Chapter 1. Theory of Interest

1. The measurement of interest

1.1 Introduction

Interest may be defined as the compensation that a borrower of capital pays to lender of capital for its use. Thus, interest can be viewed as a form of rent that the borrower pays to the lender to compensate for the loss of use of capital by the lender while it is loaded to the borrower. In theory, capital and interest need not be expressed in terms of the same commodity.

Outline

A. **Effective rates** of interest and discount;

B. **Present value**;

C. **Force** of interest and discount;

1.2. The accumulation and amount functions

Principal: The initial of money (capita) invested;

Accumulate value: The total amount received after a period of time;

Amount of Interest: The difference between the accumulated value and the principle.

Measurement period: The unit in which time is measured.

Accumulation function $a(t)$: The function gives the accumulated value at time $t \geq 0$ of an original investment of 1.

1. $a(0) = 1$.

2. $a(t)$ is generally an increasing function if the interest is not negative.

3. $a(t)$ will be continuous if interest accrues continuously.

Amount function $A(t)$: The accumulated value at time $t \geq$ of an original investment of k.

$$A(t) = k \cdot a(t) \text{ and } A(0) = k$$

Amount of Interest I_n earned during the nth period from the date of invest:

$$I_n = A(n) - A(n-1), n \geq 1.$$

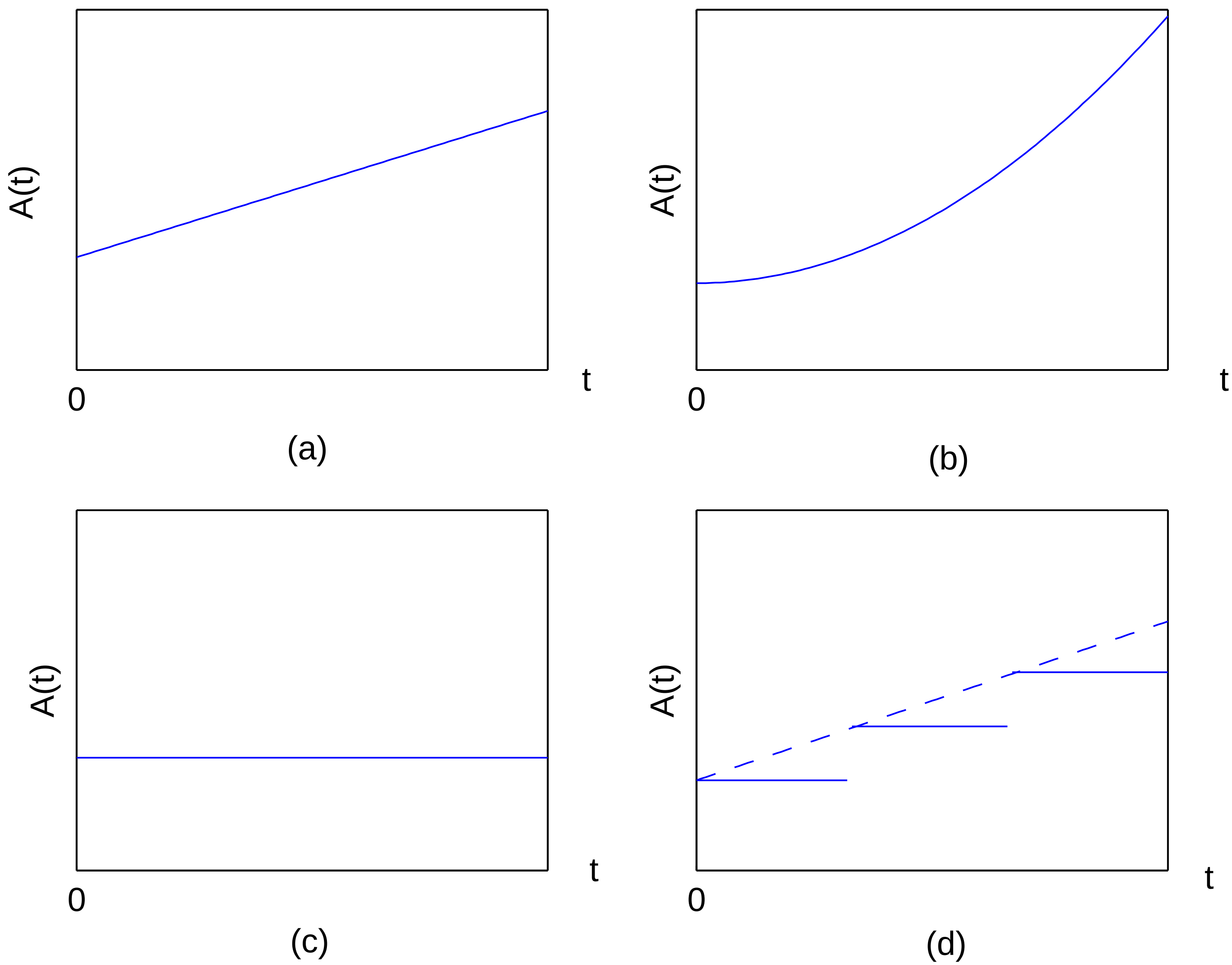

Figure 1: Four illustrative amount functions

1.3. The effective rate of interest

Precise definition: The effective rate of interest i is the amount of money that one unit invested at the beginning of a period will earning during the period, where interest is paid at the end of the period.

This definition is equivalent to

$$i = a(1) - a(0) \text{ or } a(1) = 1 + i.$$

Alternative definition:

$$i = \frac{(1+i) - 1}{1} = \frac{a(1) - a(0)}{a(0)} = \frac{A(1) - A(0)}{A(0)} = \frac{I_1}{A(0)}.$$

Let i_n be the effective rate of interest during the nth period from the date of investment. Then we have

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)}, \; n \geq 1.$$

- The use of the word "**effective**" is not intuitively clear.
- The effective rate of interest is often expressed as a **percentage**, e.g. $i = 8\%$.
- The amount of principal remains **constant** throughout the period.
- The effective rate of interest is a measure in which interest is paid at **end of period**.

1.4. Simple Interest and Compound Interest

Simple Interest: The accruing of interest according to the following patter is called simple interest.

