A numerical search for a singularity of 2D inviscid Boussinesq approximation equations

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- 2D inviscid incompressible Boussinesq approximation equations without mean temperature gradient are (the temperature T, the velocity $\boldsymbol{u}$) :

$$\partial_t\, T + (\boldsymbol{u}\cdot\nabla)T = 0, \tag{1}$$

$$\partial_t\, \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{\nabla p}{\rho_0} + \begin{pmatrix} 0 \\ \alpha g T \end{pmatrix}, \tag{2}$$

$$\nabla\cdot\boldsymbol{u} = 0. \tag{3}$$

  The vorticity $\omega \equiv \partial_x v - \partial_y u$ obeys (the stream function ψ)

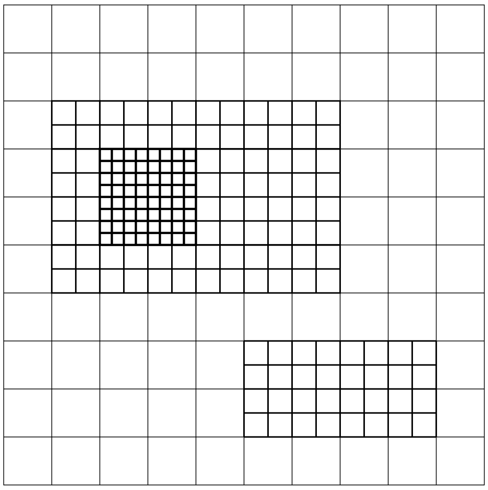
$$\partial_t\, \omega + \frac{\partial(\psi,\omega)}{\partial(x,y)} = \alpha g\, \partial_x T, \tag{4}$$

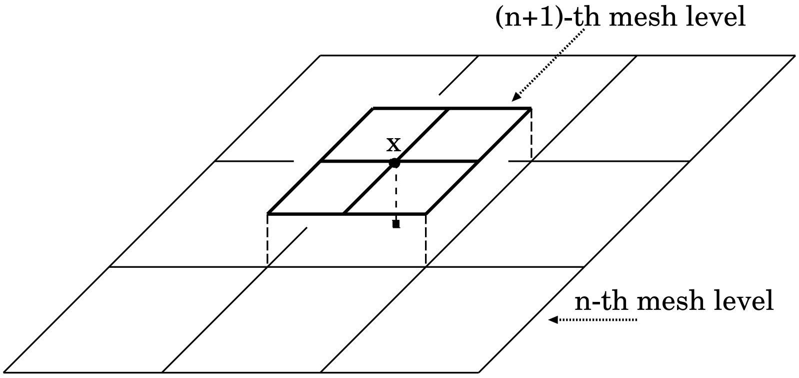
$$\nabla^2\psi = -\omega. \tag{5}$$

- We solve Eqs. (1) and (4) in a doubly periodic domain $[0,2\pi]\times[0,2\pi]$ with an *adaptive mesh refinement* (AMR) technique based on a finite-difference scheme.

- Our goal here is to capture a possible finite-time blow-up of the vorticity and the temperature gradient.

- This problem has been investigated by several authors:
  - Pumir and Siggia (1992), *Phys. Fluids A* **4**, 1472.
    An adaptive mesh refinement method, blow-up: positive.
  - E and Shu (1994), *Phys. Fluids* **6**, 49.
    Two spatial discretization schemes, blow-up: negative.
  - Ceniceros and Hou (2001), *J. Comput. Phys.* **172**, 609.
    A moving non-uniform mesh method, blow-up: negative.



