Lecture Note 1: Appendix

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- **Example 1.** Solution:

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)} = e^{(\lambda_1+\lambda_2)(e^t-1)}$$

Hence, $X + Y$ is Poisson distributed with mean $\lambda_1 + \lambda_2$

- **Example 2.** Solution:

$$\begin{aligned}\phi_{X+Y}(t) &= \phi_X(t)\phi_Y(t) = \exp\left\{\frac{\sigma_1^2 t^2}{2} + \mu_1 t\right\}\exp\left\{\frac{\sigma_2^2 t^2}{2} + \mu_2 t\right\} \\ &= \exp\left\{\frac{(\sigma_1^2+\sigma_2^2)t^2}{2} + (\mu_1+\mu_2)t\right\}\end{aligned}$$

which is the moment generating function of a normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. Hence, the result follows since the moment generating function uniquely determines the distribution.

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- **Example 1.** Solution: Since the binomial is a discrete random variable, and the normal a continuous random variable, it leads to a better approximation to write the desired probability as

$$\begin{aligned} P\{X=20\} &= P\{19.5 < X < 20.5\} \\ &= P\left\{\frac{19.5-20}{\sqrt{10}} < \frac{X-20}{\sqrt{10}} < \frac{20.5-20}{\sqrt{10}}\right\} \\ &= P\left\{-0.16 < \frac{X-20}{\sqrt{10}} < 0.16\right\} \\ &\approx \Phi(0.16) - \Phi(-0.16) = 2\Phi(0.16) - 1 = 0.1272 \end{aligned}$$

The exact result is

$$P\{X=20\} = \begin{pmatrix} 40 \\ 20 \end{pmatrix} \left(\frac{1}{2}\right)^{40} = 0.1254.$$

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- **Example 2.** Solution: Since $\mathrm{E}[X_i] = \frac{1}{2}$, $\mathrm{Var}(X_i) = \frac{1}{12}$ we have by the central limit theorem that

$$\begin{aligned} P\left\{\sum_1^{10} X_i > 7\right\} &= P\left\{\frac{\sum_1^{10} X_i - 5}{\sqrt{10\left(\frac{1}{12}\right)}} > \frac{7-5}{\sqrt{10\left(\frac{1}{12}\right)}}\right\} \\ &\approx 1 - \Phi(2.19) = 0.0143. \end{aligned}$$

- **Example 3.** Solution: If we let X_i denote the lifetime of the ith battery to be put in use, then we desire $p = P\{X_1 + \cdots + X_{25} > 1100\}$, which is approximated as follows:

$$\begin{aligned} p &= P\left\{\frac{X_1 + \cdots + X_{25} - 1000}{20\sqrt{25}} > \frac{1100 - 1000}{20\sqrt{25}}\right\} \\ &\approx P(N(0,1) > 1) = 1 - \Phi(1) \approx 0.1587 \end{aligned}$$

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- **Example 1.** Solution: With $q = 1 - p$

$$
\begin{aligned}
& p\{X_1 = k | X_1 + X_2 = m\} \\
= \quad & \frac{P\{X_1 = k, X_1 + X_2 = m\}}{P\{X_1 + X_2 = m\}} \\
= \quad & \frac{P\{X_1 = k, X_2 = m - k\}}{P\{X_1 + X_2 = m\}} = \frac{P\{X_1 = k\}P\{X_2 = m - k\}}{P\{X_1 + X_2 = m\}} \\
= \quad & \frac{\begin{pmatrix} n_1 \\ k \end{pmatrix} p^k q^{n_1 - k} \begin{pmatrix} n_2 \\ m - k \end{pmatrix} p^{m-k} q^{n_2 - m + k}}{\begin{pmatrix} n_1 + n_2 \\ m \end{pmatrix} p^m q^{n_1 + n_2 - m}}
\end{aligned}
$$

where we have used that $X_1 + X_2$ is a binomial random variable with parameters $(n_1 + n_2, p)$. Thus, the conditional probability mass function of X_1, given that $X_1 + X_2 = m$, is

$$
P\{X_1 = k | X_1 + X_2 = m\} = \frac{\begin{pmatrix} n_1 \\ k \end{pmatrix} \begin{pmatrix} n_2 \\ m - k \end{pmatrix}}{\begin{pmatrix} n_1 + n_2 \\ m \end{pmatrix}}
$$

