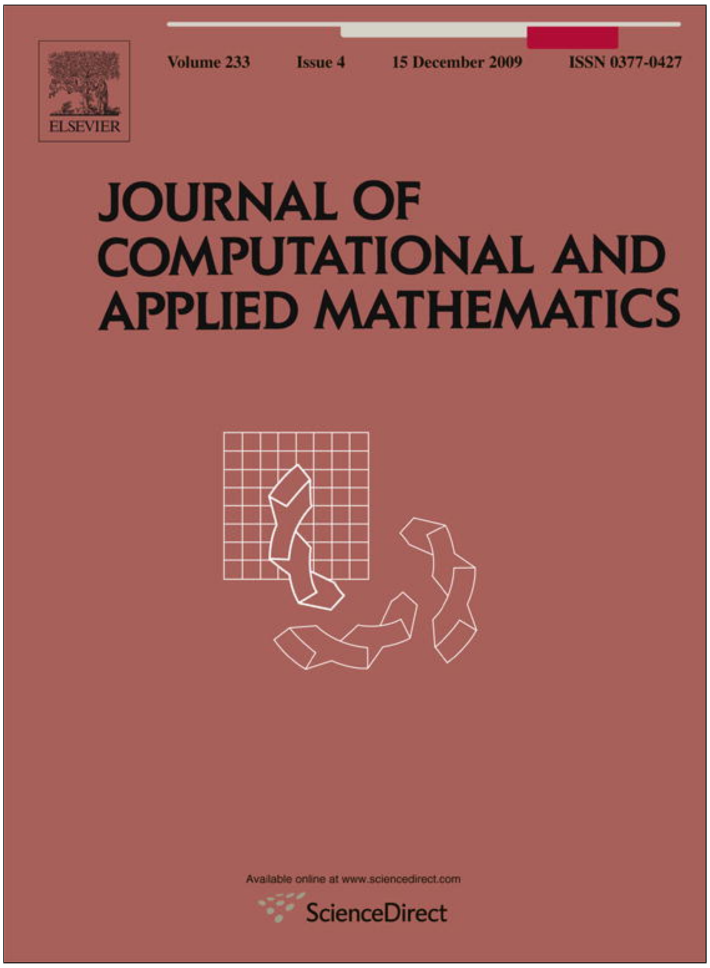
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Spectral methods for weakly singular Volterra integral equations with smooth solutions

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ABSTRACT

We propose and analyze a spectral Jacobi-collocation approximation for the linear Volterra integral equations (VIEs) of the second kind with weakly singular kernels. In this work, we consider the case when the underlying solutions of the VIEs are sufficiently smooth. In this case, we provide a rigorous error analysis for the proposed method, which shows that the numerical errors decay exponentially in the infinity norm and weighted Sobolev space norms. Numerical results are presented to confirm the theoretical prediction of the exponential rate of convergence.

1. Introduction

We consider the linear Volterra integral equations (VIEs) of the second kind, with weakly singular kernels

$$y(t) = g(t) + \int_0^t (t-s)^{-\mu} K(t,s) y(s) \mathrm{d}s, \quad t \in I, \tag{1.1}$$

where $I = [0, T]$, the function $g \in C(I)$, $y(t)$ is the unknown function, $\mu \in (0, 1)$ and $K \in C(I \times I)$ with $K(t, t) \neq 0$ for $t \in I$. Several numerical methods have been proposed for (1.1) (see, e.g., [1–11]).

The numerical treatment of the VIEs (1.1) is not simple, mainly due to the fact that the solutions of (1.1) usually have a weak singularity at $t = 0$, even when the inhomogeneous term $g(t)$ is regular. As discussed in [4], the first derivative of the solution $y(t)$ behaves like $y'(t) \sim t^{-\mu}$. In [12], a Jacobi-collocation spectral method is developed for (1.1). To handle the non-smoothness of the underlying solutions, both function transformation and variable transformation are used to change the equation into a new Volterra integral equation defined on the standard interval $[-1, 1]$, so that the solution of the new equation possesses better regularity and the Jacobi orthogonal polynomial theory can be applied conveniently. However, the function transformation (see also [9]) generally makes the resulting equations and approximations more complicated. We point out that for (1.1) without the singular kernel (i.e., $\mu = 0$), spectral methods and the corresponding error analysis

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have been provided recently [13,14]; see also [15,16] for spectral methods to pantograph-type delay differential equations. In both cases, the underlying solutions are smooth.

In this work, we will consider a special case, namely, the exact solutions of (1.1) are smooth. This case may occur when the source function g in (1.1) is non-smooth; see, e.g., Theorem 6.1.11 in [4]. In this case, the Jacobi-collocation spectral method can be applied directly; and the main purpose of this work is to carry out an error analysis for the spectral method. It is known that most systems of weakly singular VIEs that arise in many application areas are of large dimension. There have been also some recent developments for solving systems of weakly singular VIEs of high dimensions, see, e.g., [5–8]. It is pointed out that although problem (1.1) under consideration is scalar, the proposed methods can be applied to systems of large dimension in a quite straightforward way. We will demonstrate this for a two-dimensional case in Section 2.

This paper is organized as follows. In Section 2, we introduce the spectral approaches for the Volterra integral equations with weakly singular kernels. We present some lemmas useful for establishing the convergence results in Section 3. The convergence analysis is provided in Section 4. Numerical experiments are carried out in Section 5, which will be used to verify the theoretical results obtained in Section 4.

2. Jacobi-collocation methods

Throughout the paper C will denote a generic positive constant that is independent of N but which will depend on the length T of the interval $I = [0, T]$ and on bounds for the given functions $f, \widetilde{K}$ which will be defined in (2.5), and the index μ.

Let $\omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ be a weight function in the usual sense, for $\alpha, \beta > -1$. As defined in [17–20], the set of Jacobi polynomials $\{J_n^{\alpha,\beta}(x)\}_{n=0}^{\infty}$ forms a complete $L^2_{\omega^{\alpha,\beta}}(-1,1)$-orthogonal system, where $L^2_{\omega^{\alpha,\beta}}(-1,1)$ is a weighted space defined by

$$L^2_{\omega^{\alpha,\beta}}(-1,1) = \left\{ v : v \text{ is measurable and } \|v\|_{\omega^{\alpha,\beta}} < \infty \right\},$$

equipped with the norm

$$\|v\|_{\omega^{\alpha,\beta}} = \left(\int_{-1}^{1} |v(x)|^2 \omega^{\alpha,\beta}(x) \mathrm{d}x \right)^{\frac{1}{2}},$$

and the inner product

$$(u, v)_{\omega^{\alpha,\beta}} = \int_{-1}^{1} u(x) v(x) \omega^{\alpha,\beta}(x) \mathrm{d}x, \quad \forall u, v \in L^2_{\omega^{\alpha,\beta}}(-1,1).$$

For a given positive integer N, we denote the collocation points by $\{x_i\}_{i=0}^{N}$, which is the set of $(N+1)$ Jacobi–Gauss, or Jacobi–Gauss–Radau, or Jacobi–Gauss–Lobatto points, and by $\{w_i\}_{i=0}^{N}$ the corresponding weights. Let $\mathscr{P}_N$ denote the space of all polynomials of degree not exceeding N. For any $v \in C[-1,1]$, we can define the Lagrange interpolating polynomial $I_N^{\alpha,\beta} v \in \mathscr{P}_N$, satisfying

$$I_N^{\alpha,\beta} v(x_i) = v(x_i), \quad 0 \le i \le N, \tag{2.1}$$

see, e.g., [17,18,20]. The Lagrange interpolating polynomial can be written in the form

$$I_N^{\alpha,\beta} v(x) = \sum_{i=0}^{N} v(x_i) F_i(x),$$

where $F_i(x)$ is the Lagrange interpolation basis function associated with $\{x_i\}_{i=0}^{N}$.