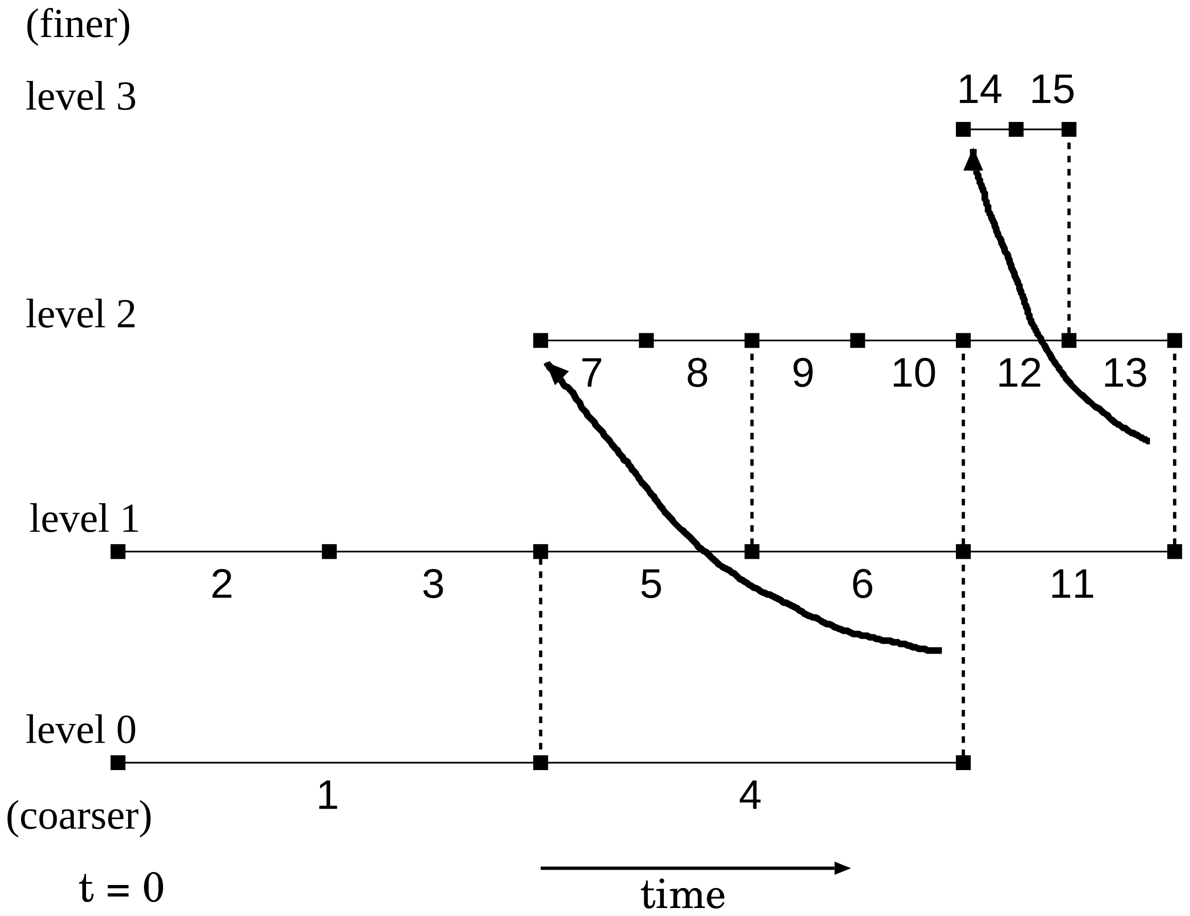


- Adaptive mesh refinement method (Berger and Oliger 1984)
  Finer meshes are adaptively placed only in the sub-regions where high resolution is needed.
  - How can the lack of the resolution be detected?
    - The difference of T^2 between two mesh levels is checked.
      If the 'error' between data at a grid point $\boldsymbol{x}$ on a fine mesh (mesh level $n+1$) and interpolated data at $\boldsymbol{x}$ from a coarser mesh data (mesh level n) exceeds some threshold ϵ :
      $$|T^2(\boldsymbol{x}) - T^2_{\text{interpolated}}(\boldsymbol{x})| \geq \epsilon, \tag{6}$$
      then we decide to put the next finer mesh (level $n+2$) to cover such points $\boldsymbol{x}$.
  - How can the data for a new finer mesh be constructed?
    The new mesh are filled with the data interpolated from the corresponding coarser mesh.
    - When should the interpolation be done?
    - Not the instance when the error exceeds the threshold.
      ⇒ Time-rewinding method
  - How can the Poisson equation ($\nabla^2\psi = -\omega$) be handled?
    Using a multigrid iteration scheme, we can solve the Poisson equation consistently in both the finer and the coarser meshes.

- Time-Rewinding (MIYASHITA and Yamada 1999)
  The data for a new adapted-mesh should not be interpolated from the coarser mesh data already lacking in resolution.

