MATH 3826 Assignment 2

1. Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state i,$i = 0, 1, 2, 3$, if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system after the nth step. Explain why $\{X_n, n = 0, 1, 2, \ldots\}$ is a Markov chain and calculate its transition probability matrix.

2. There are k players, with player i having value $v_i > 0, i = 1, \ldots, k$. In every period, two of the players play a game, while the other $k - 2$ wait in an ordered line. The loser of a game joins the end of the line, and the winner then plays a new game against the player who is first in line. Whenever i and j play, i wins with probability $\frac{v_i}{v_i+v_j}$.

3. A Markov chain $\{X_n, n \geq 0\}$ with states 0,1,2, has the transition probability matrix
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$
If $P\{X_0 = 0\} = P\{X_0 = 1\} = \frac{1}{4}$, find $\mathrm{E}[X_3]$.

4. An urn initially contains 2 balls, one of which is red and the other blue. At each stage a ball is randomly selected. If the selected ball is red, then it is replaced with a red ball with probability .7 or with a blue ball with probability .3; if the selected ball is blue, then it is equally likely to be replaced by either a red or blue ball.

   (a) Let X_n equal 1 if the nth ball selected is red, and let it equal 0 otherwise. Is $\{X_n, n \geq 1\}$ a Markov chain? If so, give its transition probability matrix.

   (b) Let Y_n denote the number of red balls in the urn immediately before the nth ball is selected. Is $\{Y_n, n \geq 1\}$ a Markov chain? If so, give its transition probability matrix.

(c) Find the probability that the second ball selected is red.

(d) Find the probability that the fourth ball selected is red.

5. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

$$\mathbf{P}_1 = \begin{Vmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Vmatrix}, \quad \mathbf{P}_2 = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}$$

$$\mathbf{P}_3 = \begin{Vmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{Vmatrix}, \quad \mathbf{P}_4 = \begin{Vmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

6. Show that if state i is recurrent and state i does not communicate with state j, then $P_{ij} = 0$. This implies that once a process enters a recurrent class of states it can never leave that class. For this reason, a recurrent class is often referred to as a closed class.

7. A transition probability matrix $\mathbf{P}$ is said to be doubly stochastic if the sum over each column equals one; that is,

$$\sum_i P_{ij} = 1, \text{for all } j$$

If such a chain is irreducible and consists of $M+1$ states $0, 1, \ldots, M$, show that the long-run proportions are given by

$$\pi_j = \frac{1}{M+1}, j = 0, 1, \ldots, M.$$

8. A DNA nucleotide has any of four values. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain model that supposes that in going from period to period the nucleotide does not change with probability $1 - 3\alpha$, and if it does change then it is equally likely to change to any of the other three values, for some $0 < \alpha < \frac{1}{3}$.

(a) Show that $P_{1,1}^n = \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n$.

(b) What is the long-run proportion of time the chain is in each state?

9. In a good weather year the number of storms is Poisson distributed with mean 1; in a bad year it is Poisson distributed with mean 3. Suppose that any year's weather conditions depends on past years only through the previous year's condition. Suppose that a good year is equally likely to be followed by either a good or a bad year, and that a bad year is twice as likely to be followed by a bad year as by a good year. Suppose that last year—call it year 0—was a good year.

    (a) Find the expected total number of storms in the next two years (that is, in years 1 and 2).

    (b) Find the probability there are no storms in year 3.

    (c) Find the long-run average number of storms per year.

    (d) Find the proportion of years that have no storms.

10. A flea moves around the vertices of a triangle in the following manner: Whenever it is at vertex i it moves to its clockwise neighbor vertex with probability p_i and to the counterclockwise neighbor with probability $q_i = 1 - p_i, i = 1, 2, 3$.

    (a) Find the proportion of time that the flea is at each of the vertices.

    (b) How often does the flea make a counterclockwise move that is then followed by five consecutive clockwise moves?