COLLOCATION METHODS FOR CAUCHY PROBLEMS OF ELLIPTIC OPERATORS VIA CONDITIONAL STABILITIES

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Abstract. Ill-posed Cauchy problems for elliptic partial differential equations appear in many engineering fields. In this paper, we focus on stable reconstruction methods for this kind of inverse problems. Using kernels that reproduce Hilbert spaces $H^m(\Omega)$, numerical approximations to solutions of elliptic Cauchy problems are formulated as solutions of nonlinear least-squares problems with quadratic inequality constraints (LSQI). A convergence analysis with respect to noise levels and fill distances of data points is provided, from which a Tikhonov regularization strategy is obtained. A nonlinear algorithm using generalized singular value decomposition of matrices and Lagrange multipliers is proposed to solve the LSQI problem. Numerical experiments of two-dimensional cases verify our proved convergence results. By comparing with solutions of MFS and FEM with the discrete Tikhonov regularization by RKHS under same Cauchy data, we demonstrate that our method can reconstruct stable and high accuracy solutions for noisy Cauchy data.

Key words. Cauchy problems, Meshfree, Kansa method, Error analysis, LSQI problem, Tikhonov regularization.

AMS subject classifications. 65D15, 65N35, 65N21.

1. Introduction. It is well known that Cauchy problems are ill-posed in the sense that their solutions do not continuously depend on data. However, Tikhonov [36] proposed that conditional stabilities of solutions for Cauchy problems can be constructed with an a priori bound to the exact solution. In [21], an interior stability for elliptic Cauchy problems was proved. A global stability was proved based on the Carleman estimate in [34] by Takeuchi and Yamamoto. There are many other interior and global conditional stability results for Cauchy problems, and for more detail, one can refer to [1, 5, 15].

Based on these conditional stabilities, efforts were made to look for stable numerical methods. The quasi-reversibility method [24] as regularization was proposed for solving Cauchy problems of Laplace equations by Klibanov in 1990 and convergence analysis for a discrete finite difference scheme was also given. In [3], a similar method with an adaptive regularization parameter selective strategy was proposed for inverse Cauchy problems. In [34], the discretized Tikhonov regularization was proposed by Takeuchi and Yamamoto. Their regularization was built on the theory of reproducing kernel Hilbert spaces (RKHS). A finite element scheme for Cauchy problems was used and convergence results of the method were also proved in the same paper. Other numerical methods with convergent analysis are found in the works [6, 19, 33].

Meshless methods are another popular numerical method for solving Cauchy problems. Generally speaking, these methods can be applied to complicated geometry and to solving high dimensional problems. The method of fundamental solution (MFS) with different regularization strategies was used to solve homogenous Cauchy problems in [16, 20, 40]. MFS combined with the method of particular solution (MPS) was used to solve nonhomogeneous cases by Li, Xiong, and Chen in [27, 38]. A meshless method called the general finite difference method (GFDM) was proposed by Fan

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in [9] to solve inverse Cauchy problems. These meshless approaches usually have good numerical performance. However, most , if not all, are ad hoc and do not have robust theoretical support.

Recently, some progress has been made in the theoretical aspects of meshless collocation methods for PDEs. The Kansa method, proposed by E. J. Kansa in 1990 [22, 23], is a typical meshless method used to solve partial differential equations (PDEs) by imposing strong form collocation conditions to PDEs. To overcome the singular problem of matrix systems by the Kansa method appearing in some cases [18], the overdetermined Kansa method was applied to solve PDEs in [29]. Partial convergence results of the overdetermined Kansa method were proved by Ling and Schaback in [30]. Recently, convergence theorems for overdetermined Kansa methods for elliptic PDEs were proved by Cheung, Ling, and Schaback in [7].

Motivated by these improvements, in this paper, we apply an overdetermined kernel-based collocation formulation to solve inverse Cauchy problems. In Section 2, we first introduce Cauchy problems considered in this paper and make some assumptions. We define discrete solutions for Cauchy problems with exact Cauchy data in some trial spaces of the symmetric positive definite kernel. The discrete solutions were defined as solutions of nonlinear optimization problems with quadratic inequality constraints. In the definition, the Tikhonov regularization strategy is used. Convergence results of discrete solutions with respect to data densities and noise levels are also proved based upon the scattered data approximation theory in RKHS [10, 37]. The value of the regularization parameter can also be fixed in the proof. After considering exact Cauchy data, we also define discrete solutions with noisy Cauchy data as solutions of nonlinear least-squares problems with quadratic constraints. The convergence theorem of the discrete solution with noisy Cauchy data is proved based on the results of the discrete solution with exact Cauchy data. In Section 3, a solver for least-squares problems with quadratic constraints is introduced based on generalized singular value decomposition (GSVD) and the Lagrange multiplier method. In Section 4, we compare the results by the least-quares optimization problem with quadratic inequality constraints (LSQI) solver we introduced with those of other nonlinear solvers and show numerically that the proposed solver can obtain high accuracy and stable solutions. Numerical experiments for two-dimensional examples are computed to verify the convergence results we proved in Section 2. The high accuracy of the numerical results can also be seen by comparing them with the numerical solutions by MFS [31] and RKHS [34].

2. Reconstruction methods and error analysis.

2.1. Cauchy problems. In this paper, we consider the following Cauchy problem for elliptic PDEs: given f, g_0^* and g_1^*, find u insider Ω or on $\partial\Omega \setminus \Gamma$ where

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in} \quad \Omega, \\ u &= g_0^* \quad \text{on} \quad \Gamma, \\ \partial_{\mathcal{L}} u &= g_1^* \quad \text{on} \quad \Gamma. \end{aligned} \tag{2.1}$$

In Eq. (2.1), domain $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ and Γ is a nonempty open subset of $\partial\Omega$. The elliptic operator $\mathcal{L}$ and the conormal derivative operator $\partial_{\mathcal{L}}$ associated with $\mathcal{L}$ can be denoted as

$$\mathcal{L}u(x) := \sum_{i,j=1}^{d} \partial_j \big(a_{ij}(x)\partial_i u(x)\big) + c(x)u(x) \quad \text{for } x \in \Omega, \tag{2.2}$$

and

$$\partial_{\mathcal{L}} u(x) := \sum_{i,j=1}^{d} a_{ij}(x)\nu_j \partial_i u(x) \quad \text{for } x \in \ \Gamma,$$

where $\nu = \nu(x)$ is the unit outer normal vector of $\partial\Omega$ at x.

We make assumptions for the domain and operator coefficients for further use.

ASSUMPTION 2.1 (Smoothness of coefficients and domain). *We assume $\Omega \subseteq \mathbb{R}^d$ is an open bounded domain with Lipschitz continuous boundary and satisfies an interior cone condition. Coefficients in Eq.* (2.2) *satisfy $c(x) \le 0$ almost everywhere in Ω, $\{a_{ij}\}_{i,j=1}^d$ and $c(x) \in W_\infty^{m-1}(\Omega)$ for $m \ge 2$. We also assume $\{a_{ij}\}_{i,j=1}^d$ are symmetric positive definite, that is there exists a constant $\alpha > 0$ such that*

$$\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \ge \alpha \sum_{i=1}^{d} \xi_i^2 \quad \textit{for all } x \in \Omega,\ \{\xi_i\}_{i=1}^d \in \mathbb{R}^d. \qquad \square$$

We assume g_0^* and g_1^* are smooth enough to admit a uniquely defined exact solution $u^* \in H^m(\Omega)$ for the Cauchy problem (2.1) [21, Thm.3.3.1]. By the trace theorem, $g_0^* \in H^{m-1/2}(\Gamma)$ and $g_1^* \in H^{m-3/2}(\Gamma)$. Conditional stabilities for Cauchy problems (2.1) can be proved under an a priori bound for u^*, based on which we construct numerical algorithms. We state the recent global conditional stability result proved by Takeuchi and Yamamoto in [34].

PROPOSITION 2.2 (Global Conditional Stability). *Let u^* be the exact solution of the Cauchy problem* (2.1) *and $u^* \in H^m(\Omega)$ with $m > \frac{d+2}{2}$. For $0 < \kappa < 1$, there exists a constant $C > 0$ such that*

$$\|u\|_{L^\infty(\partial\Omega\setminus\Gamma)} \le C\|u\|_{H^m(\Omega)} \left(\log \frac{1}{\mathcal{E}(u)} + \log \frac{1}{\|u\|_{H^m(\Omega)}} \right)^{-\kappa}, \tag{2.3}$$

with $\mathcal{E}(u) := \|u\|_{L^2(\Gamma)} + \|\partial_{\mathcal{L}} u\|_{L^2(\Gamma)} + \|\mathcal{L}u\|_{L^2(\Omega)}$. $\square$

From the above result, we can easily see that $\|u\|_{L^\infty(\partial\Omega\setminus\Gamma)}$ converges to zero whenever $\|u^*\|_{H^m(\Omega)} \le M$ and $\mathcal{E}(u)$ converges to 0. The latter suggests that kernel collocation methods similar to those for solving direct problems can be developed to minimize $\mathcal{E}(u)$.