  Instead,

  - we preserve all the field data at every error-checking time.
  - When a new finer mesh is to be adapted, the data for this adapted-mesh are interpolated from the preserved coarser-mesh data. And we redo the simulation with the finer mesh.



- In each mesh, we employ:
  - spatial 2nd order MUSCLE scheme
  - temporal 1st order forward Euler scheme
    - Time step is determined by the CFL condition in each mesh.
  - Poisson solver 2nd order scheme

- The initial condition

Two temperature fronts and associated shear.

For $x \leq \pi$,

$$\omega(x,y) = \frac{1}{10} \exp\left[-\frac{1}{2}\left(\frac{x-x_c}{w}\right)^2\right], \tag{7}$$

$$T(x,y) = \frac{1}{2}\left\{1+\frac{2}{\sqrt{\pi}}\int_0^x \exp\left[-\frac{1}{2}\left(\frac{\xi-x_c}{w}\right)^2\right] d\xi\right\}, \tag{8}$$

$$x_c = \frac{\pi}{2} - a\sin y. \tag{9}$$

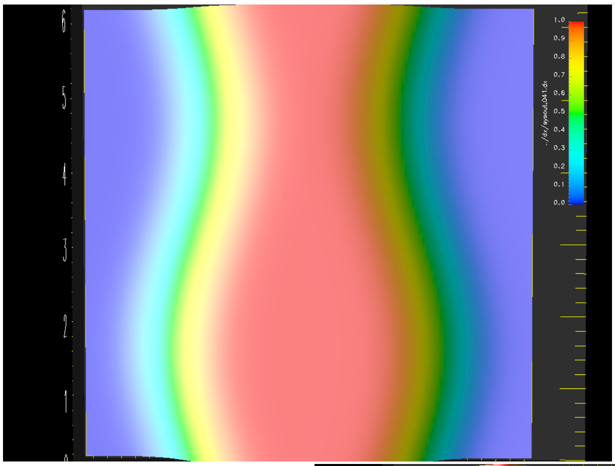
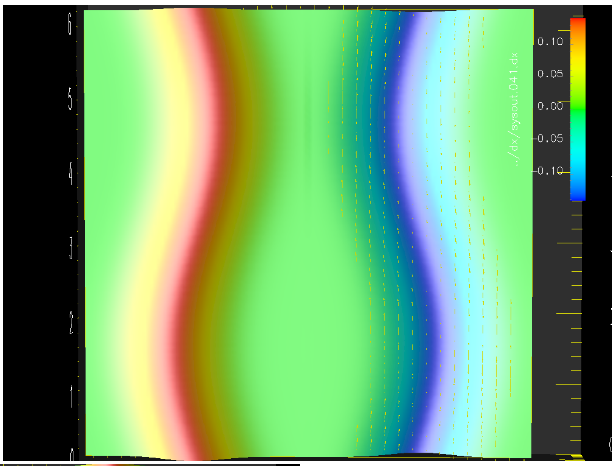
For $x \geq \pi$,

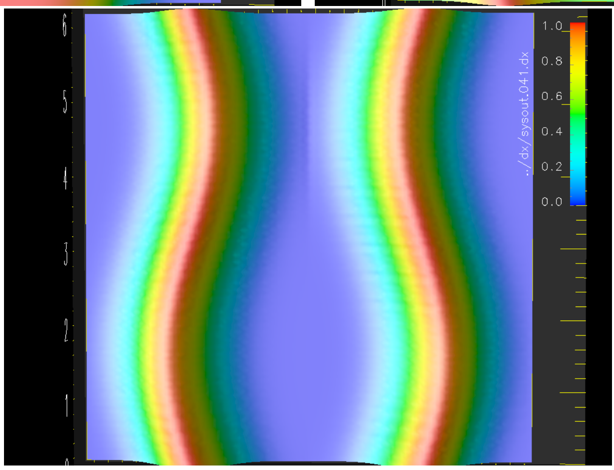
$$\omega(x,y) = -\frac{1}{10} \exp\left[-\frac{1}{2}\left(\frac{x-x_c}{w}\right)^2\right], \tag{10}$$

$$T(x,y) = \frac{1}{2}\left\{1-\frac{2}{\sqrt{\pi}}\int_0^x \exp\left[-\frac{1}{2}\left(\frac{\xi-x_c}{w}\right)^2\right] d\xi\right\}, \tag{11}$$