$$a(t) = 1 + it \quad \text{for integral } t \geq 0.$$

Let i be the rate of simple interest and let i_n be the effective rate of interest for the nth period. Then we have

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{[1 + in] - [1 + i(n-1)]}{1 + i(n-1)} = \frac{i}{1 + i(n-1)}$$

for integral $n \geq 1$.

A more **rigorous mathematical approach** for simple interest to the definition of $a(t)$ for nonintegral value of t:

$$a(t+s) = a(t) + a(s) - 1 \quad \text{for } t \geq 0 \text{ and } s \geq 0.$$

Assuming $a(t)$ is differentiable, we have

$$\begin{aligned} a'(t) &= \lim_{s \to 0} \frac{a(t+s) - a(t)}{s} \\ &= \lim_{s \to 0} \frac{[a(t) + a(s) - 1] - a(t)}{s} \\ &= \lim_{s \to 0} \frac{a(s) - 1}{s} = \lim_{s \to 0} \frac{a(s) - a(0)}{s} \\ &= a'(0) \end{aligned}$$

So $a'(t)$ is a constant, we have

$$a(t) - a(0) = \int_0^t a'(r)dr = \int_0^t a'(0)dr = t \cdot a'(0)$$

$$a(t) = 1 + t \cdot a'(0)$$

Remember that

$$a(1) = 1 + i = 1 + a'(0),$$

so

$$a(t) = 1 + it, t \geq 0.$$

Compound Interest: The word "Compound" refers to the process of interest being reinvested to earn additional interest. The theory of compound interest handles the problem by assuming that interest earned is automatically reinvested.

Accumulation function for a constant compound interest:

$$a(t) = (1+i)^t \quad \text{for integral } t \geq 0.$$

Effective interest

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = \frac{1+i-1}{1} = i.$$

which is independent of n.

Rigorous mathematical approach for compound interest to the definition of $a(t)$ for nonintegral value of t:

$$a(t+s) = a(t) \cdot a(s) \quad \text{for } t \geq 0 \text{ and } s \geq 0$$

Assuming $a(t)$ is differentiable, we have

$$\begin{aligned} a'(t) &= \lim_{s \to 0} \frac{a(t+s) - a(t)}{s} \\ &= \lim_{s \to 0} \frac{a(t) \cdot a(s) - a(t)}{s} \\ &= a(t) \lim_{s \to 0} \frac{a(s) - 1}{s} \\ &= a(t) \cdot a'(0). \end{aligned}$$

Thus

$$\frac{a'(t)}{a(t)} = \frac{d}{dt}\log_e a(t) = a'(0).$$

and

$$\log_e a(t) - \log_e a(0) = \int_0^t \frac{d}{dr}\log_e a(r)dr = \int_0^t a'(0)dr = t \cdot a'(0)$$

Since $\log_e a(0) = 0$, if we let $t = 1$ and remember that $a(1) = 1+i$, we have

$$\log_e a(1) = \log_e(1+i) = a'(0)$$

and

$$\log_e a(t) = t\log_e(1+i) = \log_e(1+i)^t \text{ or } a(t) = (1+i)^t \text{ for } t \geq 0.$$

Difference between simple and compound interest:

- Same results over one measurement period,. Over a longer period, compound interest produces a larger accumulated value than simple interest while the opposite is true over a shorter period.