2.2. Kernels and native space. We consider symmetric positive definite kernels $\Phi :\ \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and further assume that their Fourier transforms $\hat{\Phi}$ of kernel Φ decay algebraically as

$$c_1(1 + \|\omega\|_2^2)^{-m} \le \hat{\Phi}(\omega) \le c_2(1 + \|\omega\|_2^2)^{-m} \quad \text{for } m > d/2. \tag{2.4}$$

Matérn functions and Wendland's compactly supported functions are two commonly used kernels satisfying (2.4). The native space $\mathcal{N}_{\mathbb{R}^d,\Phi}$ of a kernel Φ is defined as

$$\mathcal{N}_{\mathbb{R}^d,\Phi} := \left\{ f \in L^2(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d) : \hat{f}/\sqrt{\hat{\Phi}} \in L^2(\mathbb{R}^d) \right\}$$

associated with norms

$$\|f\|_{\mathcal{N}_{\mathbb{R}^d,\Phi}}^2 := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{\hat{f}^2(\omega)}{\hat{\Phi}} d\omega.$$

It was shown in [37] that native spaces $\mathcal{N}_{\mathbb{R}^d,\Phi}$ of kernels Φ satisfying (2.4) coincide with Sobolev spaces $H^m(\mathbb{R}^d)$. Native space norms $\|\cdot\|_{\mathcal{N}_{\mathbb{R}^d,\Phi}}$ and Sobolev norms $\|\cdot\|_m$ are equivalent. From [37, Cor.10.48], if Ω has a Lipschitz boundary, we also have native spaces $\mathcal{N}_{\Omega,\Phi}$ being norm equivalent to Sobolev spaces $H^m(\Omega)$.

Let $Z = \{z_1, z_2, \dots, z_{n_Z}\}$ be a discrete set of trial centers in the domain Ω. We define the finite dimensional trial space based on the trial set Z.

Definition 2.3. *Let Z be the trial set and kernel Φ satisfy* (2.4)*, the finite dimensional trial space $\mathcal{U}_Z$ is defined as:*

$$\mathcal{U}_{Z,\Phi_m} := \operatorname{span}\{\Phi(.,z_i),\ z_i \in Z\} \subset \mathcal{N}_{\Omega,\Phi}.$$

We propose a numerical method to seek numerical approximations of the Cauchy problem (2.1) from these trial spaces. We do so by imposing collocation conditions. Let $X = \{x_1, x_2, \dots, x_{n_X}\}$, $Y_0 = \{y_1^0, y_2^0, \dots, y_{n_{Y_0}}^0\}$ and $Y_1 = \{y_1^1, y_2^1, \dots, y_{n_{Y_1}}^1\}$ be sets of discrete collocation points in Ω and on the Dirichlet and Neumann boundary.

To describe the point density of Z, we define the following quantities

$$h_Z := \sup_{z\in\Omega} \min_{z_i\in Z} \|z - z_i\|_{\ell_2(\mathbb{R}^d)}, \quad q_Z := \frac{1}{2} \min_{\substack{z_i,\, z_j \,\in\, Z \\ z_i \neq z_j}} \|z_i - z_j\|_{\ell_2(\mathbb{R}^d)} \quad \text{and} \quad \rho_Z := \frac{h_Z}{q_Z},$$

which are normally called *fill distance*, *separation distance*, and *mesh ratio* of Z respectively. We further assume the trial set Z and collocation sets X, Y_0 and Y_1 are all quasi-uniform, that is, the mesh ratio $\rho_\chi \geqslant 1$ satisfies

$$q_\chi \le h_\chi \le \rho_\chi q_\chi \quad \text{and} \quad \chi = \{X,\ Y_0,\ Y_1,\ Z\}. \tag{2.5}$$

2.3. The discrete solution with exact Cauchy data and error analysis. For easy understanding, we begin by introducing a discrete approximation with exact Cauchy data in the trial space to the solution of the Cauchy problem. We aim to develop a simple least-squares approach. Let u be a function in $H^m(\Omega)$, and v be a general notation for functions in the trial space $\mathcal{U}_{Z,\Phi_m} \subseteq \mathcal{N}_{\Omega,\Phi_m} = H^m(\Omega)$. We first introduce some preliminaries. For any $u \in \mathcal{C}(\Omega)$, we define a discrete norm of u on collocation set X as

$$\|u\|_X = \left(\sum_{x_i\in X} u(x_i)^2 \right)^{1/2}.$$

From [11], for any $u \in H^m(\Omega)$, when Assumption 2.1 holds for coefficients of an elliptic operator, one has

$$\|\mathcal{L}u\|_{H^{m-2}(\Omega)} \le C_{\Omega,\mathcal{L}} \|u\|_{H^m(\Omega)}, \tag{2.6}$$

and

$$\|\partial_{\mathcal{L}} u\|_{H^{m-1}(\Omega)} \le C_{\Omega,\partial_{\mathcal{L}}} \|u\|_{H^m(\Omega)}.$$

To define a computable scheme for the Cauchy problem, we need to discretize the continuous norms by discrete point sets in both the domain and the Cauchy boundary. To do this, we introduce sampling inequalities in the following proposition.

Proposition 2.4 (Sampling inequality of fractional order [7]). *Suppose $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain with a piecewise C^m–boundary. Then there is a constant $C_{\Omega,m,s}$ depending only on Ω, m and s such that following inequalities hold:*

$$\|u\|_{s,\Omega} \le C_{\Omega,m,s}\left(h_X^{m-s}\|u\|_{m,\Omega} + h_X^{d/2-s}\|u\|_X\right) \quad \text{for } 0 \le s \le m,$$

and

$$\|u\|_{s-1/2,\Gamma} \le C_{\Gamma,m,s}\left(h_Y^{m-s}\|u\|_{m,\Omega} + h_Y^{d/2-s}\|u\|_Y\right) \quad \text{for } 1/2 \le s \le m,$$

for any $u \in H^m(\Omega)$ with $m > d/2$ and any discrete sets $X \subset \Omega$ and $Y \subset \Gamma$ with sufficiently small mesh norm h_X and h_Y.

For the ill-posed property of the problems being solved, different regularization strategies were proposed to stabilizing the numerical solutions, for instance, the Tikhonov regularization method [26, 35], the damped singular value decomposition [8, 20], and the truncated singular value decomposition [17]. Recently, a novel regularization method for ill-posed problems called mixed regularization method was put forward by Zheng, Zhang in [41]. In this paper, we use the Tikhonov regularization method. Combining discrete norms on collocation sets X, Y_0 and Y_1, we define the discrete solution in trial space $\mathcal{U}_{Z,\Phi_m}$ of Cauchy problems with exact data as:

Definition 2.5. *The solution $u_{X,Y_0,Y_1,\sigma} \in \mathcal{U}_{Z,\Phi_m}$ with exact Cauchy data defined as the solution of the following least-squares problems with quadratic inequality constraints (LSQI) problem:*

$$\begin{aligned} u_{X,Y_0,Y_1,\sigma} &:= \operatorname*{arg\,inf}_{v\in\mathcal{U}_{Z,\Phi_m}} \ \sigma^2\|v\|^2_{H^m(\Omega)} + \|\mathcal{L}v - f\|^2_X \\ s.t. \quad & h_{Y_0}^{d-1}\|v - g_0^*\|^2_{Y_0} + h_{Y_1}^{d-1}\|\partial_{\mathcal{L}}v - g_1^*\|^2_{Y_1} \le (h_Z^{2m-d-2} + h_Z^{2m-d})\widetilde{M}^2. \end{aligned} \tag{2.7}$$

with σ being a regularization parameter and $\widetilde{M}$ being a constant.

To decide the value of regularization parameter, one can choose different experimental methods, like discrepancy principal, L-curve method, generalized cross-validation, and quasi-optimality criterion [20,39]. In our work, both the regularization parameter σ and the constant $\widetilde{M}$ in (2.7) will be chosen during the convergence proof. Let s_u denote the interpolant of the exact solution $u^* \in H^m(\Omega)$ on Z from the trial space $\mathcal{U}_{Z,\Phi_m}$. It is known that s_u can be uniquely defined for positive definite kernels. Convergence analysis of s_u to u^* in native space was well studied in [10, 32, 37]. To make use of these results in proving convergence of $u_{X,Y_0,Y_1,\sigma}$, we first show that s_u is feasible for the problem (2.7).

Lemma 2.6. *Suppose domain Ω and elliptic operator satisfy the Assumption 2.1. Let $u_{X,Y_0,Y_1,\sigma}$ be the discrete solution with exact Cauchy data and $u^* \in H^m(\Omega)$ be the exact solution. When kernel smoothness $m > 1 + d/2$, the unique interpolant s_u of u^* in $\mathcal{U}_{Z,\Phi_m}$ is a feasible solution for the problem* (2.7).