$$x_c = \frac{3\pi}{2} + a\sin y. \tag{12}$$

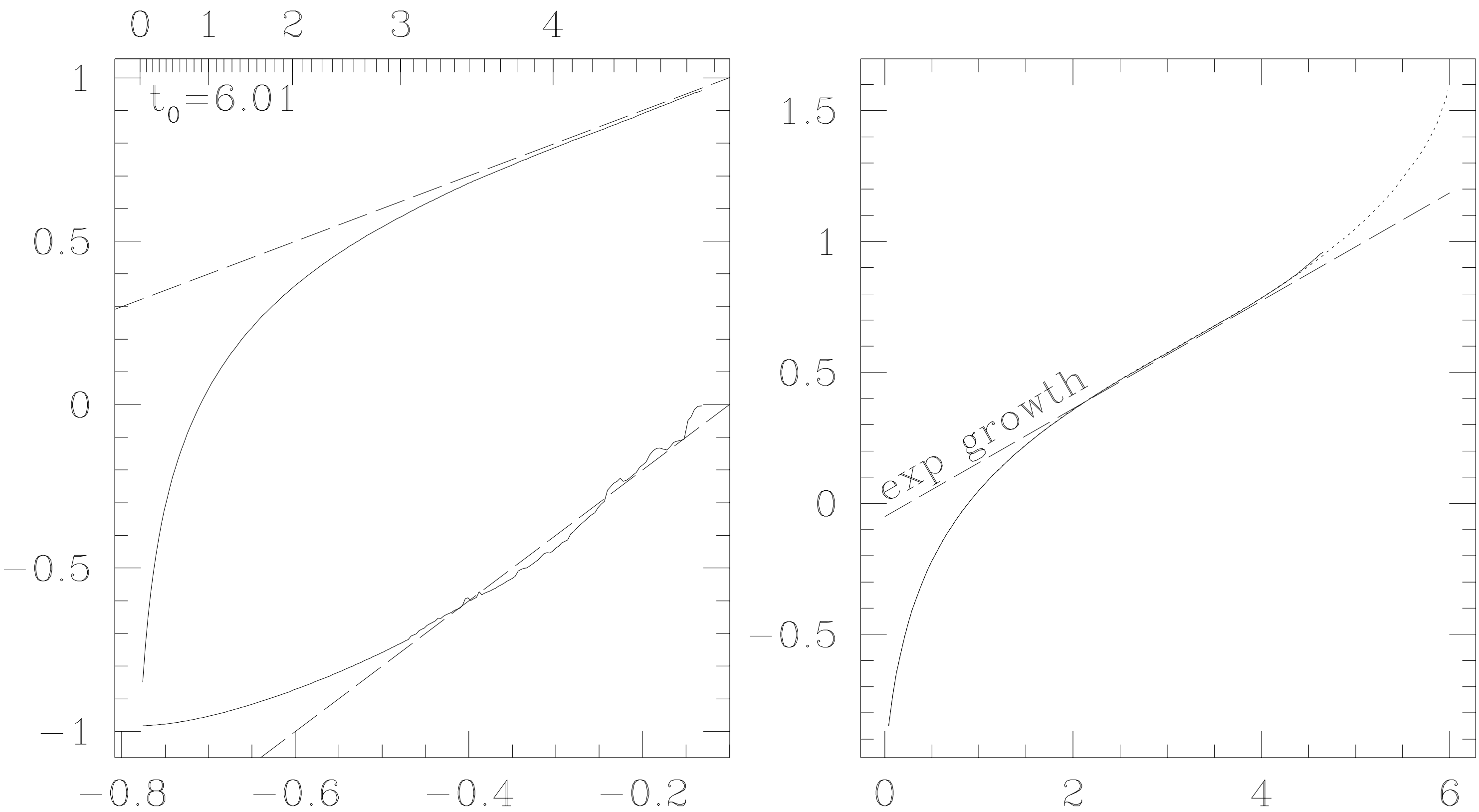
We set $a = 0.3, w = 0.4$.