In this paper, we deal with the special case that

$$\alpha = -\mu, \qquad \beta = 0, \qquad \omega^{-\mu,0}(x) = (1-x)^{-\mu}.$$

2.1. Numerical scheme in one dimension

For the sake of applying the theory of orthogonal polynomials, we use the change of variable

$$t = \frac{1}{2}T(1+x), \qquad x = \frac{2t}{T} - 1,$$

to rewrite the weakly singular VIEs problem (1.1) as follows

$$u(x) = f(x) + \int_0^{T(1+x)/2} \left(\frac{T}{2}(1+x) - s \right)^{-\mu} K\left(\frac{T}{2}(1+x), s \right) y(s) \mathrm{d}s, \tag{2.2}$$

where $x \in [-1, 1]$, and

$$u(x) = y\left(\frac{T}{2}(1+x)\right), \qquad f(x) = g\left(\frac{T}{2}(1+x)\right). \tag{2.3}$$

Furthermore, to transfer the integral interval $[0, T(1+x)/2]$ to the interval $[-1, x]$, we make a linear transformation: $s = T(1+\tau)/2$, $\tau \in [-1, x]$. Then, Eq. (2.2) becomes

$$u(x) = f(x) + \int_{-1}^{x} (x-\tau)^{-\mu} \widetilde{K}(x, \tau) u(\tau) \mathrm{d}\tau, \quad x \in [-1, 1], \tag{2.4}$$

where

$$\widetilde{K}(x, \tau) = \left(\frac{T}{2}\right)^{1-\mu} K\left(\frac{T}{2}(1+x), \frac{T}{2}(1+\tau)\right). \tag{2.5}$$

Firstly, Eq. (2.4) holds at the collocation points $\{x_i\}_{i=0}^N$ on $[-1, 1]$:

$$u(x_i) = f(x_i) + \int_{-1}^{x_i} (x_i-\tau)^{-\mu} \widetilde{K}(x_i, \tau) u(\tau) \mathrm{d}\tau, \quad 0 \le i \le N. \tag{2.6}$$

In order to obtain high order accuracy for the VIEs problem, the main difficulty is to compute the integral term in (2.6). In particular, for small values of x_i, there is little information available for $u(\tau)$. To overcome this difficulty, we first transfer the integral interval $[-1, x_i]$ to a fixed interval $[-1, 1]$

$$\int_{-1}^{x_i} (x_i-\tau)^{-\mu} \widetilde{K}(x_i, \tau) u(\tau) \mathrm{d}\tau = \left(\frac{1+x_i}{2}\right)^{1-\mu} \int_{-1}^{1} (1-\theta)^{-\mu} \widetilde{K}(x_i, \tau_i(\theta)) u(\tau_i(\theta)) \mathrm{d}\theta, \tag{2.7}$$

by using the following variable change

$$\tau = \tau_i(\theta) = \frac{1+x_i}{2}\theta + \frac{x_i-1}{2}, \quad \theta \in [-1, 1]. \tag{2.8}$$

Next, using a $(N+1)$-point Gauss quadrature formula relative to the Jacobi weight $\{w_i\}_{i=0}^N$, the integration term in (2.6) can be approximated by

$$\int_{-1}^{1} (1-\theta)^{-\mu} \widetilde{K}(x_i, \tau_i(\theta)) u(\tau_i(\theta)) \mathrm{d}\theta \sim \sum_{k=0}^{N} \widetilde{K}(x_i, \tau_i(\theta_k)) u(\tau_i(\theta_k)) w_k, \tag{2.9}$$

where the set $\{\theta_k\}_{k=0}^N$ coincides with the collocation points $\{x_i\}_{i=0}^N$ on $[-1, 1]$. We use u_i, $0 \le i \le N$ to approximate the function value $u(x_i)$, $0 \le i \le N$, and use

$$u^N(x) = \sum_{j=0}^{N} u_j F_j(x) \tag{2.10}$$

to approximate the function $u(x)$, namely, $u(x_i) \approx u_i$, $u(x) \approx u^N(x)$, and

$$u(\tau_i(\theta_k)) \approx \sum_{j=0}^{N} u_j F_j(\tau_i(\theta_k)). \tag{2.11}$$

Then, the Jacobi-collocation method is to seek $u^N(x)$ such that $\{u_i\}_{i=0}^N$ satisfies the following collocation equations:

$$u_i = f(x_i) + \left(\frac{1+x_i}{2}\right)^{1-\mu} \sum_{j=0}^{N} u_j \left(\sum_{k=0}^{N} \widetilde{K}(x_i, \tau_i(\theta_k)) F_j(\tau_i(\theta_k)) w_k\right), \quad 0 \le i \le N. \tag{2.12}$$

It follows from (2.3) that the exact solution of the VIEs problem (1.1) can be written as

$$y(t) = y\left(\frac{T}{2}(1+x)\right) = u(x), \quad t \in [0, T] \text{ and } x \in [-1, 1]. \tag{2.13}$$

Thus, we can define

$$y^N(t) = y^N\left(\frac{T}{2}(1+x)\right) = u^N(x), \quad t \in [0, T] \text{ and } x \in [-1, 1], \tag{2.14}$$

as the approximated solution of the VIEs problem (1.1). It is obvious to see that

$$(y - y^N)(t) = (u - u^N)(x) := e(x), \quad t \in [0, T] \text{ and } x \in [-1, 1]. \tag{2.15}$$

2.2. Two-dimensional extension

Problem (1.1) is considered to be scalar; however, many applications involve systems of weakly singular VIEs with high dimensions. The spectral methods proposed in the last subsection are generalizable to large systems of VIEs and to higher dimensions. To demonstrate this, we briefly outline how to solve the second-kind VIEs in two dimension:

$$y(s,t)=g(s,t)+\int_0^s\int_0^t (s-\sigma)^{-\alpha}(t-\tau)^{-\beta}K(s,t,\sigma,\tau)y(\sigma,\tau)\mathrm{d}\sigma\mathrm{d}\tau, \tag{2.16}$$

where $(s,t)\in[0,T]^2$. By using some linear transformations as in Section 2.1, Eq. (2.16) becomes

$$u(x,y)=f(x,y)+\int_{-1}^x\int_{-1}^y (x-\xi)^{-\alpha}(y-\eta)^{-\beta}\widetilde{K}(x,y,\xi,\eta)u(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta, \tag{2.17}$$

where $(x,y)\in[-1,1]^2$ and

$$\widetilde{K}(x,y,\xi,\eta)=\left(\frac{T}{2}\right)^{(2-\alpha-\beta)}K\left(\frac{T}{2}(1+x),\frac{T}{2}(1+y),\frac{T}{2}(1+\xi),\frac{T}{2}(1+\eta)\right).$$