- Under simple interest, it is the *absolute amount* of growth that is constant over equal periods of time, while under compound interest, it is the relative rate of growth that is constant.

- Compound interest is used almost exclusively for financial transaction covering a period of one year or more and is often used for shorter term transaction as well. Simple interest is occasionally used for short-term transaction and as an approximation for compound interest over fractional periods.

Example 1: Find the accumulated value of \$2000 invest for four years, if the rate of simple interest and compound rate is 8% annum respectively.

The answer is

For simple interest: $2000[1 + (.08)(4)] = 2640.$

For compound interest: $2000(1.08)^4 = 2720.98.$

1.5. Present Value

- **Accumulation factor:** $1+i$. It accumulates the value of an investment at the beginning of a period to its value at end of the period.

- **Discount factor:** $v = \frac{1}{1+i}$, it discounts the value of an investment at end of a period to its value at the beginning of the period.

- **Discount function:** $a^{-1}(t)$, since $a^{-1}(t) \cdot a(t) = 1$.

$$\text{Simple interest:} \quad a^{-1}(t) = \frac{1}{1+it}$$

$$\text{Compound interest:} \quad a^{-1}(t) = \frac{1}{(1+i)^t} = v^t$$

Accumulating and discounting are opposite processes. The term $(1+i)^t$ is said to be the ***accumulated value*** of 1 at the end of t period. The term v^t is said to ***present value*** of 1 to be paid at the end of t periods.

Example 2: Find the amount which must be invested at a rate of simple interest of 9% per annum in order to accumulate $ 1000 at end of three years.

The answer is $\frac{1000}{1+(.09)(3)} = \frac{1000}{1.27} = \$787.40.$

Example 3: Rework the example above using compound interest instead of simple interest.

The answer is $1000v^3 = \frac{1000}{(1.09)^3} = \$772.18.$

1.6. The effective rate of discount

Numerical illustration:

If A goes to a bank and borrow \$100 for one year at an effective rate of interest of 6%, then the bank will give A \$100. At the end of the year, A will repay the bank the original loan of \$100, plus interest \$6 or a total of \$106.

However, if A borrows \$ 100 for one year at an effective rate of discount of 6%. then the bank will collect its interest of 6% in advance and will give A only \$94. At end of the year, A will repay \$100.

In the case of an effective rate of interest, the 6% is taken as a percentage of the balance at the beginning of the year, while in the case of an effective rate of discount , the 6% is taken as percentage of the balance at the end of the year.

Definition of the effective rate of discount: The effective rate of discount d is the ratio of the amount of interest (sometimes called the "amount of discount" or just "discount") earned during the period to the amount invested at end of the period.

- The phrases amount of discount and amount of interest can be used interchangeably in situations involving rates of discount.

- The definition does not use the word "principal", since the definition of principal refers to the amount invest at the beginning of the period and not at the end of the period.

The key distinction between the effective rate of interest and the effective rate of discount can be summarized as follows:

a) Interest-paid at end of the period on the balance at the beginning of the period.

b) Discount-paid at the beginning of the period on the balance at the end of the period.

Effective rates d_n of discount over any **particular measurement period**:

$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{I_n}{A_n} \quad \text{for integral } n \geq 1.$$

Compound discount: If we have compound interest, in which case the effective rate of interest is constant, then the effective rate of discount is also constant. This situations are referred to as *compound discount*.

Relationship between effective rates of interest and effective rates of discount

Concept of equivalency: Two rates of interest or discount are said to be equivalent if a given amount of principal invested for the same length of time at each of the rates produces the same accumulated value.

Assume that a person borrows 1 at an effective rate of discount d. Then the effective rate of interest is

$$i = \frac{d}{1-d}$$

By simple algebra, we also get

$$d = \frac{i}{1+i},$$

$$d = iv, d = 1 - v, \text{and } i - d = id$$

- **Simple discount:** $a^{-1}(t) = 1 - dt$.
- **Compound discount:** $a^{-1}(t) = v^t = (1-d)^t$.
- A constant rate of simple discount implies an increasing effective rate of discount(and interest).