Proof: As s_u is the interpolant function of u^* in trial space $\mathcal{U}_{Z,\Phi_m}$, we need only to prove that it satisfies quadratic inequality constraints in (2.7). First, on the Dirichlet boundary, one has for $m > d/2$

$$\begin{aligned} h_{Y_0}^{(d-1)/2}\|s_u - g_0^*\|_{Y_0} &\le h_{Y_0}^{(d-1)/2} n_{Y_0}^{1/2}\|s_u - g_0^*\|_{L^\infty(\Gamma)} \\ &\le C_{\rho_{Y_0},\Gamma,\Omega}\|s_u - u^*\|_{L^\infty(\Omega)} \\ &\le C_{\Omega,\rho_{Y_0},\Phi_m,\Gamma} h_Z^{m-d/2}\|u^*\|_{H^m(\Omega)}, \end{aligned}$$

with the second inequality from $n_{Y_0} \le C_\Gamma q_{Y_0}^{-(d-1)} \le C_{\Gamma,\rho_{Y_0}} h_{Y_0}^{-(d-1)}$ and the last inequality from [10, Sec.15.1.2]. On the Neumann boundary, for $m > 1 + d/2$

$$\begin{aligned} h_{Y_1}^{(d-1)/2}\|\partial_{\mathcal{L}} s_u - g_1^*\|_{Y_1} &\le h_{Y_1}^{(d-1)/2} n_{Y_1}^{1/2}\|\partial_{\mathcal{L}} s_u - \partial_{\mathcal{L}} u^*\|_{L^\infty(\Gamma)} \\ &\le C_{\Omega,\rho_{Y_1},\Gamma}\|\partial_{\mathcal{L}} s_u - \partial_{\mathcal{L}} u^*\|_{L^\infty(\Omega)} \\ &\le C_{\Omega,\rho_{Y_1},\Phi_m,\partial_{\mathcal{L}},\Gamma} h_Z^{m-1-d/2}\|u^*\|_{H^m(\Omega)}, \end{aligned}$$

by [10, Sec.15.1.2] and inequality $n_{Y_1} \le C_\Gamma q_{Y_1}^{-(d-1)} \le C_{\Gamma,\rho_{Y_1}} h_{Y_1}^{-(d-1)}$. Squaring both sides of these two inequalities and applying $\|u^*\|_{H^m(\Omega)} \le M$, the lemma was proved. □

From the proof of Lemma 2.6, besides the fill distance of trial center Z and upper bound M of $\|u^*\|_{H^m(\Omega)}$, the right hand side value of the inequality constraints in problem (2.7) also affected by constant C depends on domain Ω, Cauchy boundary Γ, and the mesh ratio $\rho_{Y_0},\ \rho_{Y_1}$ of boundary collocation sets $Y_0,\ Y_1$ and kernel Φ_m. As the constant cannot be evaluated exactly, we write $\widetilde{M} = C_{\Omega,\rho_{Y_0},\rho_{Y_1},\Phi_m,\Gamma} M$.

With Lemma 2.6, we can prove the error of objective functions in LSQI problem (2.7). Let functional $J_\sigma : H^m \to \mathbb{R}$ be defined as

$$J_\sigma(v) := \left(\sigma^2\|v\|_{H^m(\Omega)}^2 + \|\mathcal{L}v - f\|_X^2\right)^{1/2}. \tag{2.8}$$

The discrete solution $u_{X,Y_0,Y_1,\sigma}$ satisfies $J_\sigma(u_{X,Y_0,Y_1,\sigma}) \le J_\sigma(s_u)$ for its optimal property, and for $J_\sigma(s_u)$, we have

$$\begin{aligned} J_\sigma(s_u) &\le \|\mathcal{L}s_u - f\|_X + \sigma\|s_u\|_{H^m(\Omega)} \\ &\le \|\mathcal{L}s_u - \mathcal{L}u^*\|_X + \sigma(\|s_u - u^*\|_{H^m(\Omega)} + \|u^*\|_{H^m(\Omega)}). \end{aligned}$$

For the interpolant s_u, by [10, Cor. 18.1], we have in native space $\mathcal{N}_{\Omega,\Phi_m}$

$$\|s_u - u^*\|_{\mathcal{N}_{\Omega,\Phi_m}}^2 \le \|s_u - u^*\|_{\mathcal{N}_{\Omega,\Phi_m}}^2 + \|s_u\|_{\mathcal{N}_{\Omega,\Phi_m}}^2 = \|u^*\|_{\mathcal{N}_{\Omega,\Phi_m}}^2.$$

Then by the norm equivalent property of $\mathcal{N}_{\Omega,\Phi_m}$ and $H^m(\Omega)$ for kernel Φ_m, we obtain $\|s_u - u^*\|_{H^m(\Omega)} \le C_{\Omega,\Phi_m}\|u^*\|_{H^m(\Omega)}$. Error estimation of s_u to u^* in [25, theorem 2.3] suggests that for kernel smoothness $m \ge 2 + d/2$

$$\|\mathcal{L}s_u - f\|_X \le C_{\Omega,\Phi_m,\mathcal{L}} \rho_X^{d/2} n_X^{1/2} h_Z^{m-2}\|u^*\|_{H^m(\Omega)}.$$

By inequality $n_X \le C_\Omega q_X^{-d} \le C_{\Omega,\rho_X} h_X^{-d}$, we can obtain the error estimation for $J_\sigma(u_{X,Y_0,Y_1,\sigma})$ as

$$J_\sigma(u_{X,Y_0,Y_1,\sigma}) \le C_{\Omega,\Phi_m,\mathcal{L},\rho_X}(\sigma + h_X^{-d/2} h_Z^{m-2})\|u^*\|_{H^m(\Omega)}. \tag{2.9}$$

This observation combined with sampling inequalities and Lemma 2.6 allows us to study the convergence of $u_{X,Y_0,Y_1,\sigma}$.

THEOREM 2.7. *(Convergence of $u_{X,Y_0,Y_1,\sigma}$) Suppose the domain and elliptic operators satisfy the Assumption 2.1 and conditional stability in the Proposition 2.3 holds for the Cauchy problem. Let kernel Φ have smoothness order $m \ge 2 + \frac{d}{2}$. The exact solution is denoted as $u^* \in H^m(\Omega)$. When the regularization parameter is taken as*

$$\sigma^* = h_X^{-\frac{d}{2}} h_Z^{m-2}, \tag{2.10}$$

convergence results for $u_{X,Y_0,Y_1,\sigma}$ *defined in* (2.7) *hold as:*

$$\|u_{X,Y_0,Y_1,\sigma}-u^*\|_{L^\infty(\partial\Omega\backslash\Gamma)} \leq CM\left(\log\frac{1}{\left(h_Z^{m-2}+h_Z^{m-1-d/2}+h_{Y_0}^{m-1/2}+h_{Y_1}^{m-3/2}\right)M^2}\right)^{-\kappa}, \tag{2.11}$$

with the constant C *depending on* $\mathcal{L}$, $\partial_{\mathcal{L}}$, Φ_m, Ω, Γ, ρ_X, ρ_{Y_0}, *and* ρ_{Y_1}.

Proof: Tikhonov regularization parameter σ is chosen to ensure the boundness of $\|u_{\sigma,X,Y_0,Y_1}\|_{H^m(\Omega)}$ which is necessary for its convergence. Because the regularization term is contained in the definition of functional $J_\sigma(u_{\sigma,X,Y_0,Y_1})$, we have $\|u_{\sigma,X,Y_0,Y_1}\|_{H^m(\Omega)} \leq J_\sigma(u_{\sigma,X,Y_0,Y_1})/\sigma$. From error estimation of $J_\sigma(u_{\sigma,X,Y_0,Y_1})$, we have

$$\begin{aligned}\|u_{\sigma,X,Y_0,Y_1}\|_{H^m(\Omega)} &\leq C_{\mathcal{L},\Phi_m,\Omega,\rho_X}\frac{1}{\sigma}\left(h_X^{-d/2}h_Z^{m-2-d/2}+\sigma\right)M\\ &\leq C_{\mathcal{L},\Phi_m,\Omega,\rho_X}M,\end{aligned} \tag{2.12}$$

with σ taken as in Eq. (2.10). Then, we consider the convergence of $\mathcal{E}(u_{\sigma,X,Y_0,Y_1}-u^*)$. It contains three terms that represent the L^2 norm of the difference between u_{σ,X,Y_0,Y_1} and u^* in the domain, on the Dirichlet Cauchy boundary, and on the Neumann Cauchy boundary. For simplicity, we consider terms in the domain and on the Cauchy boundary separately. For boundary terms, using sampling inequalities on the boundary in Proposition 2.4 and inequality $a+b\leq C(a^2+b^2)^{1/2}$, we have

$$\begin{aligned}&\|u_{\sigma,X,Y_0,Y_1}-g_0^*\|_{L^2(\Gamma)}+\|\partial_{\mathcal{L}}u_{\sigma,X,Y_0,Y_1}-g_1^*\|_{L^2(\Gamma)}\\ &\leq C_{\Omega,\Gamma,\partial_{\mathcal{L}}}\Big(\left(h_{Y_0}^{d-1}\|u_{\sigma,X,Y_0,Y_1}-g_0^*\|_{Y_0}^2+h_{Y_1}^{d-1}\|\partial_{\mathcal{L}}u_{\sigma,X,Y_0,Y_1}-g_1^*\|_{Y_1}^2\right)^{1/2}\\ &\qquad+\left(h_{Y_0}^{m-1/2}+h_{Y_1}^{m-3/2}\right)\|u_{\sigma,X,Y_0,Y_1}-u^*\|_{H^m(\Omega)}\Big).\end{aligned}$$

Because u_{σ,X,Y_0,Y_1} satisfy constraint inequalities in (2.7) and $\|u_{\sigma,X,Y_0,Y_1}\|_{H^m(\Omega)}$ is bounded, we obtain

$$\begin{aligned}&\|u_{\sigma,X,Y_0,Y_1}-g_0^*\|_{L^2(\Gamma)}+\|\partial_{\mathcal{L}}u_{\sigma,X,Y_0,Y_1}-g_1^*\|_{L^2(\Gamma)}\\ &\qquad\leq C\left(h_Z^{m-1-d/2}+h_{Y_0}^{m-1/2}+h_{Y_1}^{m-3/2}\right)M,\end{aligned}$$

with C depending on Ω, Γ, ρ_{Y_0}, ρ_{Y_0}, Φ_m and $\partial_{\mathcal{L}}$. By sampling inequality in the domain, we can get

$$\begin{aligned}\|\mathcal{L}u_{\sigma,X,Y_0,Y_1}-f\|_{L^2(\Omega)} \leq C_\Omega h_X^{d/2}\Big(&\|\mathcal{L}u_{\sigma,X,Y_0,Y_1}-f\|_X+\sigma\|u_{\sigma,X,Y_0,Y_1}-u^*\|_{m,\Omega}\\ &+\left(h_X^{m-2-d/2}-\sigma\right)_+\|u_{\sigma,X,Y_0,Y_1}-u^*\|_{m,\Omega}\Big),\end{aligned}$$

with $(x)_+=\max\{0,x\}$. When σ takes as in Eq. (2.10), we have $(h_X^{m-2-d/2}-\sigma)_+=0$ under the condition $h_X\leq h_Z$. Applying the boundness property of $\|u^*_{\sigma,X,Y_0,Y_1}\|_{H^m(\Omega)}$ in Eq. (2.12), for $m\geq 2+d/2$, we have

$$\begin{aligned}\|\mathcal{L}u_{\sigma,X,Y_0,Y_1}-f\|_{L^2(\Omega)} &\leq C_\Omega h_X^{d/2}\left(J_\sigma(u_{\sigma,X,Y_0,Y_1})+\sigma\|u^*\|_{H^m(\Omega)}\right)\\ &\leq C_{\Omega,\Phi_m,\mathcal{L},\rho_X}h_Z^{m-2}M.\end{aligned}$$