- **Example 2.** Solution: Let us firstly calculate the conditional probability mass function of X given that $X+Y=n$. We obtain

$$
\begin{aligned}
P\{X=k|X+Y=n\} &= \frac{P\{X=k, X+Y=n\}}{P\{X+Y=n\}} = \frac{P\{X=k, Y=n-k\}}{P\{X+Y=n\}} \\
&= \frac{P\{X=k\}P\{Y=n-k\}}{P\{X+Y=n\}}
\end{aligned}
$$

where the last equality follows from the assumed independence of X and Y. Recalling that $X+Y$ has a Poisson distribution with mean $\lambda_1+\lambda_2$, the preceding equation equals

$$
\begin{aligned}
P\{X=k|X+Y=n\} &= \frac{e^{-\lambda_1}\lambda_1^k}{k!}\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\left[\frac{e^{-(\lambda_1+\lambda-2)}(\lambda_1+\lambda_2)^n}{n!}\right]^{-1} \\
&= \frac{n!}{(n-k)!k!}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} \\
&= \left(\begin{array}{c} n \\ k \end{array}\right)\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}
\end{aligned}
$$

In other words, the conditional distribution of X given that $X+Y=n$ is the binomial distribution with parameters n and $\lambda_1/(\lambda_1+\lambda_2)$. Hence,

$$
\mathrm{E}\{X|X+Y=n\} = n\frac{\lambda_1}{\lambda_1+\lambda_2}.
$$

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- **Example 3.** Solution: We first compute the conditional density

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} = \frac{6xy(2-x-y)}{\int_0^1 6xy(2-x-y)dx} \\ &= \frac{6xy(2-x-y)}{y(4-3y)} = \frac{6x(2-x-y)}{4-3y} \end{aligned}$$

Hence,

$$\mathrm{E}[X|Y=y] = \int_0^1 \frac{6x^2(2-x-y)dx}{4-3y} = \frac{(2-y)2-\frac{6}{4}}{4-3y} = \frac{5-4y}{8-6y}$$

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- **Example 4.** Solution: To begin, let us first note that if $f(x,y)$ is the joint density of X, Y, then the joint density of X and $X+Y$ is

$$f_{X,X+Y}(X,t) = f(x, t-x)$$

which is easily seen by noting that the Jacobian of the transformation

$$g_1(x,y) = x, g_2(x,y) = x+y$$

is equal to 1. Hence

$$\begin{aligned} f_{X_1|X_1+X_2(x|t)} &= \frac{f_{X_1,X_1+X_2}(x,t)}{f_{X_1+X_2}(t)} \\ &= \frac{\mu_1 e^{-\mu_1 x}\mu_2 e^{-\mu_2(t-x)}}{f_{X_1+X_2}(t)} = Ce^{(\mu_1-\mu_2)x}, 0 \leq x \leq t \end{aligned}$$

where $C = \frac{\mu_1\mu_2 e^{-\mu_2 t}}{f_{X_1+X_2}(t)}$.

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- **Example 4.** Solution continuous:
  Now if $\mu_1 = \mu_2$, then $f_{X_1|X_1+X_2}(x|t) = C, 0 \leq x \leq t$ yield that $C = 1/t$, and that X_1 given $X_1 + X_2 = t$ is uniformly distributed on $(0, t)$.

  On the other hand, if $\mu_1 \neq \mu_2$, then we use

$$1 = \int_0^t f_{X_1|X_1+X_2}(x|t)dx = \frac{C}{\mu_1 - \mu_2}\left(1 - e^{-(\mu_1-\mu_2)t}\right)$$

  to obtain

$$C = \frac{\mu_1 - \mu_2}{1 - e^{-(\mu_1-\mu_2)t}}$$

  thus yield the result:

$$f_{X_1|X_1+X_2}(x|t) = \frac{(\mu_1 - \mu_2)e^{-(\mu_1-\mu_2)x}}{1 - e^{-(\mu_1-\mu_2)t}}$$

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- **Example 1.** Solution: Let X be the number of misprints. Because it would be easy to compute $\mathrm{E}[X]$ if we know which book Sam chooses, let

$$Y = \begin{cases} 1, & \text{if Sam choose his history book} \\ 2, & \text{if chooses his probability book} \end{cases}$$

Conditioning on Y yields

$$\mathrm{E}[X] = \mathrm{E}[X|Y=1]P(Y=1) + \mathrm{E}[X|Y=2]P(Y=2) = 5\frac{1}{2} + 2\frac{1}{2} = \frac{7}{2}$$

- **Example 2.** Solution: Letting N denote the number of accidents and X_i the number injured in the ith accident, $i = 1, 2, \ldots$, then the total number of injuries can be expressed as $\sum_{i=1}^{N} X_i$. Hence, we need to compute the expected value of the sum of a random number of random variables. Because it is easy to compute the expected value of the sum of a fixed number of random variables, let us try conditioning on N.