For the weight function $\omega^{-\alpha,0}(x)$, we denote the collocation points by $\{x_i\}_{i=0}^N$, which is the set of $(N+1)$ Jacobi–Gauss, or Jacobi–Gauss–Radau, or Jacobi–Gauss–Lobatto points, and by $\{w_i^{(1)}\}_{i=0}^N$ the corresponding weights. Similarly, for the weight function $\omega^{-\beta,0}(y)$, we denote the collocation points by $\{y_j\}_{j=0}^N$, which is the set of $(N+1)$ Jacobi–Gauss, or Jacobi–Gauss–Radau, or Jacobi–Gauss–Lobatto points, and by $\{w_j^{(2)}\}_{j=0}^N$ the corresponding weights. Assume that Eq. (2.17) holds at the Jacobi-collocation point-pairs (x_i,y_j). Using the linear transformation and tricks used in one-dimensional case yields

$$u_{i,j}=f(x_i,y_j)+\left(\frac{1+x_i}{2}\right)^{1-\alpha}\left(\frac{1+y_j}{2}\right)^{1-\beta}\sum_{k=0}^N\sum_{l=0}^N u_{k,l}a_{k,l}, \tag{2.18}$$

where

$$a_{k,l}=\sum_{m=0}^N\sum_{n=0}^N \widetilde{K}(x_i,y_j,\xi_i(\theta_m),\eta_j(\theta_n))F_k(\xi_i(\theta_m))F_l(\eta_j(\theta_n))w_m^{(1)}w_n^{(2)},$$

$$\xi_i(\theta_m)=\frac{1+x_i}{2}\theta_m+\frac{x_i-1}{2},\qquad \eta_j(\theta_n)=\frac{1+y_j}{2}\theta_n+\frac{y_j-1}{2},$$

for any $0\le i,j\le N$.

3. Some useful lemmas

We first introduce some weighted Hilbert spaces. For simplicity, denote $\partial_x v(x)=(\partial/\partial x)v(x)$, etc. For non-negative integer m, define

$$H^m_{\omega^{\alpha,\beta}}(-1,1):=\left\{v:\partial_x^k v\in L^2_{\omega^{\alpha,\beta}}(-1,1),\ 0\le k\le m\right\},$$

with the semi-norm and the norm as

$$|v|_{m,\omega^{\alpha,\beta}}=\|\partial_x^m v\|_{\omega^{\alpha,\beta}},\qquad \|v\|_{m,\omega^{\alpha,\beta}}=\left(\sum_{k=0}^m |v|^2_{k,\omega^{\alpha,\beta}}\right)^{1/2},$$

respectively. Let $\omega(x)=\omega^{-\frac{1}{2},-\frac{1}{2}}(x)$ denote the Chebyshev weight function. In bounding some approximation error of Chebyshev polynomials, only some of the L^2-norms appearing on the right-hand side of above norm enter into play. Thus, it is convenient to introduce the semi-norms

$$|v|_{H^{m;N}_{\omega}(-1,1)}=\left(\sum_{k=\min(m,N+1)}^m \|\partial_x^k v\|^2_{L^2_\omega(-1,1)}\right)^{\frac{1}{2}}.$$

For bounding some approximation error of Jacobi polynomials, we need the following non-uniformly weighted Sobolev spaces:

$$H^m_{\omega^{\alpha,\beta},*}(-1,1):=\left\{v:\partial_x^k v\in L^2_{\omega^{\alpha+k,\beta+k}}(-1,1),\ 0\le k\le m\right\},$$

equipped with the inner product and the norm as

$$(u, v)_{m,\omega^{\alpha,\beta},*} = \sum_{k=0}^{m}(\partial_x^k u, \partial_x^k v)_{\omega^{\alpha+k,\beta+k}}, \qquad \|v\|_{m,\omega^{\alpha,\beta},*} = \sqrt{(v, v)_{m,\omega^{\alpha,\beta},*}}.$$

Furthermore, we introduce the orthogonal projection $\pi_{N,\omega^{\alpha,\beta}} : L^2_{\omega^{\alpha,\beta}}(-1, 1) \to \mathcal{P}_N$, which is a mapping such that for any $v \in L^2_{\omega^{\alpha,\beta}}(-1, 1)$,

$$(v - \pi_{N,\omega^{\alpha,\beta}}v, \phi)_{\omega^{\alpha,\beta}} = 0, \quad \forall \phi \in \mathcal{P}_N.$$

It follows from Theorem 1.8.1 in [20] and (3.18) in [18] that

Lemma 3.1. *Let $\alpha, \beta > -1$. Then for any function $v \in H^m_{\omega^{\alpha,\beta},*}(-1, 1)$ and any non-negative integer m, we have*

$$\|\partial_x^k(v - \pi_{N,\omega^{\alpha,\beta}}v)\|_{\omega^{\alpha+k,\beta+k}} \leq CN^{k-m}\|\partial_x^m v\|_{\omega^{\alpha+m,\beta+m}}, \quad 0 \leq k \leq m. \tag{3.1}$$

In particular,

$$\|v - \pi_{N,\omega^{\alpha,\beta}}v\|_{\omega^{\alpha,\beta}} \leq CN^{-1}|v|_{1,\omega^{\alpha+1,\beta+1}}. \tag{3.2}$$

Applying Theorem 1.8.4 in [20] Theorem 4.3, 4.7, and 4.10 in [21], we have the following optimal error estimate for the interpolation polynomials based on the Jacobi–Gauss points, Jacobi–Gauss–Radau points, and Gauss–Lobatto points.

Lemma 3.2. *For any function v satisfying $\partial_x v \in H^m_{\omega^{\alpha,\beta},*}(-1, 1)$, we have, for $0 \leq k \leq m$,*

$$\|\partial_x^k(v - I_N^{\alpha,\beta}v)\|_{\omega^{\alpha+k,\beta+k}} \leq CN^{k-m}\|\partial_x^m v\|_{\omega^{\alpha+m,\beta+m}}, \tag{3.3}$$

$$|v - I_N^{\alpha,\beta}v|_{1,\omega^{\alpha,\beta}} \leq C\,(N(N+\alpha+\beta))^{1-m/2}\,\|\partial_x^m v\|_{\omega^{\alpha+m,\beta+m}}. \tag{3.4}$$

Let us define a discrete inner product. For any $u, v \in C[-1, 1]$, define

$$(u, v)_N = \sum_{j=0}^{N} u(x_j)v(x_j)w_j. \tag{3.5}$$

Due to (5.3.4) in [17], and Lemmas 3.1 and 3.2, we can have the integration error estimates from the Jacobi–Gauss polynomials quadrature.