Then, the convergence of $\mathcal{E}(u_{\sigma,X,Y_0,Y_1} - u^*)$ becomes

$$\mathcal{E}(u_{\sigma,X,Y_0,Y_1} - u^*) \leq C\big(h_Z^{m-2} + h_Z^{m-1-d/2} + h_{Y_0}^{m-1/2} + h_{Y_1}^{m-3/2}\big)M, \qquad (2.13)$$

with C depending on Ω, Φ_m, $\mathcal{L}$, ρ_X, Γ, ρ_{Y_0}, ρ_{Y_0}, and $\partial_{\mathcal{L}}$.

Substituting estimation for $\mathcal{E}(u_{\sigma,X,Y_0,Y_1} - u^*)$ in Eq. (2.13), boundness of H^m norm of u_{σ,X,Y_0,Y_1} to the conditional stability of the Cauchy problem in Eq. (2.3) results in the convergence of u_{σ,X,Y_0,Y_1} obtained as in Eq. (2.11). □

From convergence results of the discrete solution with exact Cauchy data in Eq. (2.11), we can see u_{σ,X,Y_0,Y_1} converge to u^* at log rate with respect to fill distance of the trial centers $h_Z^{m-1-d/2}$, boundary collocation sets $h_{Y_0}^{m-1/2}$ and $h_{Y_1}^{m-3/2}$. After knowing that there is a good approximation in $H^m(\Omega)$ with exact Cauchy data, we can now seek a good comparison function in trial spaces with noisy Cauchy data.

2.4. Discrete solution with noisy Cauchy data and error analysis. When considering Cauchy data with noise, we need only to consider boundary terms. We denote noisy Cauchy data as g_0^δ and g_1^δ for the Dirichlet and Neumann boundaries, respectively, and assume noise level $\Delta > 0$ such that

$$\big(h_{Y_0}^{d-1}\|g_0^\delta - g_0^*\|_{Y_0}^2 + h_{Y_1}^{d-1}\|g_1^\delta - g_1^*\|_{Y_1}^2\big)^{1/2} \leq \Delta.$$

With noisy Cauchy data with noise level Δ contained in the definition of the solution, similar to the discrete solution in the noise-free case in Definition 2.5, we can define the solutions with noisy Cauchy data as:

Definition 2.8. *The discrete solution $u_{X,Y_0,Y_1,\sigma}^\delta \in H^m(\Omega)$ with noisy data defined as solutions of the following LSQI problem:*

$$\begin{aligned} &u_{X,Y_0,Y_1,\sigma}^\delta := \underset{v\in\mathcal{U}_{Z,\Phi_m}}{\arg\inf}\ \sigma^2\|v\|_{H^m(\Omega)}^2 + \|\mathcal{L}v - f\|_{\ell^2(X)}^2, \\ &s.t. \qquad h_{Y_0}^{d-1}\|v - g_0^\delta\|_{Y_0}^2 + h_{Y_1}^{d-1}\|\partial_{\mathcal{L}}v - g_1^\delta\|_{Y_1}^2 \leq \big(h_Z^{2m-d-2} + h_Z^{2m-d}\big)\widetilde{M}^2 + \Delta^2. \end{aligned} \qquad (2.14)$$

Unlike the definition in the noise-free case, noise Cauchy data g_0^δ and g_1^δ are used in the left side of the inequality constraint, and an additional noise level term is added to the right side of the inequality constraint. It is easy to show that the discrete solution in the noise-free case $u_{X,Y_0,Y_1,\sigma}$ is a feasible solution of problem (2.14). By triangle inequalities, we have

$$\|u_{X,Y_0,Y_1,\sigma} - g_0^\delta\|_{Y_0}^2 \leq \|u_{X,Y_0,Y_1,\sigma} - g_0^*\|_{Y_0}^2 + \|g_0^\delta - g_0^*\|_{Y_0}^2,$$

and

$$\|\partial_{\mathcal{L}}u_{X,Y_0,Y_1,\sigma} - g_1^\delta\|_{Y_1}^2 \leq \|\partial_{\mathcal{L}}u_{X,Y_0,Y_1,\sigma} - g_1^*\|_{Y_1}^2 + \|g_1^\delta - g_1^*\|_{Y_1}^2.$$

Since $u_{X,Y_0,Y_1,\sigma}$ satisfies quadratic constraints in definition 2.5, we have proved it to be a feasible solution of problem (2.14). By the optimal property of $u_{X,Y_0,Y_1,\sigma}^\delta$ and the convergence results of $J_\sigma(u_{X,Y_0,Y_1,\sigma})$ in Eq. (2.9), we get

$$J(u_{X,Y_0,Y_1,\sigma}^\delta) \leq J(u_{X,Y_0,Y_1,\sigma}) \leq C_{\Omega,\Phi_m,\mathcal{L},\rho_X}\big(h_X^{-d/2}h_Z^{m-2} + \sigma\big)M.$$

Then, we are ready to prove the convergence of the discrete solution with noisy Cauchy data.

THEOREM 2.9. *Suppose the domain and elliptic operators satisfy the Assumption 2.1 and conditional stability in the Proposition 2.3 holds for the Cauchy problem. Let kernel Φ have smoothness order $m \geq 2 + \frac{d}{2}$ with Theorem 2.7 for $u^*_{X,Y_0,Y_1,\sigma}$ holding. The exact solution is denoted as $u^* \in H^m(\Omega)$. When the regularization parameter takes the value*

$$\sigma = h_X^{-d/2} h_Z^{m-2}, \tag{2.15}$$

the convergence result for discrete solution $u^\delta_{X,Y_0,Y_1,\sigma}$ with noisy Cauchy data is

$$\begin{aligned} &\|u^\delta_{X,Y_0,Y_1,\sigma} - u^*\|_{L^\infty(\partial\Omega\setminus\Gamma)} \\ &\qquad \leq CM \left(\log \frac{1}{\left((h_Z^{m-2} + h_Z^{m-1-d/2} + h_{Y_0}^{m-1/2} + h_{Y_1}^{m-3/2})M + \Delta\right)M} \right)^{-\kappa}, \end{aligned} \tag{2.16}$$

with the constant C depending on $\mathcal{L}$, $\partial_{\mathcal{L}}$, Φ_m, Ω, Γ, ρ_X, ρ_{Y_0}, and ρ_{Y_1}.

Proof: For convergence of $u^\delta_{\sigma,X,Y_0,Y_1}$, we need to prove the convergence of functional $\mathcal{E}(u^\delta_{\sigma,X,Y_0,Y_1} - u^*)$ and the boundness of $\|u^\delta_{\sigma,X,Y_0,Y_1}\|_{m,\Omega}$. We first consider the boundness condition. From the definition of functional J_σ in Eq. (2.8), we have

$$\|u^\delta_{X,Y_0,Y_1,\sigma}\|_{m,\Omega} \leq \frac{J_\sigma(u^\delta_{X,Y_0,Y_1,\sigma})}{\sigma} \leq C_{\Omega,\Phi_m,\mathcal{L},\rho_X} M,$$

with σ as in Eq. (2.15). Next, we analyze the convergence of $\mathcal{E}(u^\delta_{\sigma,X,Y_0,Y_1} - u^*)$. By applying sampling inequalities on boundary terms and then inserting noisy Cauchy data g_0^δ and g_1^δ, we get

$$\begin{aligned} \|u^\delta_{X,Y_0,Y_1,\sigma} - g_0^*\|_{L^2(\Gamma)} &\leq C_{\Omega,\Gamma,\Phi_m,\mathcal{L},\rho_X} \left(h_{Y_0}^{\frac{d-1}{2}} \|u^\delta_{X,Y_0,Y_1,\sigma} - g_0^*\|_{Y_0} + h_{Y_0}^{m-\frac{1}{2}} M \right) \\ &\leq C\Big(h_{Y_0}^{\frac{d-1}{2}} (\|u^\delta_{X,Y_0,Y_1,\sigma} - g_0^\delta\|_{Y_0} + \|g_0^\delta - g_0^*\|_{Y_0}) + h_{Y_0}^{m-\frac{1}{2}} M \Big), \end{aligned}$$