- Simple and compound discount produce the same result over one measument period. Over a longer period, simple discount produce a smaller present value than compound discount, while the opposite is true over a shorter period.

- Simple discount is used only for short-term transactions and as an approximation for compound discount over fractional periods.

Example 4 Rework Example 2 and Example 3 using simple and compound discount instead of simple and compound interest.
The answer is
For simple discount: $1000[1-(0.9)(3)] = \$730.$
For compound discount: $1000(0.91)^3 = \$753.57.$

1.7. Nominal Rates of Interest and Discount

Effective: used for the rates of interest and discount in which interest is paid once per measurement period.

Nominal: Rates of interest and discount in which interest is paid more frequently than once per measurement period.

Nominal rate of interest $i^{(m)}$: payable m times per period. The effective rate of interest is $i^{(m)}/m$ for each *m*th of a period.

Relationship between the nominal rate and effective rate of interest

$$1 + i = \left[1 + \frac{i^{(m)}}{m}\right]^m$$

$$i = \left[1 + \frac{i^{(m)}}{m}\right]^m - 1 \quad \text{and} \quad i^{(m)} = m[(1+i)^{\frac{1}{m}} - 1].$$

Nominal rate of discount $d^{(m)}$: payable m times per period. The effective rate of discount is $d^{(m)}/m$ for each *m*th of a period.

Relationship between the nominal rate and effective rate of discount

$$1 - d = \left[1 - \frac{d^{(m)}}{m}\right]^m$$

$$d = 1 - \left[1 - \frac{d^{(m)}}{m}\right]^m$$ and $$i^{(m)} = m[1 - (1-d)^{\frac{1}{m}}] = m[1 - v^{\frac{1}{m}}].$$

Relationship between nominal rates of interest and nominal rates of discount

$$\left[1+\frac{i^{(m)}}{m}\right]^{m}=\left[1-\frac{d^{(p)}}{p}\right]^{-p},$$

$$\frac{i^{(m)}}{m}-\frac{d^{(m)}}{m}=\frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m}.$$

Example 5 Find the accumulated value of \$500 invested for five years at 8% annum convertible quarterly
The answer is

$$500\left[1+\frac{.08}{4}\right]^{4\cdot 5}=500(1.02)^{20}$$

It should be noted that this situation is equivalent to one in which \$500 is invested at a rate of interest of 2% for 20 years.

Example 6 Find the present value of $1000 to be paid at end of six years at 6% per annum payable in advance and convertible semiannually.
The answer is

$$1000\left[1-\frac{.06}{2}\right]^{2\cdot 6}=1000(.97)^{12}$$

It should be noted that this situation is equivalent to one in which the present value of $1000 to be paid at the end of 12 years is calculated at a rate of discount of 3%.

Example 7 Find the nominal rate of interest convertible quarterly which is equivalent to a nominal rate of discount of 6% per annum convertible monthly.
The answer is

$$\left[1+\frac{i^{(4)}}{4}\right]^4=\left[1-\frac{.06}{12}\right]^{-12}$$

$$1+\frac{i^{(4)}}{4}=(.995)^{-3}, i^{(4)}=4[(.995)^{-3}-1].$$

1.8. Forces of Interest and Discount

Force of interest The measure of interest at individual moments of time. The force of interest at time t, denoted by δ_t is defined as

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{a'(t)}{a(t)}$$

- δ_t is measure of the intensity of interest at exact time t, independent of the amount in the fund.

- δ_t expresses this measurement as a rate of per measurement period.

Expressions for the value of A(t) and a(t) in terms of the function δ_t

$$\delta_t = \frac{d}{dt} \log_e A(t) = \frac{d}{dt} \log_e a(t).$$

$$\int_0^t \delta_r dr = \int_0^t \frac{d}{dr} \log_e A(r) dr = \log_e \frac{A(t)}{A(0)}$$

$$e^{\int_0^t \delta_r dr} = \frac{A(t)}{A(0)} = \frac{a(t)}{a(0)} = a(t)$$

$$\int_0^n A(t)\delta_t dt = \int_0^n A'(t) dt = A(n) - A(0).$$

Force of discount δ'_t

$$\delta'_t = -\frac{\frac{d}{dt}a^{-1}(t)}{a^{-1}(t)}.$$

The force of discount bears a relationship to nominal and effective rates of discount similar to the relationship that force of itnerest bears to norminal and effective rate interest. By simple computation, We have

$$\delta'_t = -\frac{\frac{d}{dt}a^{-1}(t)}{a^{-1}(t)} = \frac{a^{-2}(t)\frac{d}{dt}a(t)}{a^{-1}(t)} = \frac{a^{-2}(t)a'(t)}{a^{-1}(t)} = \delta_t.$$