Lemma 3.3. *Let v be any continuous function on $[-1, 1]$ and ϕ be any polynomial of $\mathcal{P}_N$. For the Jacobi–Gauss and Jacobi–Gauss–Radau integration, we have*

$$\begin{aligned}|(v, \phi)_{\omega^{\alpha,\beta}} - (v, \phi)_N| &\leq \|v - I_N^{\alpha,\beta}v\|_{\omega^{\alpha,\beta}}\|\phi\|_{\omega^{\alpha,\beta}}\\ &\leq CN^{-m}\|\partial_x^m v\|_{\omega^{\alpha+m,\beta+m}}\|\phi\|_{\omega^{\alpha,\beta}}.\end{aligned} \tag{3.6}$$

For the Jacobi–Gauss–Lobatto integration, we have

$$\begin{aligned}|(v, \phi)_{\omega^{\alpha,\beta}} - (v, \phi)_N| &\leq C\left(\|v - \pi_{N-1,\omega^{\alpha,\beta}}v\|_{\omega^{\alpha,\beta}} + \|v - I_N^{\alpha,\beta}v\|_{\omega^{\alpha,\beta}}\right)\|\phi\|_{\omega^{\alpha,\beta}}\\ &\leq CN^{-m}\|\partial_x^m v\|_{\omega^{\alpha+m,\beta+m}}\|\phi\|_{\omega^{\alpha,\beta}}.\end{aligned} \tag{3.7}$$

From [22], we have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials.

Lemma 3.4. *Assume $\alpha = -\mu, \beta = 0$ and assume that $F_j(x)$ is the corresponding Nth Lagrange interpolation polynomials associated with the Gauss, or Gauss–Radau, or Gauss–Lobatto points of the Jacobi polynomials. Then*

$$\|I_N^{-\mu,0}\|_\infty := \max_{x\in(-1,1)}\sum_{j=0}^{N}|F_j(x)| = \mathcal{O}(\sqrt{N}). \tag{3.8}$$

The following generalization of Gronwall's Lemma for singular kernels, whose proof can be found, e.g. in [23] (Lemma 7.1.1), will be essential for establishing our main results.

Lemma 3.5. *Suppose $L \geq 0$, $0 < \mu < 1$ and $v(t)$ is a non-negative, locally integrable function defined on $[0, T]$ satisfying*

$$u(t) \leq v(t) + L \int_0^t (t-s)^{-\mu} u(s) \mathrm{d}s. \tag{3.9}$$

Then there exists a constant $C = C(\mu)$ such that

$$u(t) \leq v(t) + CL \int_0^t (t-s)^{-\mu} v(s) \mathrm{d}s \quad \text{for } 0 \leq t < T. \tag{3.10}$$

From now on, for $r \geq 0$ and $\kappa \in [0, 1]$, $C^{r,\kappa}([-1, 1])$ will denote the space of functions whose rth derivatives are Hölder continuous with exponent κ, endowed with the usual norm

$$\|v\|_{r,\kappa} = \max_{0 \leq k \leq r} \max_{x \in [-1,1]} |\partial_x^k v(x)| + \max_{0 \leq k \leq r} \sup_{\substack{x,y \in [-1,1] \\ x \neq y}} \frac{|\partial_x^k v(x) - \partial_x^k v(y)|}{|x-y|^\kappa}.$$

When $\kappa = 0$, $C^{r,0}([-1, 1])$ denotes the space of functions with r continuous derivatives on $[-1, 1]$, which is also commonly denoted by $C^r([-1, 1])$, and with norm $\| \cdot \|_r$.

We shall make use of a result of Ragozin [24,25] (see also [26]), which states that, for non-negative integers r and $\kappa \in (0, 1)$, there exists a constant $C_{r,\kappa} > 0$ such that for any function $v \in C^{r,\kappa}([-1, 1])$, there exists a polynomial function $\mathcal{T}_N v \in \mathcal{P}_N$ such that

$$\|v - \mathcal{T}_N v\|_\infty \leq C_{r,\kappa} N^{-(r+\kappa)} \|v\|_{r,\kappa}. \tag{3.11}$$

Actually, as stated in [24,25], $\mathcal{T}_N$ is a linear operator from $C^{r,\kappa}([-1, 1])$ into $\mathcal{P}_N$.

We further define a linear, weakly singular integral operator $\mathcal{M}$:

$$\mathcal{M} v = \int_{-1}^x (x-\tau)^{-\mu} \widetilde{K}(x, \tau) v(\tau) \mathrm{d}\tau. \tag{3.12}$$

Below we will show that $\mathcal{M}$ is compact as an operator from $C([-1, 1])$ to $C^{0,\kappa}([-1, 1])$ provided that the index κ satisfies $0 < \kappa < 1 - \mu$. A similar result can be found in Theorem 3.4 of [27]. The proof of the following lemma can be found in [12].

Lemma 3.6. *Let $\kappa \in (0, 1)$ and $\mathcal{M}$ be defined by (3.12) under the assumption that $0 < \kappa < 1 - \mu$. Then, for any function $v \in C([-1, 1])$, there exists a positive constant C, which is dependent of $\|\widetilde{K}\|_{0,\kappa}$, such that*

$$\frac{|\mathcal{M} v(x') - \mathcal{M} v(x'')|}{|x' - x''|^\kappa} \leq C \max_{x \in [-1,1]} |v(x)|, \tag{3.13}$$

for any $x', x'' \in [-1, 1]$ and $x' \neq x''$. This implies that

$$\|\mathcal{M} v\|_{0,\kappa} \leq C \|v\|_\infty, \tag{3.14}$$

where $\| \cdot \|_\infty$ is the standard norm in $C([-1, 1])$.

4. Convergence analysis

4.1. Error estimate in L^∞

Theorem 4.1. *Let u be the exact solution to the Volterra integral equation (2.4), which is assumed to be sufficiently smooth. Let the approximated solution u^N be obtained by using the spectral collocation scheme (2.12) together with a polynomial interpolation (2.10). If μ associated with the weakly singular kernel satisfies $0 < \mu < \frac{1}{2}$ and $u \in H_\omega^m(-1, 1) \cap H_{\omega^{-\mu,0},*}^m(-1, 1)$, then*

$$\|u - u^N\|_\infty \leq C N^{1-m} |u|_{H_\omega^{m;N}(-1,1)} + C N^{1/2-m} \cdot K^* \|u\|_\infty, \tag{4.1}$$

for N sufficiently large, where $\tau_i(\theta)$ is defined by (2.8) and

$$K^* := \max_{0 \leq i \leq N} \|\partial_\theta^m \widetilde{K}(x_i, \tau_i(\cdot))\|_{\omega^{m-\mu,m}}.$$

Proof. First, we use the weighted inner product to rewrite (2.6) as

$$u(x_i) = f(x_i) + \left(\frac{1 + x_i}{2}\right)^{1-\mu} \left(\widetilde{K}(x_i, \tau_i(\cdot)), u(\tau_i(\cdot))\right)_{\omega^{-\mu,0}}, \quad 0 \leq i \leq N. \tag{4.2}$$

By using the discrete inner product (3.5), we set

$$\left(\widetilde{K}(x_i,\tau_i(\cdot)),\phi(\tau_i(\cdot))\right)_N=\sum_{k=0}^{N}\widetilde{K}(x_i,\tau_i(\theta_k))\phi(\tau_i(\theta_k))w_k.$$