with C depending on Ω, Γ, Φ_m, $\mathcal{L}$, ρ_X, and

$$\begin{aligned} \|\partial_{\mathcal{L}} u^\delta_{\sigma,X,Y_0,Y_1} - g_1^*\|_{L^2(\Gamma)} &\leq C\Big(h_{Y_1}^{\frac{d-1}{2}} \|\partial_{\mathcal{L}} u^\delta_{\sigma,X,Y_0,Y_1} - g_1^*\|_{Y_1} + h_{Y_1}^{m-\frac{3}{2}} M \Big) \\ &\leq C\Big(h_{Y_1}^{\frac{d-1}{2}} (\|\partial_{\mathcal{L}} u^\delta_{\sigma,X,Y_0,Y_1} - g_1^\delta\|_{Y_1} + \|g_1^\delta - g_1^*\|_{Y_1}) \\ &\qquad + h_{Y_1}^{m-\frac{3}{2}} M \Big), \end{aligned}$$

with C depending on Ω, Γ, Φ_m, $\mathcal{L}$, ρ_X, and $\partial_{\mathcal{L}}$. Combining these two inequalities and using constraint conditions for $u^\delta_{\sigma,X,Y_0,Y_1}$ in definition 2.8, we obtain

$$\begin{aligned} &\|u^\delta_{X,Y_0,Y_1,\sigma} - g_0^*\|_{L^2(\Gamma)} + \|\partial_{\mathcal{L}} u^\delta_{X,Y_0,Y_1,\sigma} - g_1^*\|_{L^2(\Gamma)} \\ &\qquad \leq C_{\Omega,\Gamma,\Phi_m,\mathcal{L},\rho_X,\rho_{Y_0},\rho_{Y_1}\partial_{\mathcal{L}}} \left((h_Z^{m-1-d/2} + h_{Y_0}^{m-1/2} + h_{Y_1}^{m-3/2})M + \Delta \right). \end{aligned}$$

In the domain, when σ takes values as in Eq. (2.15), by the same argument used in the proof of the theorem 2.7, the residual has an error estimation as

$$\begin{aligned} \|\mathcal{L} u^\delta_{\sigma,X,Y_0,Y_1} - f\|_{L^2(\Omega)} &\leq C_\Omega h_X^{d/2} (J_\sigma(u^\delta_{\sigma,X,Y_0,Y_1}) + \sigma \|u^*\|_{H^m(\Omega)}) \\ &\leq C_{\Omega,\Phi_m,\mathcal{L},\rho_X} h_Z^{m-2} M. \end{aligned}$$

By combining the error in the domain with that on the Cauchy boundary, the error estimation for $\mathcal{E}(u^{\delta}_{\sigma,X,Y_0,Y_1} - u^*)$ becomes

$$\mathcal{E}(u^{\delta}_{\sigma,X,Y_0,Y_1} - u^*) \le C\big((h_Z^{m-2} + h_Z^{m-1-d/2} + h_{Y_0}^{m-1/2} + h_{Y_1}^{m-3/2})M + \Delta\big). \quad (2.17)$$

with C depending $\Omega,\ \Gamma,\ \Phi_m,\ \mathcal{L},\ \rho_X,\ \rho_{Y_0},\ \rho_{Y_1}$, and $\partial_{\mathcal{L}}$. Substituting estimation in Eq. (2.17) and the boundness of $\|u^{\delta}_{\sigma,X,Y_0,Y_1}\|_{H^m(\Omega)}$ to the conditional stability of the Cauchy problem in Eq. (2.3), the convergence of $u^{\delta}_{\sigma,X,Y_0,Y_1}$ holds as Eq. (2.16). □

From Theorem 2.9, with an *a prior* bound to the H^m norm of exact solution u^*, the discrete solution with noisy Cauchy data $u^{\delta}_{\sigma,X,Y_0,Y_1}$ converges at log-rate with respect to noise levels Δ, the fill distance of trial centers h_Z^{m-2}, and collocation set $h_{Y_0}^{m-1/2}$ and $h_{Y_1}^{m-3/2}$. After defining the solution for the Cauchy problem (2.1) and proving its convergence with respect to the exact solution, we can find numerical methods to solve problem (2.14) in Definition 2.8.

3. Numerical algorithms. In this section, the LSQI problem will first be written in matrix form by RBF collocation methods. A numerical solver by combining GSVD and the Lagrange multiplier method is introduced for the LSQI problem. The approximated solution $u^{\delta}_{\sigma,X,Y_0,Y_1}$ can be represented by the radial basis function expansion analogous to that used for scattered data interpolation as

$$u^{\delta}_{\sigma,X,Y_0,Y_1} = \sum_{j=1}^{n_Z} \lambda_j \Phi(\cdot, z_j) \quad \text{for } z_j \in Z.$$

By overdetermined Kansa methods, collocation conditions in the domain Ω are imposed at set X with elliptic operator $\mathcal{L}$ acting on the collocation matrix as

$$\mathcal{L}u^{\delta}_{\sigma,X,Y_0,Y_1} = \sum_{i=1}^{n_X}\sum_{j=1}^{n_Z} \lambda_j \mathcal{L}\Phi(x_i, z_j) := \mathcal{L}K(X,Z)\lambda \quad \text{for } x_i \in X,\ z_j \in Z,$$

with $\lambda = [\lambda_1, \ldots, \lambda_{n_Z}]^{\mathrm{T}} \in \mathbb{R}^{n_Z}$. By the same argument, we impose collocation conditions on the Dirichlet and Neumann boundaries at sets Y_0 and Y_1 as

$$u^{\delta}_{\sigma,X,Y_0,Y_1} = \sum_{i=1}^{n_Y}\sum_{j=1}^{n_Z} \lambda_j \Phi(y_i, z_j) := K(Y,Z)\lambda \quad \text{for } y_i \in Y_0,\ z_j \in Z,$$

and

$$\partial_{\mathcal{L}} u^{\delta}_{\sigma,X,Y_0,Y_1} = \sum_{i=1}^{n_Y}\sum_{j=1}^{n_Z} \lambda_j \partial_{\mathcal{L}}\Phi(y_i, z_j) := \partial_{\mathcal{L}}K(Y,Z)\lambda \quad \text{for } y_i \in Y_1,\ z_j \in Z.$$

Furthermore, by the norm equivalence property with native space norm from [37, Sec.10.1], norm $\|u^{\delta}_{\sigma,X,Y_0,Y_1}\|_m$ in the Tikhonov regularization term can be expressed as

$$\|u^{\delta}_{\sigma,X,Y_0,Y_1}\|_m^2 = \sum_{i=1}^{n_Z}\sum_{j=1}^{n_Z} \lambda_j \lambda_i \Phi(z_i, z_j) := \lambda^T K(Z,Z)\lambda \quad \text{for } z_i,\ z_j \in Z.$$

Combining the above representations, the problem (2.14) with quadratic constraints can be written in matrix form as:

$$\arg\inf_{\lambda\in\mathbb{R}^{n_Z}} \quad \|A\lambda - b\|_2 \quad \text{s.t.} \quad \|B\lambda - d\|_2 \le E, \tag{3.1}$$

with expressions and sizes for matrices and vectors are

$$\begin{aligned}
A &= \begin{bmatrix} \mathcal{L}K(X,Z) \\ \sigma(K(Z,Z))^{1/2} \end{bmatrix} \in \mathbb{R}^{(n_X+n_Z)\times n_Z}, \qquad b = \begin{bmatrix} f(X) \\ 0 \end{bmatrix} \in \mathbb{R}^{(n_X+n_Z)}, \\
B &= \begin{bmatrix} h_{Y_0}^{(d-1)/2} K(Y_0,Z) \\ h_{Y_1}^{(d-1)/2} \partial_{\mathcal{L}} K(Y_1,Z) \end{bmatrix} \in \mathbb{R}^{(n_{Y_0}+n_{Y_1})\times n_Z},\ d = \begin{bmatrix} h_{Y_0}^{(d-1)/2} g_0^\delta|_{Y_0} \\ h_{Y_1}^{(d-1)/2} g_1^\delta|_{Y_1} \end{bmatrix} \in \mathbb{R}^{n_{Y_0}+n_{Y_1}}, \\
E &= \left((h_Z^{2m-d-2} + h_Z^{2m-d})\widetilde{M}^2 + \Delta^2\right)^{1/2} \in \mathbb{R}.
\end{aligned}$$

Nonlinear optimization solvers such as SDPT3 solver in Matlab CVX toolbox [13,14], and Mosek solver [2] can be used to solve the quadratic constraints quadratic problem (3.1). Furthermore, a faster algorithm presented in [12] can be modified to solve problem (3.1) and we introduce it here. First, the problem is simplified using the GSVD of matrix A and B in problem (3.1). The full GSVD of A and B are

$$U^T A X = D_A, \quad V^T B X = D_B, \quad U^T U = I_{n_X+n_Z}, \quad \text{and } V^T V = I_{n_{Y_0}+n_{Y_1}}, \tag{3.2}$$

with the size of each matrix being $U \in \mathbb{R}^{(n_X+n_Z)\times(n_X+n_Z)}$, $X \in \mathbb{R}^{n_Z\times n_Z}$, $D_A \in \mathbb{R}^{(n_X+n_Z)\times n_Z}$, $D_B \in \mathbb{R}^{(n_{Y_0}+n_{Y_1})\times n_Z}$, and $V \in \mathbb{R}^{(n_{Y_0}+n_{Y_1})\times(n_{Y_0}+n_{Y_1})}$. Matrices D_A and D_B have representations as:

$$D_A = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{n_Z} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad D_B = \begin{bmatrix} \beta_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \beta_{n_{Y_0}+n_{Y_1}} & \cdots & 0 \end{bmatrix}.$$