When force of interest is constant δ over an interval of time, we have

- The effective rate of interest will also be constant over that interval for compound interest,

$$e^{\int_0^n \delta_t dt} = e^{n\delta} = a(n) = (1+i)^n, i = e^{\delta} - 1 \text{ and } \delta = \log_e(1+i).$$

- For simple interest, we have

$$\delta_t = \frac{\frac{d}{dt}(1+it)}{1+it} = \frac{i}{1+it}, t \geq 0.$$

and

$$\delta_t = \delta_t' = -\frac{\frac{d}{dt}(1-dt)}{1-dt} = \frac{d}{1-dt}, 0 \leq t < 1/d.$$

$$\left[1+\frac{i^{(m)}}{m}\right]^m = 1+i = v^{-1} = (1-d)^{-1} = \left[1-\frac{d^{(p)}}{p}\right]^{-p} = e^{\delta}$$

$$i^{(m)} = m(e^{\frac{\delta}{m}} - 1).$$

By series expansion

$$i^{(m)} = m\left[\frac{\delta}{m} + \frac{1}{2!}\left[\frac{\delta}{m}\right]^2 + \frac{1}{3!}\left[\frac{\delta}{m}\right]^3 + \cdots\right]$$

and let $m \to \infty$ we have

$$\lim_{m\to\infty} i^{(m)} = \delta.$$

Similarly, we also have

$$\lim_{m\to\infty} d^{(m)} = \delta.$$

Example 8 Find the accumulated value of \$1000 invested for ten years if the force of interest is 5%.
The answer is

$$1000e^{(.05)(10)} = 1000e^{.5}.$$

1.9. Varying Interest

- Continuously varying force of interest.

$$a(t) = e^{\int_0^t \delta_r dr}.$$

- Changes in the effective rate of interest over a period of time. Let it_n be the effective rate of interest during the nth period from the date of investment. Then we have

$$a(t) = (1 + i_1)(1 + i_2) \cdots (1 + i_t) = \prod_{k=1}^{t} (1 + i_k),$$

$$a^{-1}(t) = (1+i_1)^{-1}(1+i_2)^{-1} \cdots (1+i_t)^{-1} = \prod_{k=1}^{t} (1+i_k)^{-1} = \prod_{k=1}^{t} v_k.$$

Example 9 Find the accumulated value of 1 at the end of n years if $\delta_t = \frac{1}{1+t}$.

The answer is

$$e^{\int_0^n \delta_t dt} = e^{\int_0^n \frac{1}{1+t} dt} = e^{\log_e(1+t)|_0^n} = 1 + n.$$

Example 10 Find the accumulated value of \$1000 at the end of 15 years if the effective rate of interest is 5% for first 5 year, 4 1/2% for the second 5 years, and 4% for the third 5 years.

The answer is $1000(1.05)^5(1.045)^5(1.04)^5$.

Table 1: **Summary of Relationships in Chapter 1**

Rate of interest or discount	The accumulated value of 1 at time t, $a(t)$	The present value of 1 at time t, $a^{-1}(t)$
Compound interest		
i	$(1+i)^t$	$v^t = (1+i)^{-t}$
$i^{(m)}$	$\left[1+\frac{i^{(m)}}{m}\right]^{mt}$	$\left[1+\frac{i^{(m)}}{m}\right]^{-mt}$
d	$(1-d)^{-t}$	$(1-d)^t$
$d^{(m)}$	$\left[1-\frac{d^{(m)}}{m}\right]^{-mt}$	$\left[1-\frac{d^{(m)}}{m}\right]^{mt}$
δ	$e^{\delta t}$	$e^{-\delta t}$
Simple interest		
i	$1+it$	$(1+it)^{-1}$
Simple discount		
d	$(1-dt)^{-1}$	$(1-dt)$

2. Solution of problems in interest

2.1. Introduction

This Chapter discusses general principles to be followed in the solution of problem interest. The purpose of this chapter is to develop a systematic approach by which the basic principles from Chapter 1 can be applied to more complex financial transaction.

2.2. Obtaining Numerical Results

Naturally , in practice work, actual numerical answers are usually desired, and the purpose of this section is to discuss the various the various possible methods of obtains such answer.

