Math3806 Lecture Note 6 Appendix

Heng Peng

March 30, 2020

P4.

$$\mathrm{Corr}(U,V) = \frac{\mathbf{a}^T\Sigma_{12}\mathbf{b}}{\sqrt{\mathbf{a}^T\Sigma_{11}\mathbf{a}}\sqrt{\mathbf{b}^T\Sigma_{11}\mathbf{b}}} = \frac{\mathbf{a}^T\Sigma_{11}^{1/2}\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1/2}\Sigma_{22}^{1/2}\mathbf{b}}{\sqrt{\mathbf{a}^T\Sigma_{11}\mathbf{a}}\sqrt{\mathbf{b}^T\Sigma_{11}\mathbf{b}}}$$
$$= \mathbf{a}^*\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1/2}\mathbf{b}^*$$

with $\|\mathbf{a}_*^T\| = \|\mathbf{b}_*^T\| = 1$.
Hence by Cauchy inequality

$$\mathrm{Corr}(U,V) \le \mathbf{b}_*^T\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1/2}\mathbf{b}_* \le \rho_1^{*2}$$

or

$$\mathrm{Corr}(U,V) \le \mathbf{a}_*^T\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}\mathbf{a}_* \le \rho_1^{*2}$$

and

$$\Sigma^{1/2}\mathbf{a} = \mathbf{e}_1, \text{or} \quad \Sigma^{1/2}\mathbf{b} = \mathbf{f}_1$$

- For standardized data $\mathbf{Z}^{(1)} = [Z_1^{(1)}, \ldots, Z_p^{(1)}]^T$ and $\mathbf{Z}^{(2)} = [Z_1^{(2)}, \ldots, Z_q^{(2)}]^T$, the canonical coefficients are unchanged (Why ?)

P6. Example 6-1.1.

$$\boldsymbol{\rho}_{11}^{-1/2} = \begin{bmatrix} 1.0681 & -.2229 \\ -.2229 & 1.0681 \end{bmatrix}, \quad \boldsymbol{\rho}_{22}^{-1} = \begin{bmatrix} 1.0417 & -.2083 \\ -.2083 & 1.0417 \end{bmatrix}$$

and

$$\boldsymbol{\rho}_{11}^{-1/2}\boldsymbol{\rho}_{12}\boldsymbol{\rho}_{22}^{-1}\boldsymbol{\rho}_{21}\boldsymbol{\rho}_{11}^{-1/2} = \begin{bmatrix} .4371 & .2178 \\ .2178 & .1096 \end{bmatrix}$$

The eigenvalues ρ_1^{*2}, ρ_2^{*2} of $\boldsymbol{\rho}_{11}^{-1/2}\boldsymbol{\rho}_{12}\boldsymbol{\rho}_{22}^{-1}\boldsymbol{\rho}_{21}\boldsymbol{\rho}_{11}^{-1/2}$ are obtained from

$$\begin{aligned} 0 &= \begin{vmatrix} .4371 - \lambda & .2178 \\ .2178 & .1096 - \lambda \end{vmatrix} \\ &= (.4371 - \lambda)(.1096 - \lambda) - (2.178)^2 = \lambda^2 - .5467\lambda + .0005 \end{aligned}$$

yielding $\rho_1^{*2} = .5458, \rho_2^{*2} = .0009$. The eigenvector $\mathbf{e}_1$ follows from the vector equation

$$\begin{bmatrix} .4371 & .2178 \\ .2178 & .1096 \end{bmatrix} \mathbf{e}_1 = (.5458)\mathbf{e}_1$$

Thus $\mathbf{e}_1^T = [.8947, .4466]$ and $\mathbf{a}_1 = \boldsymbol{\rho}_{11}^{-1/2}\mathbf{e}_1 = [.8561, .2776]^T$.

- Example 6-1.1. Continuous. By Result 6.1, $\mathbf{f}_1 \propto \boldsymbol{\rho}_{22}^{-1/2}\boldsymbol{\rho}_{21}\boldsymbol{\rho}_{11}^{-1/2}\mathbf{e}_1$ and $\mathbf{b}_1 = \boldsymbol{\rho}_{22}^{-1/2}\mathbf{f}_1$. Consequently,

$$\mathbf{b}_1 \propto \boldsymbol{\rho}_{22}^{-1}\boldsymbol{\rho}_{21}\mathbf{a}_1 = \begin{bmatrix} 1.0417 & -.2083 \\ -.2083 & 1.0417 \end{bmatrix} \begin{bmatrix} .8561 \\ .2776 \end{bmatrix} = \begin{bmatrix} .4026 \\ .5443 \end{bmatrix}$$

Scale $\mathbf{b}_1$ so that

$$\text{Var}(V_1) = \text{Var}(\mathbf{b}_1^T \mathbf{Z}^{(2)}) = \mathbf{b}_1^T \boldsymbol{\rho}_{22}\mathbf{b}_1 = 1$$

This gives

$$[.4026, .5443] \begin{bmatrix} 1.0 & .2 \\ .2 & 1.0 \end{bmatrix} \begin{bmatrix} .4026 \\ .5443 \end{bmatrix} = .5460$$

and $\sqrt{.5460} = .7389$ so

$$\mathbf{b}_1 = \frac{1}{.7389} \begin{bmatrix} .4026 \\ .5443 \end{bmatrix} = \begin{bmatrix} .5448 \\ .7366 \end{bmatrix}$$

- Example 6-1.1. Continuous.
  The first pair of canonical variates is

  $$U_1 = \mathbf{a}_1^T \mathbf{Z}^{(1)} = .86 Z_1^{(1)} + .28 Z_2^{(1)}, \quad V_1 = \mathbf{b}_1^T \mathbf{Z}^{(2)} = .54 Z_1^{(2)} + .74 Z_2^{(2)}$$

  and their canonical correlation is $\rho_1^* = \sqrt{\rho_1^{*2}} = \sqrt{.5458} = .74$.
- The second canonical correlation $\rho_2^* = \sqrt{.0009} = .03$ is very small, and conveys very little information about the association between sets.

P6. Example 6-1.2. With $p = 1$

$$\mathbf{A}_z = [.86, .28], \;\; \mathbf{B}_z = [.54, .74]$$

so

$$\boldsymbol{\rho}_{U_1,\mathbf{Z}^{(1)}} = \mathbf{A}_z\boldsymbol{\rho}_{11} = [.86, .28]\begin{bmatrix} 1.0 & .4 \\ .4 & 1.0 \end{bmatrix} = [.97, .62]$$

and

$$\boldsymbol{\rho}_{V_1,\mathbf{Z}^{(2)}} = \mathbf{B}_z\boldsymbol{\rho}_{22} = [.54, .74]\begin{bmatrix} 1.0 & .2 \\ .2 & 1.0 \end{bmatrix} = [.69, .85]$$

We also obtain the correlations

$$\boldsymbol{\rho}_{U_1,\mathbf{Z}^{(2)}} = \mathbf{A}_z\boldsymbol{\rho}_{12} = [.86, .28]\begin{bmatrix} .5 & .6 \\ .3 & .4 \end{bmatrix} = [.51, .63]$$

and

$$\boldsymbol{\rho}_{V_1,\mathbf{Z}^{(1)}} = \mathbf{B}_z\boldsymbol{\rho}_{21} = [.54, .74]\begin{bmatrix} .5 & .3 \\ .6 & .4 \end{bmatrix} = [.71, .46]$$