After computing the GSVD of matrices A and B in Eq. (3.2), we can convert the LSQI problem to

$$\arg\inf_{\Lambda_Z\in\mathbb{R}^{n_Z}} \quad \|D_A\widetilde{\Lambda}_Z - \widetilde{b}\|_2 \quad \text{s.t.} \quad \|D_B\widetilde{\Lambda}_Z - \widetilde{d}\|_2 \le E, \tag{3.3}$$

with $\widetilde{\Lambda}_Z = X^{-1}\Lambda_Z \in \mathbb{R}^{n_Z}$, $\widetilde{b} = U^T b \in \mathbb{R}^{n_X+n_Z}$ and $\widetilde{d} = V^T d \in \mathbb{R}^{n_{Y_0}+n_{Y_1}}$. We can write the problem (3.3) in scaler form as

$$\arg\inf_{\Lambda_Z\in\mathbb{R}^{n_Z}} \quad \sum_{i=1}^{n_Z}(\alpha_i\widetilde{\lambda}_i - \widetilde{b}_i)^2 + \sum_{i=n_Z+1}^{n_X+n_Z}\widetilde{b}_i^2 \quad \text{s.t.} \quad \sum_{j=1}^{n_{Y_0}+n_{Y_1}}(\beta_j\widetilde{\lambda}_j - \widetilde{d}_j)^2 \le E^2,$$

with $\widetilde{\Lambda}_Z = \{\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_{n_Z}\}$. The minimization without regards to constraints given as

$$\widetilde{\lambda}_i = \begin{cases} \widetilde{b}_i/\alpha_i, & \alpha_i \ne 0, \\ \widetilde{d}_i/\beta_i, & \alpha_i = 0. \end{cases}$$

If the above unconstrained solution does not satisfy the constraint, the solution of the LSQI problem occurs on the boundary of the feasible set. Therefore, we need only find the solution of the least-squares problem with the equality constraint condition

$$\arg\inf_{\Lambda_Z\in\mathbb{R}^{n_Z}} \quad \|D_A\widetilde{\Lambda}_Z-\widetilde{b}\|_2 \quad \text{s.t.} \quad \|D_B\widetilde{\Lambda}_Z-\widetilde{d}\|_2=E.$$

To solve the above optimization problem, we use the method of Lagrange multipliers. The Lagrange function is defined as:

$$h(\eta,\widetilde{\Lambda})=\|D_A\widetilde{\Lambda}_Z-\widetilde{b}\|_2^2+\eta(\|D_B\widetilde{\Lambda}_Z-\widetilde{d}\|_2^2-E^2).$$

By making derivatives of h with respect to $\widetilde{\lambda}_i,\ i=1,\ldots,n_Z$ equal zero, we obtain the following equation system:

$$(D_A^TD_A+\eta D_B^TD_B)\widetilde{\Lambda}_Z=D_A^T\widetilde{b}+D_B^T\widetilde{d}.$$

The solution of $\widetilde{\lambda}$ can obtained with respect to Lagrange parameter η by solving the above equations system

$$\widetilde{\lambda}_i(\eta)=\begin{cases}\dfrac{\alpha_i\widetilde{b}_i+\eta\beta_i\widetilde{d}_i}{\alpha_i^2+\eta\beta_i^2}, & 1\le i\le n_{Y_0}+n_{Y_1},\\ \dfrac{\widetilde{b}_i}{\alpha_i}, & n_{Y_0}+n_{Y_1}+1\le i\le n_Z.\end{cases}$$

We are left to evaluate the Lagrange parameter η, which can be obtained by solving the scaler secular equation:

$$\phi(\eta)=\|D_B(\widetilde{\Lambda}_Z(\eta)-(D_B)^{-1}\widetilde{d})\|_2^2=E^2.$$

It was shown in [12] that the above scaler secular equation has a unique solution η^* and that it can be obtained by, say, Newton iteration with a Hebden model as in [4]. Finally, the coefficients Λ_Z for the LSQI problem (3.1) can evaluated by the relation $\Lambda_Z=X\widetilde{\Lambda}_Z$.

4. Numerical experiments. In this section, we test the accuracy and efficiency of the proposed method in Section 3 for solving Cauchy problems by comparing them with other nonlinear solvers. We study convergence behavior of numerical results with respect to noise levels and the fill distance of trial centers. By comparing our numerical results with the MFS and the finite element method (FEM), we further show the effectiveness of our method.

Noisy Cauchy data are utilized to test the robustness of the algorithm proposed in Section 3. Cauchy data with noise are generated by the same method as in [31] and [34]

$$g_i^\delta=g_i^*+\delta\max_{y\in\Gamma}|g_i^*|rand(\xi)\quad\text{for } i=0,1,$$

where $rand(\xi)$ is a uniformly random number in $[-1,1]$ for each component and δ is the level of noise. In all numerical experiments, we compute relative errors over the domain as

$$E_r(u^\delta_{\sigma,X,Y_0,Y_1})=\frac{\|u^*-u^\delta_{\sigma,X,Y_0,Y_1}\|_{L^2(\Omega)}}{\|u^*\|_{L^2(\Omega)}}$$

and pointwise relative errors on evaluation points as:

$$E(u^\delta_{\sigma,X,Y_0,Y_1})(i) = \frac{|u^*(i) - u^\delta_{\sigma,X,Y_0,Y_1}(i)|}{\max\{|u^*|\}}$$

as in [31] and [34] for the sake of comparison. The unscaled Whittle-Matérn-Sobolev kernel

$$\Phi_m(x) := \|x\|_2^{m-d/2}\mathcal{K}_{m-d/2}(\|x\|_2) \quad \text{for all } x \in \mathbb{R}^d,$$

satisfying (2.4) is used in all numerical examples, and K_ν is the Bessel function of the second kind.

When solving the LSQI problem (2.14) numerically, the value of $\widetilde{M}$, which appears in the upper bound of the inequality constraint, is required. As its value cannot be evaluated exactly from Lemma 2.6, we take $\widetilde{M} = 1$ in all numerical tests. Boundary collocation sets Y_0 and Y_1 are given as $h_{Y_0} = h_{Y_1}$ and we use h_Y as a simple notation. The elliptic operator is chosen to be the Laplacian operator in all examples.

4.1. Robustness of the proposed solver. Besides the LSQI solver introduced in section 3, other nonlinear solvers can also be applied to problem (3.1). We show numerical solutions by different solvers in this part. Cauchy data were generated from exact solutions

$$u^* := x^3 - 3xy^3 + e^{2y}\sin(2x) - e^y\cos(x).$$

The problem is solved in the domain $\Omega := [-1, 1] \times [0, 1]$ under the Cauchy boundary $\Gamma := \partial\Omega \setminus [-1, 1] \times \{1\}$. We use $h_Y \in \{0.07, 0.08, 0.10\}$. Regularly distributed trial centers Z and collocation points X are constructed such that $h_X = h_Z = h_Y$. Relative errors are approximated by using 60^2 uniform grid. Kernel smoothness is required as $m \geq 2 + d/2$, and we test $m \in \{3, 4\}$. Three nonlinear solvers are used to solve the quadratic constraint least-squares problems (3.1):

1 LSQI solver introduced in Section 3,
2 SDPT3 solver in MATLAB CVX toolbox [13, 14], and
3 MOSEK [2].

To be consistent with other papers, we use the value of δ in Eq. (4.1) to measure noise and a logarithmically spaced noise level vector δ with 10 elements between 10^{-6} to 10^{-1} is used. L^2 errors obtained by the three solvers are shown in Figure 4.1. As the exact same optimization problem is solved by different solvers when kernel smoothness m and sets X, Z and Y are fixed, the same solutions should be obtained if numerical errors are ignored. From Figure 4.1, almost identical L^2 errors are obtained by all three solvers when the problems are solved with kernel smoothness $m = 3$. For higher kernel smoothness $m = 4$, MOSEK solver failed to solve the problem for some δ when $h_Y \in \{0.07, 0.08\}$, and SDPT3 solver could not obtain solutions for most tested cases except for one successful case when $h_Y = 0.10$ and $\delta = 10\%$.

In cases when SPDT3 and MOSEK converged, the three solvers yielded the same L^2 errors. When considering CPU times, the LSQI solver was the fastest of the three under the same problem setting as only a nonlinear scaler equation needs to be solved. SDPT3 solver consumed the most CPU time. Thus, in the following numerical tests, we use only the LSQI solver for solving problem (3.1).

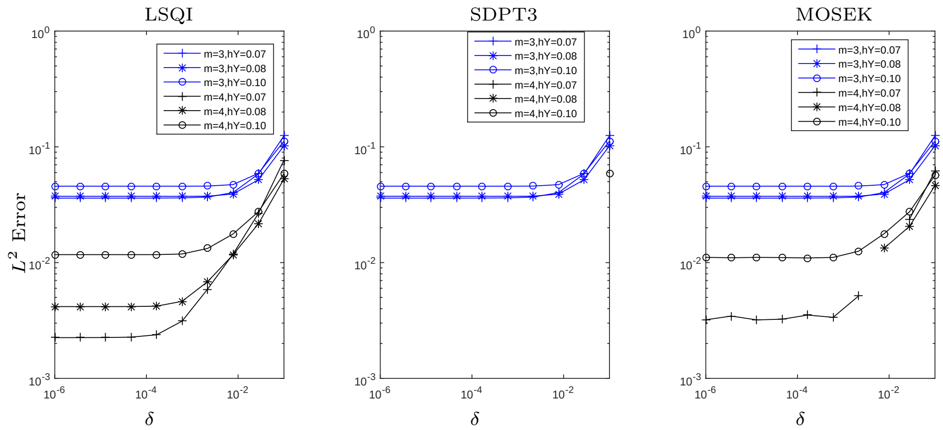


Fig. 4.1: L^2 errors by three nonlinear solvers for example $u^* = x^3 - 3xy^3 + e^{2y}\sin(2x) - e^y\cos(x)$ with Cauchy boundary $\partial\Omega\backslash[-1,1]\times\{1\}$, $h_Y \in \{0.07, 0.08, 0.10\}$ and $m \in \{3, 4\}$

4.2. Convergence with respect to δ and h_Z. From Theorem 2.9, numerical solutions converge to exact solutions with respect to a log-rate of noise levels δ, fill distances h_Z^{m-2} and $h_Y^{m-3/2}$. For convergence tests, we use an example with exact solutions $u^* := x^3 - 3xy^3 + e^{2y}\sin(2x) - e^y\cos(x)$ in domain $\Omega := [-1,1]\times[0,1]$ under two kinds of Cauchy boundaries

$$\Gamma_1 : \partial\Omega \setminus [-1,1]\times\{1\},$$

and

$$\Gamma_2 : [-1,1]\times\{0\}.$$

Regularly distributed collocation points and trial centers satisfying $h_X = h_Z = h_Y$ are used.