Direct calculation by Personal computers, inexpensive pocket calculators with exponential and logarithmic functions.

Compound interest tables Use of the compound interest tables is a convenient approach if required values appear in the tables.

Direct calculation by hand. This may require the us e of series expansions. Here are two examples: One example wound be to evaluate $(1+i)^k$ using the binomial expansion theorem.

$$(1+i)^k = 1 + ki + \frac{k(k-1)}{2!}i^2 + \frac{k(k-1)(k-2)}{3!}i^3 + \cdots,$$

A second example would be to evaluate $e^{k\delta}$ as

$$e^{k\delta} = 1 + k\delta + \frac{(k\delta)^2}{2!} + \frac{(k\delta)^3}{3!}i^3 + \cdots$$

It should be emphasized that using series expansions for calculation purposes is cumbersome and should be unnecessary except in unusual circumstances.

One method of **crediting interest**

- Using compound interest for integral periods of time .
- Using simple interest for any fractional period.
- Using first two terms of the binomial expansion assuming $0 < k < 1$.
- Such method is commonly encountered in practice.

Simple interest for a final fractional period is equivalent to performing a linear interpolation between $(1+i)^n$ and $(1+i)^{n+1}$

$$\begin{aligned}(1+i)^{n+k} &\approx (1-k)(1+i)^n + k(1+i)^{n+1} \\ &= (1+i)^n(1+ki)\end{aligned}$$

Analogous fashion for simple discount over the final fractional period by linear interpolation

$$\begin{aligned}v^{n+k} = (1-d)^{n+k} &\approx (1-k)(1-d)^n + k(1-d)^{n+1} \\ &= (1-d)^n(1-kd).\end{aligned}$$

Example 2.1 Find the accumulated value of \$5000 at the end of 30 years and 4 months at 6% er annum convertible semiannually: (1) assuming compound interest throughout, and (2) assuming simple interest during the final fractional period.

1. Assuming compound interest throughout, the answer is

$$5000(1.03)^{60\frac{2}{3}} = \$30,044.27$$

by direct calculation.

2. Assuming simple interest during the final fractional period, the answer is

$$5000(1.03)^{60}(1.02) = \$30,047.18.$$

The answer to 2 is larger that 1, illustrating that simple interest produces larger accumulated values over fractional periods than compound interest does, although the difference is quite small.

2.3. Determining Time periods

Although there would appear to be no ambiguity in this process, different methods of counting the days in a period of investment have arisen in practice.

Three methods are commonly encountered.

1. Using exact number of days for the period of investment and to use 365 days in a year.

2. Assuming each calendar month has 30 days and the entire calendar year has 360 days. Formula for computing the number of days between two given days is

$$360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1)$$

3. A hybrid method, uses exact number of days for period of investment, but uses 360 days in a year.

For **simple interest**, those method are called

1. Exact simple interest, “actal/actal”.

2. Ordinary simple interest, “$30/360$”

3. Banker’s rule, “actual/360”

- Banker's Rule is more favorable to a lender than is ordinary interest.

- A further complication arises in a leap year. In most cases, Feb. 29 is counted a day and the year has 366 days.

- The three commonly encountered calculation bases also are used for calculations on a compound interest basis.

- It is assumed, unless stated otherwise, that in counting days interest is not credited for both the date of deposit and the date of withdrawal, but for only one of these two dates.

- Many financial transaction are handled on a monthly , quarterly, semiannual or annual basis.

Example 2. Find the amount of interest that \$2000 deposited on June 17 will earn, if the money is withdrawn on September 10 in the same year and if the rate of interest is 8%, on the following bases: (1) exact simple interest, (2) ordinary simple interest, (3) the Banker's Rule.

1. September 10 is day 253 and June 17 is day 168. The actual number of days in the period of investment is $253 - 168 = 85$. Thus the answer is

$$2000(0.08)(85/365) = \$37.26$$

. Assuming that the year in question is not a leap year.

2. Using the formula above to compute the number of days is

$$360(0) + 30(9 - 6) + (10 - 17) = 83.$$

Thus, the answer is

$$2000(0.08)(83/360) = \$36.89.$$

3. The answer is

$$2000(0.08)(85/360) = \$37.78.$$

Not surprising, the answer using the **Banker's Rule** is greater than using either **exact simple interest** or **ordinary simple interest**.

2.4*. *The Basic Problem*

- The principal originally invested.
- The length of the investment period.
- The rate of interest.
- The accumulated value of the principle at the end of the investment period.