- Configuration of adapted meshes

- Result: $|\omega|_{\max}$, $|\nabla T|_{\max}$ versus t



- At the early stage, $|\omega|_{\max}$ and $|\nabla T|_{\max}$ grow exponentially.
- Then, $t > 4$, they increase more rapidly. The fitting indicates

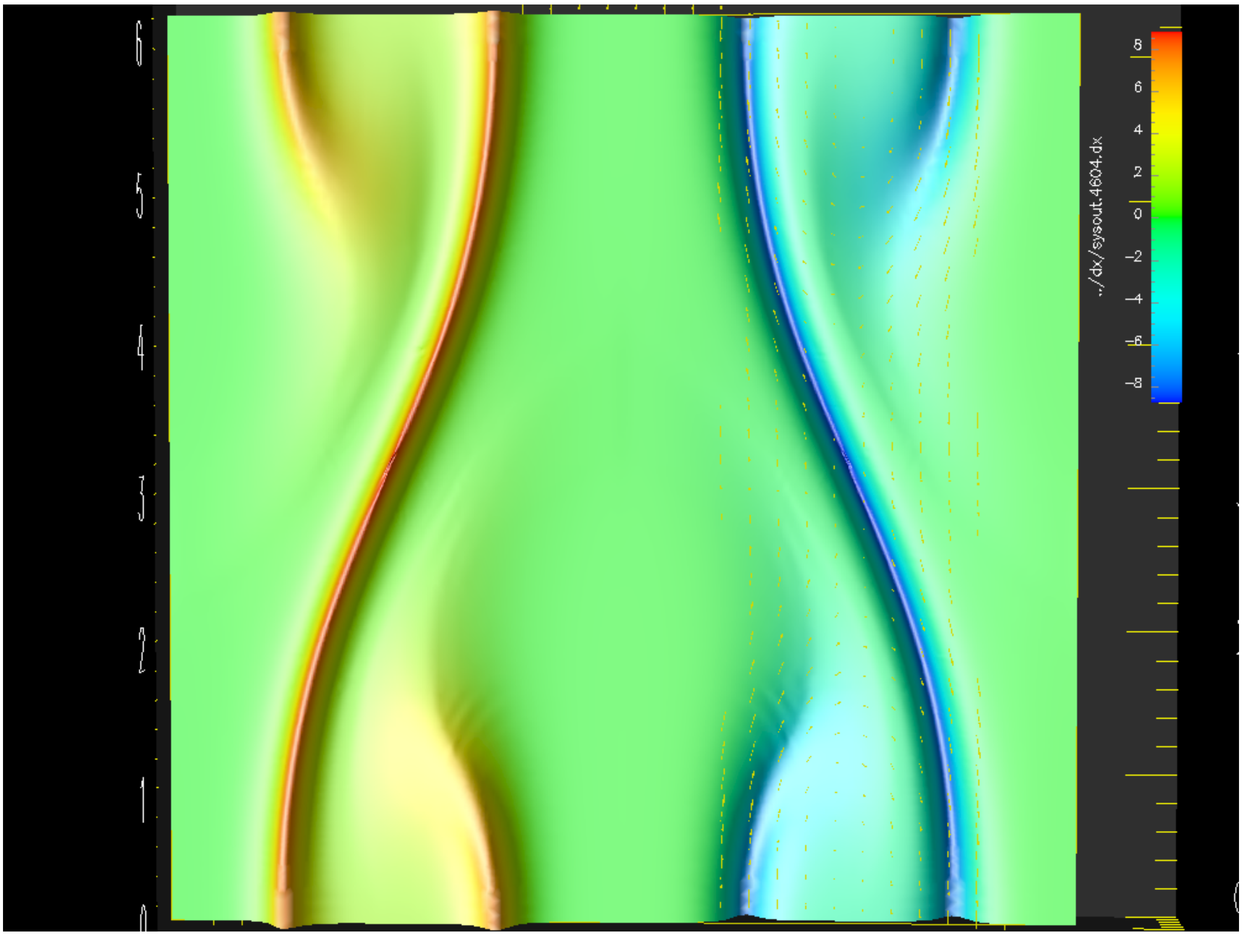
$$|\omega|_{\max} \propto (t_0 - t)^{-1}, \tag{13}$$
$$|\nabla T|_{\max} \propto (t_0 - t)^{-2}, \tag{14}$$

