s_{11}}{n}},\ \bar{x}_1 + \sqrt{\frac{p(n-1)}{(n-p)}F_{p,n-p}(.05)}\sqrt{\frac{s_{11}}{n}}\right) \\
= & \left(.564 - \sqrt{\frac{2(41)}{40}3.23}\sqrt{\frac{0.0144}{42}}, .564 + \sqrt{\frac{2(41)}{40}3.23}\sqrt{\frac{0.0144}{42}}\right) \\
= & (.516, .612)
\end{aligned}
$$

$$
\begin{aligned}
& \left(\bar{x}_2 - \sqrt{\frac{p(n-1)}{(n-p)}F_{p,n-p}(.05)}\sqrt{\frac{s_{22}}{n}},\ \bar{x}_2 + \sqrt{\frac{p(n-1)}{(n-p)}F_{p,n-p}(.05)}\sqrt{\frac{s_{22}}{n}}\right) \\
= & \left(.603 - \sqrt{\frac{2(41)}{40}3.23}\sqrt{\frac{0.0146}{42}}, .603 + \sqrt{\frac{2(41)}{40}3.23}\sqrt{\frac{0.0146}{42}}\right) \\
= & (.555, .651)
\end{aligned}
$$

P14. Example 4.5. $n = 87, p = 3$,

$$\bar{\mathbf{x}} = \begin{bmatrix} 526.29 \\ 54.69 \\ 25.13 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 5808.06 & 597.84 & 222.03 \\ 597.84 & 126.05 & 23.39 \\ 222.03 & 23.39 & 23.11 \end{bmatrix}$$

$$\frac{p(n-1)}{n-p} F_{p,n-p}(\alpha) = \frac{3(87-1)}{87-3} F_{3,84}(.05) = \frac{3(86)}{84}(2.7) = 8.29.$$

Then we obtain the simultaneous confidence statements

$$526.29 - \sqrt{8.29}\sqrt{\frac{5808.06}{87}} \le \mu_1 \le 526.29 + \sqrt{8.29}\sqrt{\frac{5808.06}{87}},$$

$$\text{or } 503.06 \le \mu_1 \le 550.12$$

$$54.69 - \sqrt{8.29}\sqrt{\frac{126.05}{87}} \le \mu_2 \le 54.69 + \sqrt{8.29}\sqrt{\frac{126.05}{87}},$$

$$\text{or } 51.22 \le \mu_2 \le 58.16$$

$$25.13 - \sqrt{8.29}\sqrt{\frac{23.11}{87}} \le \mu_3 \le 25.13 + \sqrt{8.29}\sqrt{\frac{23.11}{87}},$$

$$\text{or } 23.65 \le \mu_3 \le 26.61$$

- The simultaneous T^2-intervals above are wider than univariate intervals because all three must hold with 95% confidence.

P18. Example 4.6, $n = 96, p = 7$. From the result 4.5, simultaneous 90% confidence limits are given by $\bar{x}_i \pm \sqrt{\chi^2_7(.10)}\sqrt{\frac{s_{ii}}{n}}, i = 1, \ldots, 7$ where $\chi^2_7(.10) = 12.02$. Thus, with approximately 90% confidence,

$$28.1 \pm \sqrt{12.02}\frac{5.76}{\sqrt{96}}, \quad \text{contains } \mu_1, \quad \text{or} \quad 26.06 \leq \mu_1 \leq 30.14$$

$$26.6 \pm \sqrt{12.02}\frac{5.85}{\sqrt{96}}, \quad \text{contains } \mu_2, \quad \text{or} \quad 24.53 \leq \mu_2 \leq 28.67$$

$$35.4 \pm \sqrt{12.02}\frac{3.82}{\sqrt{96}}, \quad \text{contains } \mu_3, \quad \text{or} \quad 34.05 \leq \mu_3 \leq 36.75$$

$$34.2 \pm \sqrt{12.02}\frac{5.12}{\sqrt{96}}, \quad \text{contains } \mu_4, \quad \text{or} \quad 32.39 \leq \mu_4 \leq 36.01$$

$$23.6 \pm \sqrt{12.02}\frac{3.76}{\sqrt{96}}, \quad \text{contains } \mu_5, \quad \text{or} \quad 22.27 \leq \mu_5 \leq 24.93$$

$$22.0 \pm \sqrt{12.02}\frac{3.93}{\sqrt{96}}, \quad \text{contains } \mu_6, \quad \text{or} \quad 20.61 \leq \mu_6 \leq 23.39$$

$$22.7 \pm \sqrt{12.02}\frac{4.03}{\sqrt{96}}, \quad \text{contains } \mu_7, \quad \text{or} \quad 21.27 \leq \mu_7 \leq 24.13$$

P23. Example 4.7. The T^2-statistic for testing $H_0 : \boldsymbol{\delta}^T = [\delta_1, \delta_2] = [0, 0]$ is constructed from the difference of paired observation

$$\begin{array}{lccccccccccc} d_{j1} = x_{1j1} - x_{2j1}: & -19, & -22, & -18, & -27, & -4, & -10, & -14, & 17, & 9, & 4, & -19 \\ d_{j2} = x_{1j2} - x_{2j2}: & 12, & 10, & 42, & 15, & -1, & 11, & -4, & 60, & -2, & 10, & -7 \end{array}$$