IF any three of these quantites are known, then the fourth quantity can be determined.

The following observations may prove helpful in the solution of problems interest

- Assessing the tools that will be available in performing the financial calculations, such as interest tables, pocket calculators, personal computers.
- The length of the investment period is measured in time units.
- An interest problem can be viewed from two perspectives, from borrower and lender.
- In practice applications involving interest the terminology can become confusing.

Equations of Value

Fundamental Principle, ***Recognition of the Time value of money:*** The value of an amount of money at any given point in time depends upon the time elapsed since the money was paid in the past or upon time which will elapse in the future before it paid.

Recognition of the time value of money reflects the effect of interest, but not the effect of inflation which reduces the purchasing power of money over time.

Two or more amount of money payable at a different points in time cannot be compared until all the amount are accumulated or discounted to a common date (**Comparison Date**), and the equation which accumulates or discounts each payment to the comparison date is called the **equation of value**.

Example 4 In return for a promise to receive \$600 at the end of 8 years, a person agrees to pay \$100 at once, \$200 at end of 5 years and to make a further payment at end of 10 years. Find the payment at end of 10 years if the nominal rate of interest is 8% convertible semiannually.

Since interest is convertible semiannually, we will count time period in half-years. There are two equation from different comparison date.

$$100 + 200v^{10} + Xv^{20} = 600v^{16} \quad \text{at } 4\%$$

$$100(1.04)^{20} + 200(1.04)^{10} + X = 600(1.04)^4.$$

Same answer is obtained by these two equations, $X = 186.76$.

Unknown Time

Let amount $s_1, s_2, \ldots, s_n$ be paid at times $t_1, t_2, \ldots, t_n$ respectively. The problem is to find time t such that $s_1+s_2+\cdots+s_n$ paid at time t is equivalent to the payments of $s_1, s_2, \ldots, s_n$ made separately.

The fundamental equation of value is

$$(s_1 + s_2 + \cdots + s_n)v^t = s_1 v^{t_1} + s_2 v^{t_2} + \cdots + s_n v^{t_n}$$

A first approximation, the method of equated time

$$\bar{t} = \frac{s_1 t_1 + s_2 t_2 + \ldots + s_n t_n}{s_1 + s_2 + \cdots + s_n}$$

It is possible to prove that the value of $\bar{t}$ is always great than the true value of t.

Another interesting question often asked is how long it takes money to double at a given rate of interest.

$$(1+i)^n = 2 \quad \text{or} \quad n \log_e(1+i) = \log_e 2.$$

Then we have

$$n = \frac{\log_e 2}{\log_e(1+i)} = \frac{.6931}{i} \cdot \frac{i}{\log_e(1+i)}.$$

Rule of 72

The last factor above equation evaluated for $i = 8\%$ is 1.0395. Thus we have

$$n \approx \frac{.6831}{i}(1.0395) = \frac{.72}{i}.$$

Table 2: Length of Time It Takes Money Double

Rate of interest	Rule of 72	Exact value
4 %	18	17.67
6	12	11.90
8	9	9.01
10	7.2	7.27
12	6	6.12
18	4	4.19

Example 5 Find the length of time necessary for \$1000 to accumulate to \$1500 if invested at 6% per annum compounded semiannually: (1) by use of logarithms, and (2) by interpolating in the interest tables.

1. $$\log_e 1000(1.03)^n = \log_e 1500 \Rightarrow n = \frac{\log_e 1.5}{\log_e 1.03} = 13.717.$$

Thus, the number of years is 6.859.

2. From the interest tables, $(1.03)^{13} = 1.46853$ and $(1.03)^{14} = 1.51259$. so $13 < n < 14$. Then by a linear interpolation

$$n = 13 + \frac{1.5 - 1.46853}{1.51259 - 1.46853} = 13.714.$$

Thus, the number of years is 6.857. This method is equivalent to the assumption of simple interest during the final fraction of an interest conversion period.

Example 6 Payments of \$100, \$200, and \$500 are due at the ends of years 2,3 and 8, respectively. Assuming an effective rate of interest of 5% per annum, find the point in time at which a payment of \$800 would be equivalent: (1) by the method of equated time, and (2) by an exact method.