Then, the numerical scheme (2.12) can be written as

$$u_i=f(x_i)+\left(\frac{1+x_i}{2}\right)^{1-\mu}\left(\widetilde{K}(x_i,\tau_i(\cdot)),u^N(\tau_i(\cdot))\right)_N,\quad 0\le i\le N, \tag{4.3}$$

where u^N is defined by (2.10). Subtracting (4.3) from (4.2) gives the error equation:

$$\begin{aligned} u(x_i)-u_i &= \left(\frac{1+x_i}{2}\right)^{1-\mu}\left(\widetilde{K}(x_i,\tau_i(\cdot)),e(\tau_i(\cdot))\right)_{\omega^{-\mu,0}}+\left(\frac{1+x_i}{2}\right)^{1-\mu}I_{i,2}\\ &=\int_{-1}^{x_i}(x_i-\tau)^{-\mu}\widetilde{K}(x_i,\tau)e(\tau)\mathrm{d}\tau+\left(\frac{1+x_i}{2}\right)^{1-\mu}I_{i,2}, \end{aligned} \tag{4.4}$$

for $0\le i\le N$, where $e(x)=u(x)-u^N(x)$ is the error function,

$$I_{i,2}=\left(\widetilde{K}(x_i,\tau_i(\cdot)),u^N(\tau_i(\cdot))\right)_{\omega^{-\mu,0}}-\left(\widetilde{K}(x_i,\tau_i(\cdot)),u^N(\tau_i(\cdot))\right)_N,$$

and the integral transformation (2.7) was used here. Using the integration error estimates from Jacobi–Gauss polynomials quadrature in Lemma 3.3, we have

$$\left|\left(\frac{1+x_i}{2}\right)^{1-\mu}I_{i,2}\right|\le CN^{-m}\|\partial_\theta^m\widetilde{K}(x_i,\tau_i(\cdot))\|_{\omega^{m-\mu,m}}\|u^N(\tau_i(\cdot))\|_{\omega^{-\mu,0}}. \tag{4.5}$$

Multiplying $F_i(x)$ on both sides of the error equation (4.4) and summing up from $i=0$ to $i=N$ yield

$$I_N^{-\mu,0}u-u^N=I_N^{-\mu,0}\left(\int_{-1}^{x}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)\mathrm{d}\tau\right)+\sum_{i=0}^{N}\left(\frac{1+x_i}{2}\right)^{1-\mu}I_{i,2}F_i(x). \tag{4.6}$$

Consequently,

$$e(x)=\int_{-1}^{x}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)\mathrm{d}\tau+I_1+I_2+I_3, \tag{4.7}$$

where

$$I_1=u-I_N^{-\mu,0}u,\qquad I_2=\sum_{i=0}^{N}\left(\frac{1+x_i}{2}\right)^{1-\mu}I_{i,2}F_i(x), \tag{4.8a}$$

$$I_3=I_N^{-\mu,0}\left(\int_{-1}^{x}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)\mathrm{d}\tau\right)-\int_{-1}^{x}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)\mathrm{d}\tau. \tag{4.8b}$$

It follows from the Gronwall inequality (Lemma 3.5)

$$\|e\|_\infty\le C\Big(\|I_1\|_\infty+\|I_2\|_\infty+\|I_3\|_\infty\Big). \tag{4.9}$$

Let $I_N^c u\in\mathcal{P}_N$ denote the interpolant of u at any of the three families of Chebyshev–Gauss points. From (5.5.28) in [17], the interpolation error estimate in the maximum norm is given by

$$\|u-I_N^c u\|_\infty\le CN^{1/2-m}|u|_{H_\omega^{m;N}(-1,1)}. \tag{4.10}$$

Note that

$$I_N^{-\mu,0}p(x)=p(x),\quad \text{i.e., } (I_N^{-\mu,0}-I)p(x)=0,\quad \forall p(x)\in\mathcal{P}_N. \tag{4.11}$$

By using (4.11), Lemma 3.4 and (4.10), we obtained that

$$\begin{aligned} \|I_1\|_\infty &= \|u-I_N^{-\mu,0}u\|_\infty\\ &=\|u-I_N^c u+I_N^{-\mu,0}(I_N^c u)-I_N^{-\mu,0}u\|_\infty\\ &\le \|u-I_N^c u\|_\infty+\|I_N^{-\mu,0}(I_N^c u-u)\|_\infty \end{aligned}$$

$$\le \left(1+\|I_N^{-\mu,0}\|_\infty\right)\|u-I_N^c u\|_\infty$$
$$\le \left(1+\sqrt{N}\right)\cdot N^{1/2-m}|u|_{H_\omega^{m;N}(-1,1)}$$
$$\le CN^{1-m}|u|_{H_\omega^{m;N}(-1,1)}. \tag{4.12}$$

Next, using the estimate (4.5) and Lemma 3.4, we have

$$\|I_2\|_\infty \le CN^{1/2-m}\max_{0\le i\le N}\|\partial_\theta^m\widetilde{K}(x_i,\tau_i(\cdot))\|_{\omega^{m-\mu,m}}\cdot\max_{0\le i\le N}\|u^N(\tau_i(\cdot))\|_{\omega^{-\mu,0}}$$
$$\le CN^{1/2-m}\max_{0\le i\le N}\|\partial_\theta^m\widetilde{K}(x_i,\tau_i(\cdot))\|_{\omega^{m-\mu,m}}\cdot(\|e\|_\infty+\|u\|_\infty)$$
$$\le \frac{1}{3}\|e\|_\infty + CN^{1/2-m}\max_{0\le i\le N}\|\partial_\theta^m\widetilde{K}(x_i,\tau_i(\cdot))\|_{\omega^{m-\mu,m}}\cdot\|u\|_\infty, \tag{4.13}$$

provided that N is sufficiently large. We now estimate the third term I_3. It follows from (3.11) and (4.11), and Lemma 3.4 that

$$\|I_3\|_\infty = \|(I_N^{-\mu,0}-I)\mathcal{M}e\|_\infty = \|(I_N^{-\mu,0}-I)(\mathcal{M}e-\mathcal{T}_N\mathcal{M}e)\|_\infty$$
$$\le (1+\|I_N^{-\mu,0}\|_\infty)\cdot\|\mathcal{M}e-\mathcal{T}_N\mathcal{M}e\|_\infty$$
$$\le C\sqrt{N}\cdot\|\mathcal{M}e-\mathcal{T}_N\mathcal{M}e\|_\infty \le CN^{\frac{1}{2}-\kappa}\|\mathcal{M}e\|_{0,\kappa}$$
$$\le CN^{\frac{1}{2}-\kappa}\|e\|_\infty, \tag{4.14}$$

where in the last step we have used Lemma 3.6 under the assumption $0<\kappa<1-\mu$. It is clear that if $\kappa>\frac{1}{2}$ (which is equivalent to $\mu<\frac{1}{2}$), then

$$\|I_3\|_\infty \le \frac{1}{3}\|e\|_\infty, \tag{4.15}$$

provided that N is sufficiently large. Combining (4.9), (4.12), (4.13) and (4.15) gives the desired estimate (4.1). □

4.2. Error estimate in weighted L^2 norm

To prove the error estimate in weighted L^2 norm, we need the generalized Hardy's inequality with weights (see, e.g., [28–30]).

Lemma 4.1. *For all measurable function $f\ge 0$, the following generalized Hardy's inequality*

$$\left(\int_a^b |(Tf)(x)|^q u(x)\mathrm{d}x\right)^{1/q} \le C\left(\int_a^b |f(x)|^p v(x)\mathrm{d}x\right)^{1/p}$$

holds if and only if

$$\sup_{a<x<b}\left(\int_x^b u(t)\mathrm{d}t\right)^{1/q}\left(\int_a^x v^{1-p'}(t)\mathrm{d}t\right)^{1/p'}<\infty,\quad p'=\frac{p}{p-1}$$

for the case $1<p\le q<\infty$. Here, T is an operator of the form

$$(Tf)(x)=\int_a^x k(x,t)f(t)\mathrm{d}t$$

with $k(x,t)$ a given kernel, u, v weight functions, and $-\infty\le a<b\le\infty$.

