MATH 3826 Assignment 1

1. A box contains three marbles: one red, one green, and one blue. Consider an experiment that consists of taking one marble from the box then replacing it in the box and drawing a second marble from the box. What is the sample space? If, at all times, each marble in the box is equally likely to be selected, what is the probability of each point in the sample space?

2. Let E and F be mutually exclusive events in the sample space of an experiment. Suppose that the experiment is repeated until either event E or event F occurs. What does the sample space of this new super experiment look like? Show that the probability that event E occurs before event F is $P(E)/[P(E) + P(F)]$.

3. Two cards are randomly selected from a deck of 52 playing cards.
   (a) What is the probability they constitute a pair (that is, that they are of the same denomination)?
   (b) What is the conditional probability they constitute a pair given that they are of different suits?

4. An individual claims to have extrasensory perception (ESP). As a test, a fair coin is flipped ten times, and he is asked to predict in advance the outcome. Our individual gets seven out of ten correct. What is the probability he would have done at least this well if he had no ESP? (Explain why the relevant probability is $P\{X \geq 7\}$ and not $P\{X = 7\}$.)

5. Let X be binomially distributed with parameters n and p. Show that as k goes from 0 to n,$P(X = k)$ increases monotonically, then decreases monotonically, reaching its largest value
   (a) in the case that $(n + 1)p$ is an integer, when k equals either $(n + 1)p - 1$ or $(n + 1)p$,
   (b) in the case that $(n + 1)p$ is not an integer, when k satisfies $(n + 1)p - 1 < k < (n + 1)p$.

Hint: Consider $P\{X = k\}/P\{X = k-1\}$ and see for what values of k it is greater or less than 1.

6. Let the probability density of X be given by

$$f(x) = \begin{cases} c(4x - 2x^2), & 0 < x < 2 \\ 0, \text{ otherwise} \end{cases}$$

   (a) What is the value of c ?

   (b) $P\{\frac{1}{2} < x < \frac{3}{2}\} =?$

7. Let $X_1, ..., X_{10}$ be independent and identically distributed continuous random variables with distribution function F, and mean $\mu = \mathrm{E}[X_i]$. Let $X_{(1)} < X_{(2)} < \cdots < X_{(10)}$ be the values arranged in increasing order. That is, for $i = 1, \ldots, 10$, $X_{(i)}$ is the ith smallest of $X_1, \ldots, X_{10}$.

   (a) Find $\mathrm{E}[\sum_{i=1}^{10} X_{(i)}]$.

   (b) Let $N = \max\{i : X_{(i)} < x\}$. What is the distribution of N.

   (c) If m is the median of the distribution (that is, if $F(m) = .5$), find $P(X_{(2)} < m < X_{(8)})$.

8. An urn contains three white, six red, and five black balls. Six of these balls are randomly selected from the urn. Let X and Y denote respectively the number of white and black balls selected. Compute the conditional probability mass function of X given that $Y = 3$. Also compute $\mathrm{E}[X|Y = 1]$.

9. Let X be exponential with mean $1/\lambda$; that is, $f_X(x) = \lambda e^{-\lambda x}, 0 < x < \infty$ Find $E[X|X > 1]$.

10. A manuscript is sent to a typing firm consisting of typists A, B, and C. If it is typed by A, then the number of errors made is a Poisson random variable with mean 2.6; if typed by B, then the number of errors is a Poisson random variable with mean 3; and if typed by C, then it is a Poisson random variable with mean 3.4. Let X denote the number of errors in the typed manuscript. Assume that each typist is equally likely to do the work.

    (a) Find $\mathrm{E}[X]$.

    (b) Find $\mathrm{Var}(X)$.