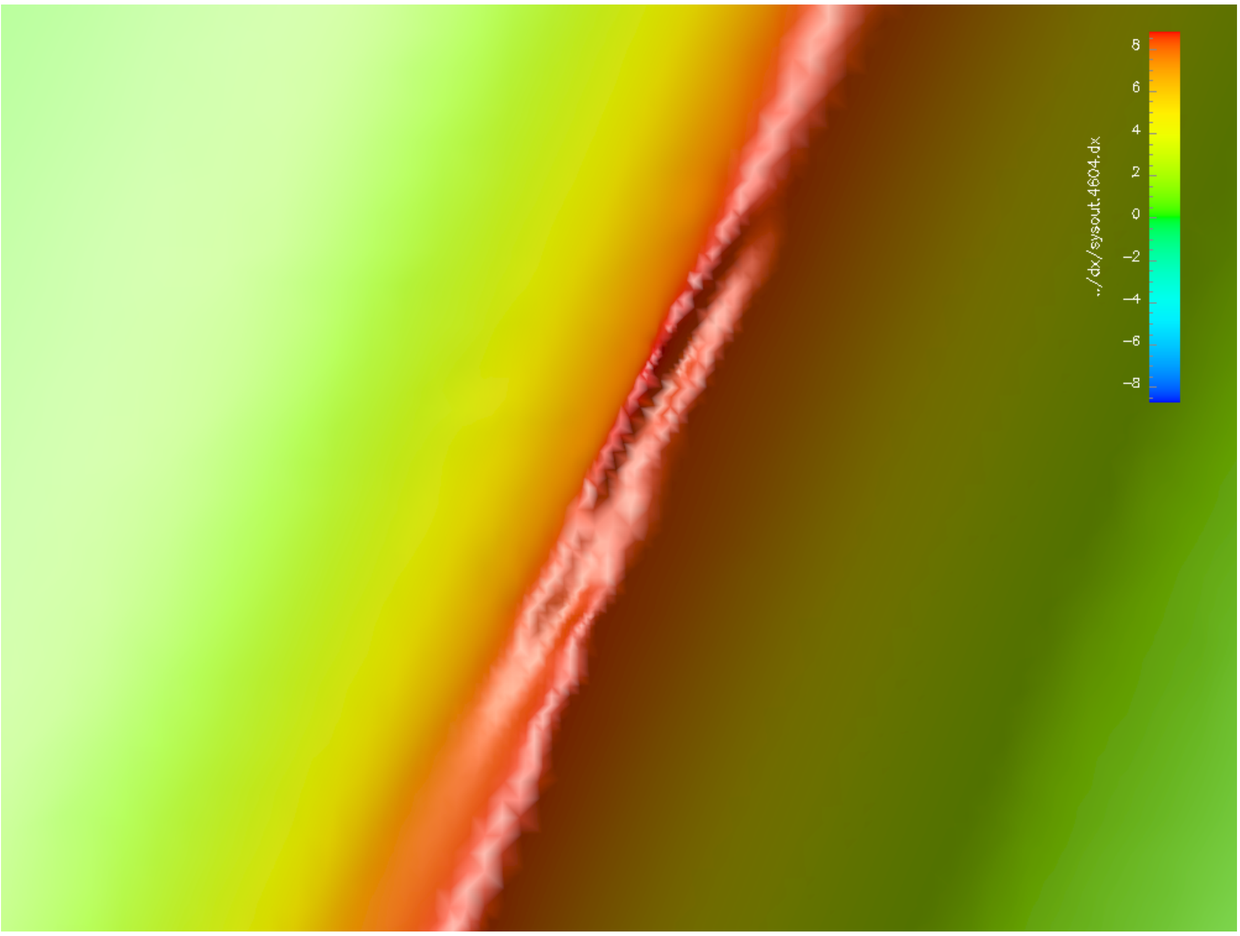
  where $t_0 \approx 6.0$.
- The smallest adapted-mesh size is $2\pi/2^{15} = 2\pi/32768$ around $t = 4.6$.
- The quantities