1. By the method of equated time,

$$\bar{t} = \frac{100 \cdot 2 + 200 \cdot 3 + +500 \cdot 8}{100 + 200 + 500} = 6 \text{ years.}$$

2. The exact equation of value is

$$800v^t = 100v^2 + 200v^3 + 500v^8 \Rightarrow v^t = .75236$$

which can be solved for t

$$t = -\frac{\log_e .75236}{\log_e 1.05} = -\frac{-.28454}{.04879} = 5.832 \text{ years}$$

Unknown rate of Interest

Four method to use in determining an unknown rate of interest

- Solve equation of value for i directly using a calculator with exponential and logarithmic function.
- Solve equation of value for i by algebraic techniques.
- Use linear interpolation in the interest tables.
- Successive approximation or iteration.

Example 7 At what interest rate convertible quarterly would \$1000 accumulate to \$1600 in six years.
Let $j = i^{(4)}/4$ so that the equation of value becomes

$$1000(1+j)^{24} = 1600 \text{ or } j = (1.6)^{1/24} - 1 \Rightarrow j = .019776.$$

The answer is $i^{(4)} = 4j = .0791$ or 7.91%.

Example 8 At what effective rate of interest rate will the present value of \$2000 at the end of two years and \$3000 at the end of four years be equal to \$4000.

An equation value is $4000 = 2000v^2 + 3000v^4$ which can be rewritten as

$$3v^4 + 2v^2 - 4 = 0.$$

So we have

$$v^2 = \frac{-2 \pm \sqrt{4 + 4 \cdot 3 \cdot 4}}{2 \cdot 3}.$$

since $v > 0$, so $v^2 = \frac{-2+\sqrt{52}}{6} = .868517$ or

$$(1+i)^2 = 1.151388 \quad \text{and } i = 0.0730 \quad \text{or } 7.3\%.$$

Example 9 At what interest rate convertible semiannually would an investment of \$1000 immediately and \$2000 3 years from now accumulate to \$5000 10 years from now.

Let $j = i^{(2)}/2$ so that the equation of value becomes $1000(1 + j)^{20} + 2000(1 + j)^{14} = 5000$. Here we use linear interpolation in the interest tables. Define

$$f(j) = 1000(1 + j)^{20} + 2000(1 + j)^{14} - 5000.$$

From the interest table, we found that $f(.0300) = -168.71$ and $f(0.035) = 227.17$. Then performing a linear interpolation

$$j = .0300 + 0.005\frac{0 + 168.71}{227.17 + 168.71} = .0321.$$

which gives $i^{(2)} = 2(.0321) = .0642$ or 6.42%.

Example 10 Obtain the answer to Example 9 to a higher level of accuracy using iteration
From Example 9 we have

$$f(j) = 1000(1+j)^{20} + 2000(1+j)^{14} - 5000.$$

and the first approximation $j = 0.0321$. Note $f(j)$ is an increasing function of $j > 0$. We have $f(.0321) = -6.114$ and $f(.0322) = 1.759$.
We use a higher value than .0321 in order a sign change for $f(j)$.
Performing another linear interpolation

$$j = .0321 + 0.001\frac{0 + 6.114}{1.759 + 6.114} = .03218.$$

Cycling again with one more decimal place $f(.03218) = .18346$ and $f(.03217) = -.60420$. Performing another linear interpolating

$$j = .03217 + 0.0001\frac{0 + .60420}{.18346 + .60420} = .32178$$

Cycling again with one more decimal place $f(.032178) = .025919$ and $f(.032177) = -.052851$. Thus.$j = .032178$, which is accurate to six decimal places. So the more accurate answer is

$$i^{(2)} = 2(.032178) = 0.6436 \quad \text{or} \quad 6.436\%.$$

The above procedure can be repeated as many times as necessary.

Practical Examples

- Advertisement frequently seen in the newspapers quotes two different rates on deposits, such as "7.91% rate/8.15% yield" ($i^{(4)} = 0.0791$ is equivalent $i = 0.0815$) and "8.00% interest rate/8.30% annual yield" ($i^{(12)} = 0.0800$ is equivalent $i = 0.083$)

- Mix of 360 day and 365 day years. Credits 6% compounded daily which produces a yield of 6.27%.

$$\left[1 + \frac{0.06}{360}\right]^{365} - 1 = 0.0627.$$

- It is important to distinguish between rates of interest and rates of discount. The rate of “T-bills” (United States Treasury issues Treasury bill) are computed as rate of discount, on the other hand longer-term Treasury securities are computed as rates of interest.

- Rates of discount are encountered in short-term commercial transactions. They are often computed by on a simple discount basis.

- Credit cards have an interesting way of charging interest.

- Investors need to be careful to consider is a penalty for early withdrawal.