In noise-free case $\delta = 0$, Figure 4.2 shows the L^2 error for $m \in \{3, 3.5, 4\}$ against the fill distance of trial centers h_Z when h_Z is a logarithmically spaced vector with 8 elements between $10^{-1.2}$ and $10^{-0.4}$. Because the ratio of the Cauchy boundary to the whole boundary influences the convergence behavior, the results of the two tested Cauchy boundaries are slightly different. For Cauchy boundary Γ_0, convergence rates are between 1 and 3 for $m \in \{3, 3.5, 4\}$. Slower rates between 0.3 to 1 are observed for the smaller boundary Γ_2. In both cases, a larger m yields a faster convergence rate.

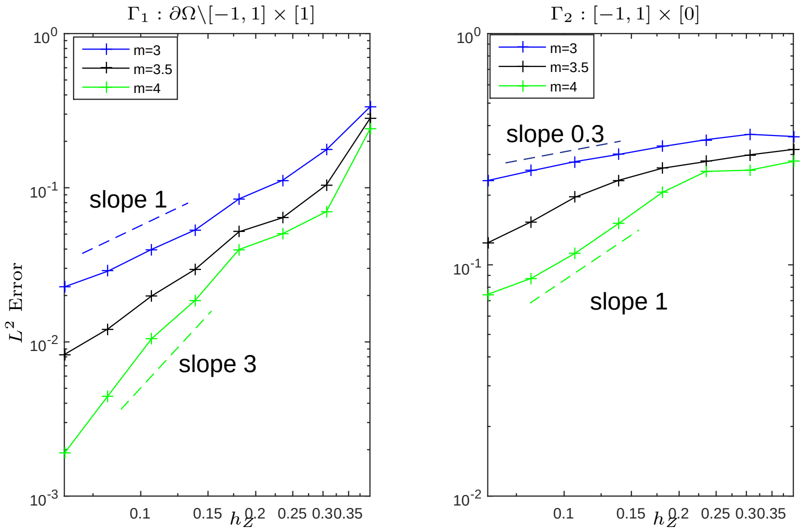
We use $h_Y \in \{0.07, 0.09\}$ and kernel smoothness $m = 4$ to test convergence behavior with respect to $10^{-6} \le \delta \le 10^{-1}$. Figure 4.3 plots the L^2 errors of our reconstructed solutions based on Cauchy boundaries Γ_1 (a) and Γ_2 (b). The L^2 errors for both Cauchy boundaries first decrease linearly at a rate of 0.5 and then stop at a rate indicating noise-free accuracy as δ approaches zero .

4.3. Comparison with other numerical methods. In this section, we consider two examples with Cauchy data generated from

$$u_1^* = x^3 - 3xy^3 + e^{2y}\sin(2x) - e^y\cos(x),$$

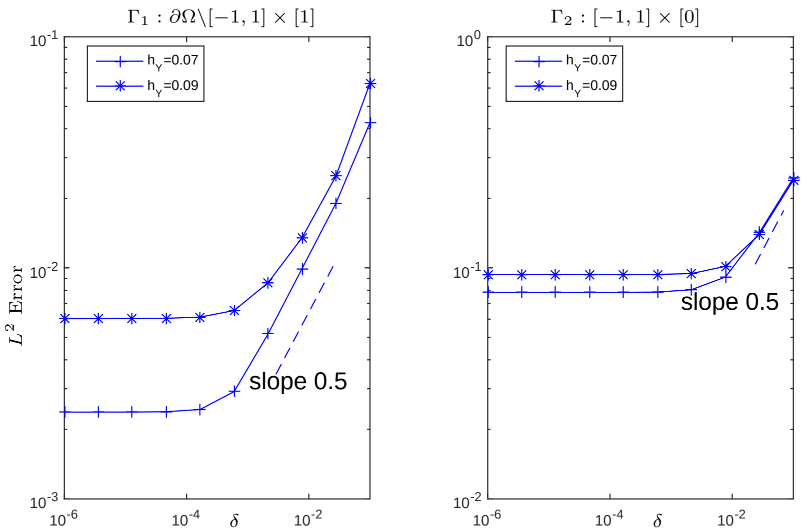
and

$$u_2^* := \cos(\pi x)\cosh(\pi y).$$



(a) (b)

Fig. 4.2: L^2 error profiles by LSQI solvers for example $u^* = x^3 - 3xy^3 + e^{2y}\sin(2x) - e^y\cos(x)$ in noisefree case when $m \in \{3, 3.5, 4\}$, Cauchy boundary $\Gamma_1 : \partial\Omega\backslash[-1,1] \times [1]$ (a) and $\Gamma_2 : [-1,1] \times [0]$ (b)



(a) (b)

Fig. 4.3: L^2 errors by LSQI solver for example $u^* = x^3 - 3xy^3 + e^{2y}\sin(2x) - e^y\cos(x)$ when $h_Y \in \{0.07, 0.09\}$, $m = 4$ and Cauchy boundary $\Gamma_1 : \partial\Omega\backslash[-1,1] \times [1]$ (a) and $\Gamma_2 : [-1,1] \times [0]$ (b)

The FEM with discrete Tikhonov regularization based on RKHS was applied to both examples in [34]. The method of fundamental solution combined with Tikhonov regularization was used to reconstruct solutions in [31]. Figure 4.4 shows the exact solution in the domain Ω of u_1^* and u_2^*. For a fair comparison, we use the same Cauchy data as in [34] ($h_Y = 0.02$) and [31] ($h_Y = 0.024$). These two examples are solved in the domain $\Omega := [-1, 1] \times [0, 1]$ under two kinds of Cauchy boundaries

$$\Gamma_1 : \partial\Omega \setminus [-1, 1] \times \{1\},$$

and

$$\Gamma_2 : [-1, 1] \times \{0\}.$$

We first consider the example with exact solution u_1^*. Fill distances of the collocation set and trial centers are taken as $h_Z = h_X = 0.06$, and the kernel smoothness is set to $m = 4$. An L^2 error comparison of accuracy obtained by our proposed solver and other numerical methods provided in Table 4.1 for Cauchy boundary Γ_1 and in Table 4.2 for Cauchy boundary Γ_2. Compared with RKHS in [34], comparable solutions are obtained by our method for all δ in both tested Cauchy boundaries. Except for the noise-free case, the same order of accuracy is obtained by our LSQI solver as that shown in the results by MFS in [31].

In the other test solution u_2^*, Cauchy data on Γ_1 and Γ_2 are flatter than data on the missing boundary at $[-1, 1] \times \{1\}$. Importantly, the Neumann data g_1^δ remains zero for all δ. These conditions make this example special and harder to solve than the other example. When we use the LSQI solver in Section 3 to solve the problem (3.1), the numerical solution may not be as accurate as the others because the Dirichlet and Neumann boundaries are considered together in inequality constraints in problem (3.1). To make use of the zero Neumann boundary for all noise levels as the other two methods did, we consider the Neumann boundary separately by imposing an equality constraint. Instead of LSQI problem (3.1), we solve the following least-squares problems with quadratic constraints on the Dirichlet boundary and an equality constraint on the Neumann boundary (LSQIEC)

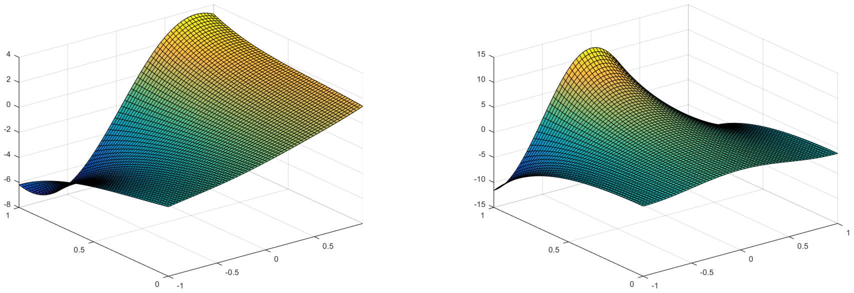
$$\begin{aligned} &\arg\inf_{\lambda \in \mathbb{R}^{nZ}} \quad \|A\lambda - b\|_2 \\ &\text{s.t.} \quad \|B_0\lambda - d_0\|_2 \le E_0 \quad \text{and} \quad B_1\lambda = d_1, \end{aligned} \tag{4.1}$$

with A and b being the same as in Eq. (3.1) and

$$\begin{aligned} B_0 &= h_Y^{(d-1)/2} K(Y, Z), \quad & B_1 &= \partial_{\mathcal{L}} K(Y, Z), \\ d_0 &= h_Y^{(d-1)/2} g_0^\delta|_Y, & d_1 &= g_1^\delta|_Y, \\ E_0 &= h_Z^{m-d/2} \widetilde{M} + \delta_0, & \delta_0 &= h_Y^{(d-1)/2} \|g_0^* - g_0^\delta\|_Y. \end{aligned}$$