$$\int T^2 d\boldsymbol{x}, \tag{15}$$
$$\int T^4 d\boldsymbol{x}, \tag{16}$$

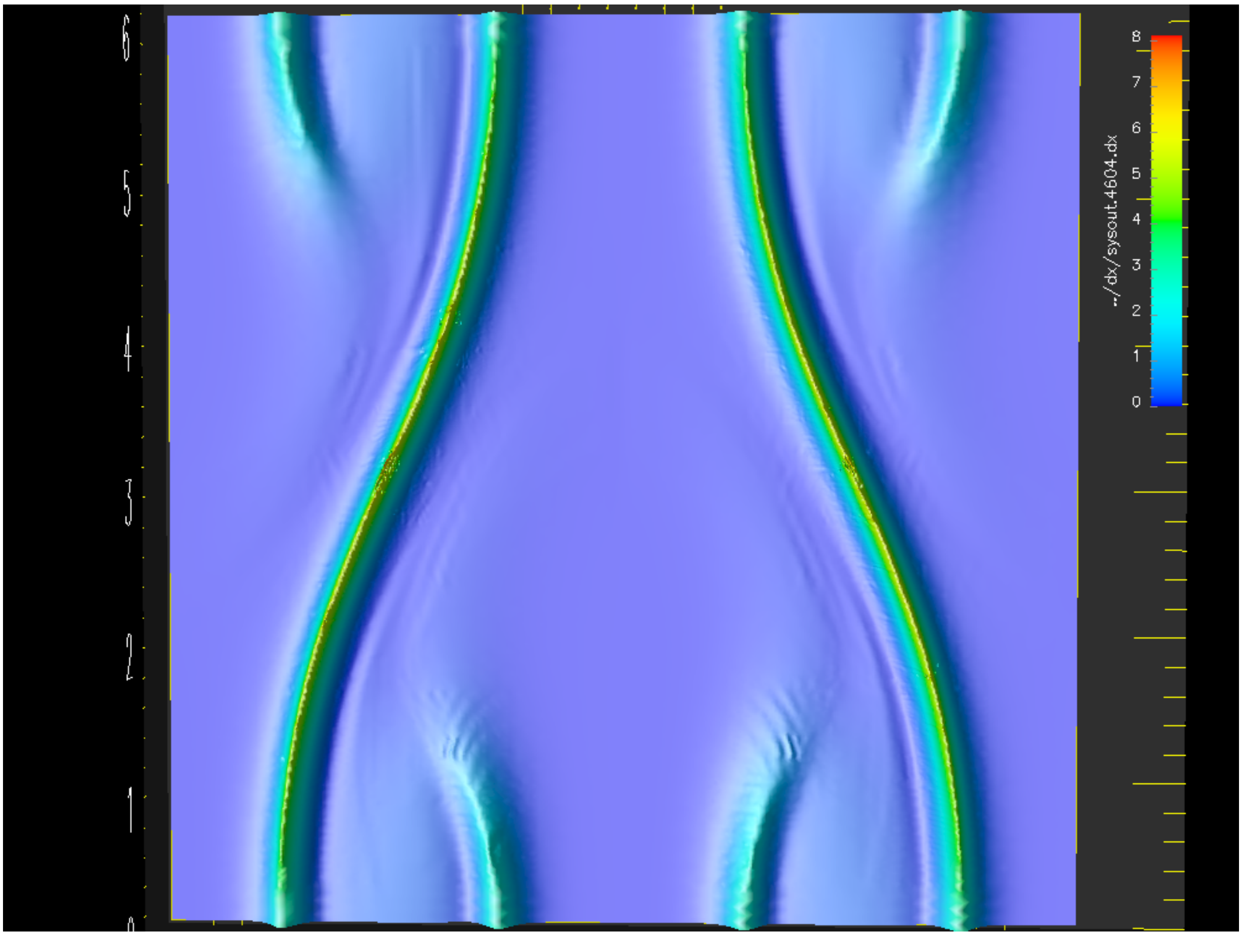
  are conserved within 0.4% error.