From Theorem 1 in [31], we have the following weighted mean convergence result of Lagrange interpolation based at the zeros of Jacobi polynomials.

Lemma 4.2. *For every bounded function $v(x)$, there exists a constant C independent of v such that*

$$\sup_N\left\|\sum_{j=0}^N v(x_j)F_j(x)\right\|_{\omega^{-\mu,0}}\le C\|v\|_\infty.$$

Theorem 4.2. *Let u be the exact solution to the Volterra integral equation (2.4), which is assumed to be sufficiently smooth. Let the approximated solution u^N be obtained by using the spectral collocation scheme (2.12) together with a polynomial interpolation (2.10). Assume that $u \in H^m_{\omega^{-\mu,0},*}(-1,1)$ and $\|\widetilde{K}(\cdot,\tau)\|_{1,\infty}$ is bounded, where*

$$\|\widetilde{K}(\cdot,\cdot)\|_{m,\infty} := \max_{0\le j\le m}\left\{\max_{-1\le\tau<x\le 1}|\partial_x^j\widetilde{K}(x,\tau)|\right\}. \tag{4.16}$$

If $0<\mu<\frac{1}{2}$, then, for N sufficiently large

$$\|u-u^N\|_{\omega^{-\mu,0}} \le CN^{-m}\|\partial_x^m u\|_{\omega^{m-\mu,m}} + CN^{-m}K^*\left(\|u'\|_{\omega}+\|u\|_{\infty}\right). \tag{4.17}$$

where $\tau_i(\theta)$ and K^* are defined in Theorem 4.1.

Proof. By the generalized Hardy's inequality Lemma 4.1, it follows from (4.7) and (3.10) that

$$\|e\|_{\omega^{-\mu,0}} \le C\Big(\|I_1\|_{\omega^{-\mu,0}}+\|I_2\|_{\omega^{-\mu,0}}+\|I_3\|_{\omega^{-\mu,0}}\Big). \tag{4.18}$$

Now, using Lemma 3.2, we obtain that

$$\|I_1\|_{\omega^{-\mu,0}} = \|u-I_N^{-\mu,0}u\|_{\omega^{-\mu,0}} \le CN^{-m}\|\partial_x^m u\|_{\omega^{m-\mu,m}}. \tag{4.19}$$

By using Lemma 4.2 and (4.5), we have

$$\begin{aligned}\|I_2\|_{\omega^{-\mu,0}} &= \left\|\sum_{i=0}^{N}\left(\frac{1+x_i}{2}\right)^{1-\mu} I_{i,2}F_i(x)\right\|_{\omega^{-\mu,0}}\\ &\le C\max_{0\le i\le N}\left|\left(\frac{1+x_i}{2}\right)^{1-\mu} I_{i,2}\right|\\ &\le CN^{-m}\max_{0\le i\le N}\|\partial_\theta^m\widetilde{K}(x_i,\tau_i(\cdot))\|_{\omega^{m-\mu,m}}\max_{0\le i\le N}\|u^N(\tau_i(\cdot))\|_{\omega^{-\mu,0}}\\ &\le CN^{-m}\max_{0\le i\le N}\|\partial_\theta^m\widetilde{K}(x_i,\tau_i(\cdot))\|_{\omega^{m-\mu,m}}\Big(\|e\|_{\infty}+\|u\|_{\infty}\Big).\end{aligned} \tag{4.20}$$

By the convergence result in Theorem 4.1, we have

$$\|e\|_{\infty} \le C\Big(|u|_{H^1_\omega(-1,1)}+\|u\|_{\infty}\Big) = C\Big(\|u'\|_{\omega}+\|u\|_{\infty}\Big), \tag{4.21}$$

for sufficiently large N. So that

$$\|I_2\|_{\omega^{-\mu,0}} \le CN^{-m}\max_{0\le i\le N}\|\partial_\theta^m\widetilde{K}(x_i,\tau_i(\cdot))\|_{\omega^{m-\mu,m}}\Big(\|u'\|_{\omega}+\|u\|_{\infty}\Big), \tag{4.22}$$

for sufficiently large N.

Next, we estimate $\|I_3\|_{\omega^{-\mu,0}}$. To this end, we first split the interval $[-1,x]$ to $[-1,x-\delta]$ and $[x-\delta,x]$ for some small $\delta>0$. Then, we can see that

$$\|I_3\|_{\omega^{-\mu,0}} \le \|I_{3,1}\|_{\omega^{-\mu,0}}+\|I_{3,2}\|_{\omega^{-\mu,0}}, \tag{4.23}$$

where

$$I_{3,1} = \left(I_N^{-\mu,0}-I\right)\int_{-1}^{x-\delta}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)\mathrm{d}\tau,$$
$$I_{3,2} = \left(I_N^{-\mu,0}-I\right)\int_{x-\delta}^{x}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)\mathrm{d}\tau.$$

Here I denotes the identical operator. It follows from Lemma 3.2 that

$$\begin{aligned}\|I_{3,1}\|_{\omega^{-\mu,0}} &\le CN^{-1}\left\|\partial_x\left(\int_{-1}^{x-\delta}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)\mathrm{d}\tau\right)\right\|_{\omega^{1-\mu,1}}\\ &= CN^{-1}\left\|I_{3,1}^{(1)}+I_{3,2}^{(2)}\right\|_{\omega^{1-\mu,1}},\end{aligned} \tag{4.24}$$

where $I_{3,1}^{(1)} := \delta^{-\mu}\widetilde{K}(x, x-\delta)e(x-\delta)$ and

$$I_{3,2}^{(2)} := \int_{-1}^{x-\delta} \left[-\mu(x-\tau)^{-\mu-1}\widetilde{K}(x,\tau) + (x-\tau)^{-\mu}\widetilde{K}_x(x,\tau)\right] e(\tau)\mathrm{d}\tau. \tag{4.25}$$

Extending $e \equiv 0$ for $x \le 0$, we can easily obtain that

$$\|I_{3,1}^{(1)}\|_{\omega^{1-\mu},1} \le \delta^{-\mu}\|\widetilde{K}(\cdot,\cdot)\|_{0,\infty}\|e(\cdot-\delta)\|_{\omega^{1-\mu},1} \le C\delta^{-\mu}\|\widetilde{K}(\cdot,\cdot)\|_{0,\infty}\|e\|_{\omega^{-\mu},0}. \tag{4.26}$$

It follows from (4.25) that

$$\|I_{3,1}^{(2)}\|_{\omega^{1-\mu},1} \le C\delta^{-\mu}\|\widetilde{K}(\cdot,\cdot)\|_{0,\infty}\|e\|_{\omega^{-\mu},0} + C\|\widetilde{K}(\cdot,\cdot)\|_{1,\infty}\|e\|_{\omega^{-\mu},0}. \tag{4.27}$$

Combining the above two estimates leads to

$$\|I_{3,1}\|_{\omega^{-\mu},0} \le C(N^{-1}\delta^{-\mu} + N^{-1})\|e\|_{\omega^{-\mu},0}. \tag{4.28}$$