The equality constraint can be handled by the null space approach in [28]. Unknown coefficients are expressed as $\lambda = \mathcal{N}_B\gamma + B_1\backslash d_1$. For the new unknown vector γ, substituting the above expression into the objective function and inequality constraint on the Dirichlet boundary yields the following problem:

$$\begin{aligned} &\arg\inf_{\lambda \in \mathbb{R}^{nZ}} \quad \|A\mathcal{N}_B\gamma - b + A(B_1\backslash d_1)\|_2 \\ &\text{s.t.} \quad \|B_0\mathcal{N}_B\gamma - d_0 + B_0(B_1\backslash d_1)\|_2 \le E_0. \end{aligned} \tag{4.2}$$



(a) (b)

Fig. 4.4: Exact solutions of two tested examples: $u^* = x^3 - 3xy^3 + e^{2y}\sin(2x) - e^y\cos(x)$ (a) and $u^* = \cos(\pi x)\cosh(\pi y)$ (b)

	Results for $h_Y = 0.020$		Results for $h_Y = 0.024$	
δ	LSQI	RKHS [34]	LSQI	MFS [31]
0.00	0.0014	0.0043	0.0012	$1.6 * 10^{-5}$
0.01	0.0064	0.0106	0.0068	0.0053
0.05	0.0170	0.0218	0.0180	0.0167
0.10	0.0258	0.0425	0.0287	0.0332

Table 4.1: Relative errors compare of $u^* = x^3 - 3xy^3 + e^{2y}\sin(2x) - e^y\cos(x)$ when Cauchy boundary given on $\partial\Omega \setminus [-1,1]\times\{1\}$, $h_X = h_Z = 0.06$ and $m = 4$

This is again an LSQI problem that can be solved by our solver. For Cauchy boundary Γ_1, fill distance as $h_Z = h_X = 0.05$ and $m = 4$, we show the L^2 error by both the LSQI and LSQIEC solutions in Table 4.3. The accuracy of solution improved for all noise levels after imposing an equality constraint on the Neumann boundary, especially for small noise level ($\delta \le 0.01$). From the third and fourth columns of Table 4.3, LSQIEC gives comparable results with those from RKHS. When compared with MFS, we again obtained better solutions by LSQIEC except in the noise-free case (see the last two columns of Table 4.3).

Table 4.4 shows the results for Cauchy boundary Γ_2. We use $h_X = h_Z = 0.04$. The Sobolev kernel with $m = 4$ is used for LSQI. For LSQIEC, we show results for both $m = 4$ and $m = 5$. The LSQIEC results are clearly improved over those of LSQI. Better results were also obtained for large noise levels compared with RKHS and MFS. For small noise levels, comparable solutions obtained by LSQIEC (m=5) with other two methods except in noise-free case by MFS.

Figure 4.5 plots the numerical solutions of $\delta \in \{0\%, 10\%\}$ and $h_Y = 0.024$ by LSQI solver recovered from Cauchy boundary Γ_2 for u_1^*. Blue points indicate the exact solution values on the missing boundary $\partial\Omega/\Gamma$. Figure 4.6 are the numerical solutions of $\delta \in \{0\%, 10\%\}$, $h_Y = 0.024$ and $m = 5$ under the Cauchy boundary Γ_2 for u_2^* obtained by LSQIEC. Although large errors appear on the missing boundary in both examples, reconstruction solutions can give reasonable approximations of the overall shape of the exact solutions.

Conclusion. We give both theoretical and numerical studies for kernel-based collocation methods for inverse Cauchy problems. We use kernels reproducing $H^m(\Omega)$

	Results for $h_Y = 0.020$		Results for $h_Y = 0.024$	
δ	LSQI	RKHS [34]	LSQI	MFS [31]
0.00	0.0690	0.0428	0.0691	$8.6 * 10^{-4}$
0.01	0.0769	0.0507	0.0784	0.0696
0.05	0.1109	0.2449	0.1144	0.1023
0.10	0.1555	0.2797	0.1631	0.1869

Table 4.2: Relative errors for Cauchy problems with $u_1^* = x^3 - 3xy^3 + e^{2y}\sin(2x) - e^y\cos(x)$ and Cauchy boundary $[-1, 1] \times \{0\}$, $h_X = h_Z = 0.06$ and $m = 4$

	Results for $h_Y = 0.020$			Results for $h_Y = 0.024$		
δ	LSQI	LSQIEC	RKHS [34]	LSQI	LSQIEC	MFS [31]
0.00	0.0181	0.0064	0.0037	0.0126	0.0063	$1.5 * 10^{-4}$
0.01	0.0188	0.0063	0.0046	0.0129	0.0067	0.0074
0.05	0.0208	0.0106	0.0198	0.0155	0.0125	0.0365
0.10	0.0223	0.0159	0.0292	0.0175	0.0176	0.0831

Table 4.3: Relative errors compare of $u^* = \cos(\pi x)\cosh(\pi y)$ when Cauchy boundary given on $\partial\Omega \setminus [-1, 1] \times \{1\}$ for $h_X = h_Z = 0.05$ and $m = 4$

and all analysis is provided in Hilbert space. A solver for LSQI problem by generalized singular value decomposition of matric and method of Lagrange multiplier is used to obtain solutions of Cauchy problems. The convergence of the algorithm respect to noise levels and fill distances of collocations sets and trial set is proved. For stable reconstruction, we use Tikhonov regularization with a priori choice of the regularization parameter.

Numerical examples verified our proved convergence results with respect to noise level and fill distance of trial centers. Robustness of our proposed method to noisy Cauchy data can be seen when compared with other numerical methods. High accuracy results show that the method can be applied to various Cauchy problems.

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δ	Results for $h_Y = 0.020$				Results for $h_Y = 0.024$			
	LSQI	LSQIEC (m=4)	LSQIEC (m=5)	RKHS [34]	LSQI	LSQIEC (m=4)	LSQIEC (m=5)	MFS [31]
0.00	0.2065	0.1932	0.0790	0.0667	0.2076	0.1934	0.0796	0.0029
0.01	0.2096	0.1964	0.1036	0.0781	0.2180	0.1994	0.0977	0.1241
0.05	0.2204	0.2090	0.1444	0.3186	0.2454	0.2210	0.1462	0.2817
0.10	0.2304	0.2253	0.1588	0.3149	0.2687	0.2498	0.1750	0.3271

Table 4.4: Relative errors comparison in Ω for Cauchy problems with $u_2^* = \cos(\pi x)\cosh(\pi y)$ and Cauchy boundary given on $[-1,1] \times \{0\}$ for $h_X = h_Z = 0.04$, $m = 4$ for LSQI and $m \in \{4,5\}$ for LSQIEC

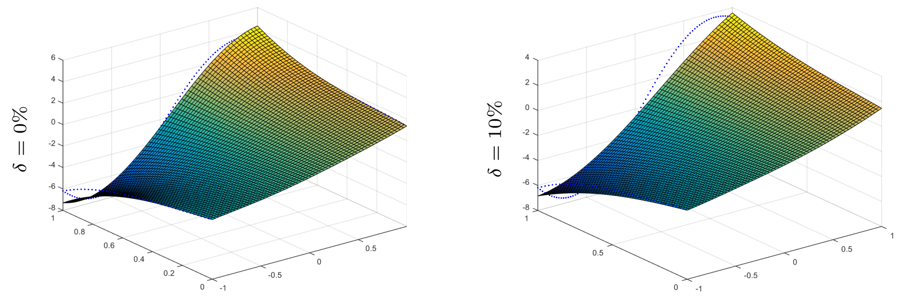

Fig. 4.5: Numerical errors for example $u^* = x^3 - 3xy^3 + e^{2y}\sin(2x) - e^y\cos(x)$ with Cauchy boundary $[-1,1] \times \{0\}$ with $h_Y = 0.024$, $h_Z = h_X = 0.06$, $m = 4$ and $\delta = \{0, 10\%\}$ by LSQI (Blue points are exact solutions on missing boundary)

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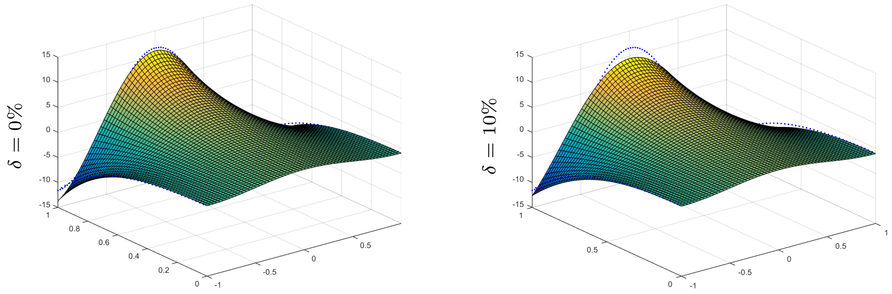

Fig. 4.6: Numerical errors for example $u^* = \cos(\pi x)\cosh(\pi y)$ with Cauchy boundary $[-1, 1] \times \{0\}$ with $h_Y = 0.024$, $h_Z = h_X = 0.04$, $m = 5$ and $\delta = \{0, 10\%\}$ by LSQIEC (Blue points are exact solutions on missing boundary)

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