Then

$$\bar{\mathbf{d}} = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \end{bmatrix} = \begin{bmatrix} -9.36 \\ 13.27 \end{bmatrix}, \quad \mathbf{S}_d = \begin{bmatrix} 199.26 & 88.38 \\ 88.38 & 418.61 \end{bmatrix}$$

and

$$T^2 = 11[-9.36, 13.27] \begin{bmatrix} .0055 & -.0012 \\ -.0012 & .0026 \end{bmatrix} \begin{bmatrix} -9.36 \\ 13.27 \end{bmatrix} = 13.6$$

Taking $\alpha = .05$, we find that $[p(n-1)/(n-p)]F_{p,n-p}(.05) = [2(10)/9]F_{2,9}(0.05) = 9.47 < T^2$. Hence we reject H_0 and conclude that there is a nonzero mean difference between the measurements of the two laboratories.

- Example 4.7 continuous. The 95% simultaneous confidence intervals for the mean differences δ_1 and δ_2 are

$$\delta_1 : \bar{d}_1 \pm \sqrt{\frac{(n-1)p}{n-p} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{d_1}^2}{n}} = -9.36 \pm \sqrt{9.47} \sqrt{\frac{199.26}{11}}$$

$$\text{or } (-22.46, 3.74)$$

$$\delta_2 : 13.27 \pm \sqrt{9.47} \sqrt{\frac{418.61}{11}}, \quad \text{or} \quad (-5.71, 32.25).$$

The 95% simultaneous confidence interval include zero, yet the hypothesis $H_0 : \boldsymbol{\delta} = 0$ was rejected at the 5% level. What are we to conclude ?

P27. Example 4.8. First, note that $\mathbf{S}_1$ and $\mathbf{S}_2$ are approximately equal, so that it is reasonable to pool them. Hence

$$\mathbf{S}_{pooled} = \frac{49}{98}\mathbf{S}_1 + \frac{49}{98}\mathbf{S}_2 = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$$

and

$$\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2 = [-1.9, .2]^T.$$

So the confidence ellipse is centred at $[-1.9, .2]^T$. The eigenvalues and eigenvectors of $\mathbf{S}_{pooled}$ are obtained from the equation

$$0 = |\mathbf{S}_{pooled} - \lambda \mathbf{I}| = \lambda^2 - 7\lambda + 9$$

consequently $\lambda_1 = 5.303$ and $\lambda_2 = 1.697$. The corresponding eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ determined from $\mathbf{S}_{pooled}\mathbf{e}_i = \lambda_i \mathbf{e}_i, i = 1, 2.$

$$\mathbf{e}_1^T = [.290, .957], \text{and} \mathbf{e}_1^T = [.957, -.290]$$

- Example 4.8.continuous. By

$$\left(\frac{1}{n_1}+\frac{1}{n_2}\right)c^2=\left(\frac{1}{50}+\frac{1}{50}\right)\frac{(98)(2)}{(97)}F_{2,97}(.05)=.25$$

since $F_{2,97}(.05)=3.1$. The confidence ellipse extends

$$\sqrt{\lambda_i}\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)c^2}=\sqrt{\lambda_i}\sqrt{.25}$$

unit along the eigenvector $\mathbf{e}_i$, or 1.15 units in the $\mathbf{e}_1$ direction and .65 units in the $\mathbf{e}_2$.

P29. Result 4.9, Type Error: $[(1/n_1+1/n_2)\,\mathbf{S}_{pooled}]$ should be

$$\frac{1}{n_1}\mathbf{S}_1+\frac{1}{n_2}\mathbf{S}_2.$$

P33. Example 4.9. $n_1 = 271, n_2 = 138, n_3 = 107$ and $|\mathbf{S}_1| = 2.783 \times 10^{-8}$, $|\mathbf{S}_2| = 89.539 \times 10^{-8}$, $|\mathbf{S}_3| = 14.579 \times 10^{-8}$, and $|\mathbf{S}_{pooled}| = 17.398 \times 10^{-8}$. Taking the nature logarithms of the determinants gives $\ln|\mathbf{S}_1| = -17.397$, $\ln|\mathbf{S}_2| = -13.926$, $\ln|\mathbf{S}_3| = -15.741$, and $\ln|\mathbf{S}_{pooled}| = -15.564$. Then we calculate

$$\begin{aligned} u &= \left[\frac{1}{270} + \frac{1}{137} + \frac{1}{106} - \frac{1}{270+137+106}\right]\left[\frac{2(4^2)+3(4)-1}{6(4+1)(3-1)}\right] \\ &= 0.0133 \end{aligned}$$

- Example 4.9. Continuous.

$$
\begin{aligned}
M &= [270+137+106](-15.564) - [270(-17.397) \\
&\quad +137(-13.926) + 106(-15.741)] \\
&= 289.3
\end{aligned}
$$

and $C = (1 - .0133)289.3 = 285.5$. Referring C to a χ^2 with the degree of freedom $\nu = 4(4+1)(3-1)/2 = 20$, it is clear that H_0 is rejected at any reasonable level of significance.