We choose $\delta \le N^{-2}$, so that $N^{-1}\delta^{-\mu} \le N^{-(1-2\mu)}$ which can be sufficiently small due to the assumption that $0 < \mu < \frac{1}{2}$. Therefore, for sufficiently large N, we have

$$\|I_{3,1}\|_{\omega^{-\mu},0} \le \frac{1}{4}\|e\|_{\omega^{-\mu},0}. \tag{4.29}$$

On the other hand, we have

$$\|I_{3,2}\|_{\omega^{-\mu},0} \le \|I_{3,2}^{(1)}\|_{\omega^{-\mu},0} + \|I_{3,2}^{(2)}\|_{\omega^{-\mu},0}, \tag{4.30}$$

where

$$I_{3,2}^{(1)} := I_N^{-\mu,0}\int_{x-\delta}^{x}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)\mathrm{d}\tau,$$
$$I_{3,2}^{(2)} := \int_{x-\delta}^{x}(x-\tau)^{-\mu}\widetilde{K}(x,\tau)e(\tau)\mathrm{d}\tau.$$

It follows from (2.25) of [18] that

$$\|I_{3,2}^{(1)}\|_{\omega^{-\mu},0}^2 = \sum_{i=0}^{N}\left|\int_{x_i-\delta}^{x_i}(x_i-\tau)^{-\mu}\widetilde{K}(x_i,\tau)e(\tau)\mathrm{d}\tau\right|^2 \cdot w_i, \tag{4.31}$$

where $\{x_i\}_{i=0}^N$ is the set of $(N+1)$ Jacobi–Gauss, or Jacobi–Gauss–Radau, or Jacobi–Gauss–Lobatto points, and $\{w_i\}_{i=0}^N$ are the corresponding weights. From (2.13) of [18] and noting that $\mu < 1/2$, we have

$$\begin{aligned}\|I_{3,2}^{(1)}\|_{\omega^{-\mu},0}^2 &\le CN^{-1}\sum_{i=0}^{N}\left|\int_{x_i-\delta}^{x_i}(x_i-\tau)^{-\mu}\widetilde{K}(x_i,\tau)e(\tau)\mathrm{d}\tau\right|^2 (1-x_i)^{-\mu+1/2}(1+x_i)^{1/2} \\ &\le CN^{-1}\sum_{i=0}^{N}\left|\int_{x_i-\delta}^{x_i}(x_i-\tau)^{-\mu}\widetilde{K}(x_i,\tau)e(\tau)\mathrm{d}\tau\right|^2.\end{aligned}$$

From Cauchy inequality and using the fact that $\widetilde{K}$ is bounded, we obtain

$$\begin{aligned}\|I_{3,2}^{(1)}\|_{\omega^{-\mu},0}^2 &\le CN^{-1}\sum_{i=0}^{N}\int_{x_i-\delta}^{x_i}(1-\tau)^{\mu}(x_i-\tau)^{-2\mu}\mathrm{d}\tau \cdot \int_{x_i-\delta}^{x_i}(1-\tau)^{-\mu}\mathrm{e}^2(\tau)\mathrm{d}\tau \\ &\le CN^{-1}\|e\|_{\omega^{-\mu},0}^2,\end{aligned} \tag{4.32}$$

where we have used the assumptions $\delta \le N^{-2} \le 1$ and $1-2\mu > 0$. Again, using Cauchy inequality and the boundedness of $\widetilde{K}$ gives

$$|I_{3,2}^{(2)}|^2 \le C\int_{x-\delta}^{x}(1-\tau)^{\mu}(x-\tau)^{-2\mu}\mathrm{d}\tau\int_{x-\delta}^{x}(1-\tau)^{-\mu}\mathrm{e}^2(\tau)\mathrm{d}\tau \le C\delta^{1-2\mu}\|e\|_{\omega^{-\mu},0}^2.$$

Consequently,

$$\|I_{3,2}^{(2)}\|_{\omega^{-\mu},0}^2 \le CN^{-2(1-2\mu)}\|e\|_{\omega^{-\mu},0}^2, \tag{4.33}$$

where we also used $\delta \le N^{-2}$ and $\mu < 1/2$. It follows from (4.32) and (4.33) that

$$\|I_{3,2}\|_{\omega^{-\mu},0} \le \|I_{3,2,1}\|_{\omega^{-\mu},0} + \|I_{3,2,2}\|_{\omega^{-\mu},0} \le \frac{1}{4}\|e\|_{\omega^{-\mu},0}, \tag{4.34}$$

provided that N is sufficiently large. Hence, by (4.23), (4.29) and (4.34), we have

$$\|I_3\|_{\omega^{-\mu},0} \le \frac{1}{2}\|e\|_{\omega^{-\mu},0}, \tag{4.35}$$

for sufficiently large N. The desired estimate (4.17) is obtained by combining (4.18), (4.19), (4.22) and (4.35). □

5. Algorithm implementation and numerical experiments

Denoting $U_N = [u_0, u_1, \dots, u_N]^{\mathrm{T}}$ and $F_N = [f(x_0), f(x_1), \dots, f(x_N)]^{\mathrm{T}}$, we can obtain an equation of the matrix form:

$$U_N = F_N + AU_N, \tag{5.1}$$

where the entries of the matrix A is given by

$$a_{ij} = \left(\frac{1+x_i}{2}\right)^{1-\mu} \sum_{k=0}^{N} \widetilde{K}(x_i, \tau_i(\theta_k)) F_j(\tau_i(\theta_k)) w_k.$$

Here, we simply introduce the computation of Gauss–Jacobi quadrature rule nodes and weights (see the detailed algorithm and download related codes in [32]). The Gauss–Jacobi quadrature formula is used to numerically calculate the integral

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\mathrm{d}x, \quad f(x) \in [-1, 1], \ \alpha, \beta > -1,$$

by using the formula

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\mathrm{d}x \sim \sum_{i=0}^{N} w_i f(x_i).$$

With the help of a change in the variables (which changes both weights w_i and nodes x_i), we can get onto the arbitrary interval $[a, b]$.

Example 5.1. We consider the following the linear Volterra integral equations of second kind with weakly singular kernels

$$y(t) = b(t) - \int_0^t (t-s)^{-\alpha} y(s)\mathrm{d}s, \quad 0 \le t \le T, \tag{5.2}$$

with

$$b(t) = t^{n+\beta} + t^{n+1+\beta-\alpha} B(n+1+\beta, 1-\alpha),$$

where $0 \le \beta \le 1$, and $B(\cdot, \cdot)$ is the Beta function defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t.$$

This problem has an unique solution: $y(t) = t^{n+\beta}$. Obviously, $y(t)$ belongs to $H_{\omega}^{n+1}(I)$. It follows from the theoretical results obtained in this work, the numerical errors will decay with a rate of $\mathcal{O}(N^{-n-1})$. The weighted function ω is chosen as $\omega = (1-x)^{-\alpha}$ with $\alpha = 0.35$. The solution interval is $t \in [0, 6]$, and the exact solution is $y(t) = t^{3.6}$, indicating that $n = 3$ and $\beta = 0.6$. In Fig. 1, numerical errors are plotted for $2 \le N \le 16$ in both L^{∞} and L^2_{ω}-norms. We also present in Table 1 the corresponding numerical errors. As expected, the errors decay algebraically as the exact solution for this example is not sufficiently smooth.