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- **Example 2.** Solution continuous: This yields

$$\mathrm{E}\left[\sum_1^N X_i\right] = \mathrm{E}\left[\mathrm{E}\left[\sum_1^N X_i|N\right]\right]$$

But

$$\mathrm{E}\left[\sum_1^N X_i|N=n\right] = \mathrm{E}\left[\sum_1^n X_i\right] = n\mathrm{E}[X]$$

So

$$\mathrm{E}\left[\sum_1^N X_i|N\right] = N\mathrm{E}[X]$$

and

$$\mathrm{E}\left[\sum_1^N X_i\right] = \mathrm{E}[N\mathrm{E}[X]] = \mathrm{E}[N]\mathrm{E}[X]$$

Therefore, in our example, the expected number of injuries during a week equals $4 \times 2 = 8$.

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- **Example 3.** Solution: Whereas we could obtain $\mathrm{E}[S]$ by conditioning on N, let us instead use the conditional variance formula. Now

$$\begin{aligned}\mathsf{Var}(S|N=n) &= \mathsf{Var}\left(\sum_{i=1}^{N} X_i|N=n\right) = \mathsf{Var}\left(\sum_{i=1}^{n} X_i|N=n\right)\\ &= \mathsf{Var}\left(\sum_{i=1}^{n} X_i\right) = n\sigma^2\end{aligned}$$

By the same reasoning

$$\mathrm{E}[S|N=n] = n\mu.$$

Therefore

$$\mathsf{Var}(S|N) = N\sigma^2, \mathrm{E}[S|N] = N\mu$$

and the conditional variance formula gives

$$\mathsf{Var}(S) = \mathrm{E}[N\sigma^2] + \mathsf{Var}(N\mu) = \sigma^2 E[N] + \mu^2\mathsf{Var}(N)$$

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- **Example 1.** Solution: Conditioning on the value of Y yields

$$
\begin{aligned}
P(X < Y) &= \int_{-\infty}^{\infty} P(X < Y | Y = y) f_Y(y) dy \\
&= \int_{-\infty}^{\infty} P(X < y | Y = y) f_Y(y) dy \\
&= \int_{-\infty}^{\infty} P(X < y) f_Y(y) dy = \int_{-\infty}^{\infty} F_X(y) f_Y(y) dy
\end{aligned}
$$

where $F_X(y) = \int_{-\infty}^{y} f_X(x) dx$.

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- **Example 2.** Solution: Let X denote the number of accidents that a randomly chosen policyholder has next year. Letting Y be the Poisson mean number of accidents for this policyholder, then conditioning on Y yields

$$\begin{aligned} P(X = n) &= \int_0^\infty P(X = n | Y = \lambda) g(\lambda) d\lambda) = \int_0^\infty e^{-\lambda} \frac{\lambda^n}{n!} \lambda e^{-\lambda} d\lambda \\ &= \frac{1}{n!} \int_0^\infty \lambda^{n+1} e^{-2\lambda} d\lambda \end{aligned}$$

However, because

$$h(\lambda) = \frac{2e^{-2\lambda}(2\lambda)^{n+1}}{(n+1)!}, \lambda > 0$$

is the density function of a gamma $(n+2, 2)$ random variable, its integral is 1. Therefore,

$$1 = \int_0^\infty \frac{2e^{-2\lambda}(2\lambda)^{n+1}}{(n+1)!} d\lambda = \frac{2^{n+2}}{(n+1)!} \int_0^\infty \lambda^{n+1} e^{-2\lambda} d\lambda$$

showing that

$$P(X = n) = \frac{n+1}{2^{n+2}}.$$

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- **Example 3.** Let $P_{n,m}$ denote the desired probability. By conditioning on which candidate receives the last vote counted we have

$$\begin{aligned} P_{n,m} &= P(\text{A always ahead}|\text{A receives last vote})\frac{n}{n+m} \\ &\quad + P(\text{A always ahead}|\text{B receives last vote})\frac{m}{n+m} \end{aligned}$$

Now, given that A receives the last vote, we can see that the probability that A is always ahead is the same as if A had received a total of $n-1$ and B a total of m votes. Because a similar result is true when we are given that B receives the last vote, we see from the preceding that

$$P_{n,m} = \frac{n}{n+m}P_{n-1,m} + \frac{m}{n+m}P_{n,m-1}$$

We can now prove that $P_{n,m} = (n-m)/(n+m)$ by induction on $n+m$. As it is true when $n+m=1$, that is, $P_{1,0}=1$, assume it whenever $n+m=k$. Then when $n+m=k+1$, we have by the equation above and the induction hypothesis that

$$P_{n,m} = \frac{n}{n+m}\frac{n-1-m}{n-1+m} + \frac{m}{n+m}\frac{n-m+1}{n+m-1} = \frac{n-m}{n+m}$$

and the result is proven.