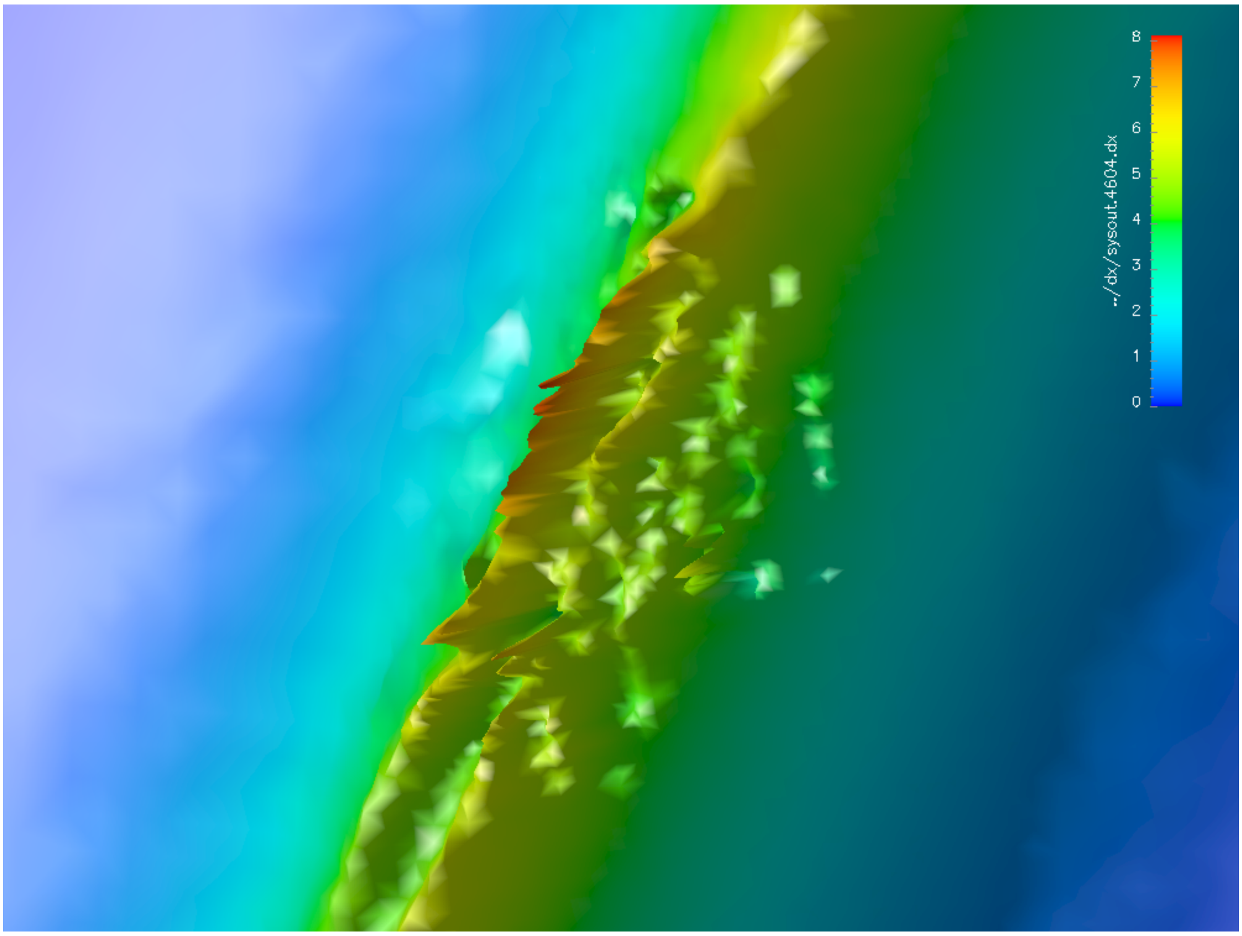
- Spatial structure around a singularity point
  ω, contour, t = 4.604 (after the exponential growth regime)





• Spatial structure around a singularity point
$|\nabla T|$ contour, t = 4.604 (after the exponential growth regime)





- Discussions
  - The instability of the temperature front leads to the finite-time blow-up.
  - Is it a real blow-up of the 2D Boussinesq equations?
    - The possibility that the blow-up is an artifact of AMR cannot be ruled out.
    - The instability could be triggered by numerical noise accompanied with successive adaptation of finer and finer meshes.
- Summary
  - We have performed an AMR simulation of 2D inviscid Boussinesq equations with a time-rewinding method and a multigrid iteration Poisson solver.
    - The smallest grid size reaches $2\pi/2^{15}$.
    - The result indicates that finite-time blow-up

$$|\omega|_{\max} \propto (t_0 - t)^{-1}, \qquad (17)$$

$$|\nabla T|_{\max} \propto (t_0 - t)^{-2}, \qquad (18)$$

      follows the initial exponential growth ($t_0 \approx 6$).
  - The instability of the temperature front plays an important role in the finite-time blow-up.