Example 5.2. Consider the following the nonlinear Volterra integral equations of second kind with weakly singular kernels

$$y(t) = b(t) + \int_0^t (t-s)^{-1/3} y^2(s)\mathrm{d}s, \quad 0 \le t \le T, \tag{5.3}$$

where

$$b(t) = (t+2)^{3/2} - \frac{243}{440}t^{11/3} - \frac{81}{20}t^{8/3} - \frac{54}{5}t^{5/3} - 12t^{2/3}.$$

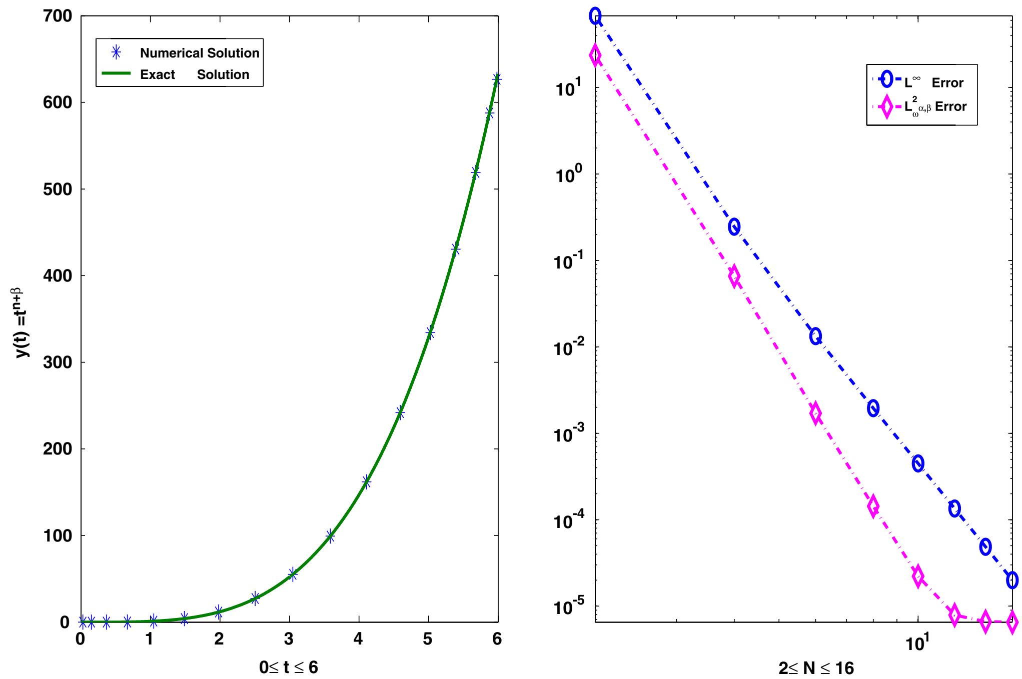

Fig. 1. Left: Numerical and exact solution of $y(t) = t^{3+0.6}$ with $n = 3$ and $\beta = 0.6$. Right: The L^∞ error and L^2_ω error versus N.

Table 1
Example 5.1: L^∞ error and L^2_ω errors, for the solution interval $t \in [0, 6]$.

N	2	4	6	8
L^∞ error	6.7887e+01	2.4594e−01	1.3307e−02	1.9500e−03
L^2_ω error	2.3692e+01	6.5956e−02	1.7042e−03	1.4331e−04
N	10	12	14	16
L^∞ error	4.4826e−04	1.3478e−04	4.8583e−05	1.9980e−05
L^2_ω error	2.2145e−05	7.8788e−06	6.6148e−06	6.5174e−06

Table 2
Example 5.2: L^∞ error and L^2_ω errors, for the solution interval $t \in [0, 10]$.

N	6	8	10	12
L^∞ error	1.0571e−03	1.1271e−04	1.3630e−05	1.7825e−06
L^2_ω error	3.8912e−04	3.6562e−05	3.9691e−06	4.7263e−07
N	14	16	18	20
L^∞ error	2.4594e−07	3.5287e−08	5.2114e−09	7.9513e−010
L^2_ω error	6.0029e−08	7.9972e−09	1.1046e−09	1.5821e−010

This example has a smooth solution $y(t) = (2 + t)^{3/2}$. As a result, we can expect an exponential rate of convergence. In Fig. 2, numerical errors are plotted for $2 \le N \le 24$ in both L^∞- and L^2_ω-norms. The weighted function ω is chosen as $\omega = (1 - x)^{-\alpha}$ with $\alpha = 1/3$. The solution interval is $t \in [0, 10]$. We also present in Table 2 the corresponding numerical errors. As expected, the errors decay exponentially which confirmed our theoretical predictions.

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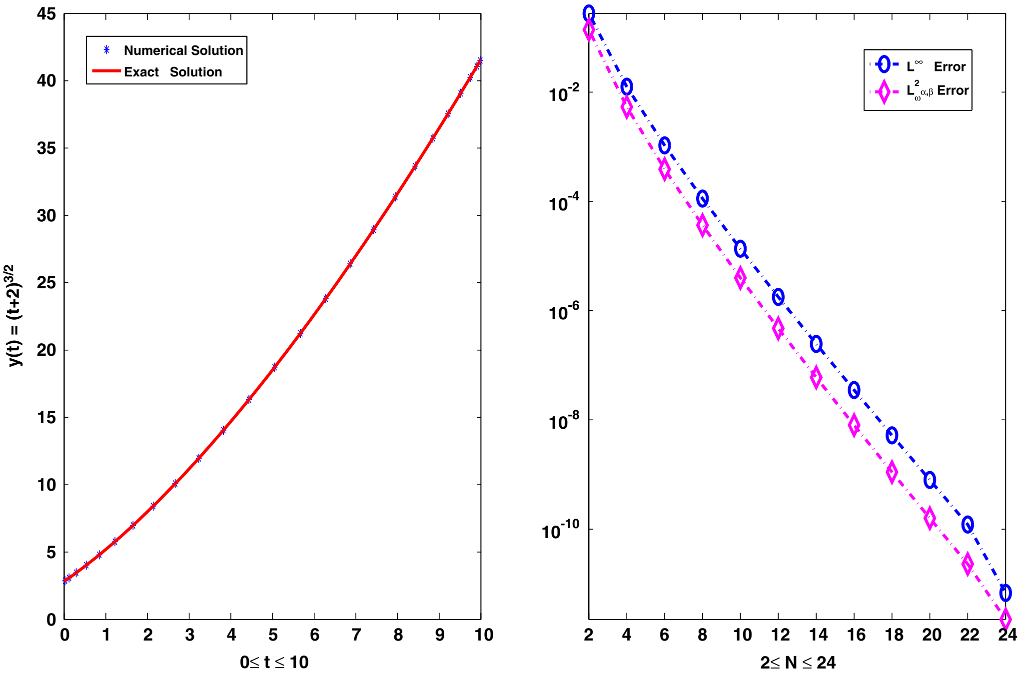

Fig. 2. Left: Numerical and exact solution of $y(t) = (t+2)^{3/2}$. Right: The L^∞ error and L^2_ω error versus N.

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