Monotone Difference Schemes for Two Dimensional Nonhomogeneous Conservation Laws

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Abstract

We study the convergence and error bounds for monotone difference approximations to nonlinear scalar conservation laws with source terms in two space dimensions. A discrete entropy inequality of monotone difference schemes for nonhomogeneous conservation laws is established. Under $\mathrm{BV}(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ assumption on initial data, it is proved that the L^1 errors with monotone schemes are bounded by $O(\sqrt{\Delta t})$. Under $L^\infty \cap L^1$ assumption on initial data, we prove that monotone difference schemes always converge to the entropy solution.

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1 Introduction

We consider the initial value problem for the scalar nonhomogeneous conservation laws in two space dimensions:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} + \frac{\partial g(u)}{\partial y} = \varrho(u), \quad (x,y) \in \mathbf{R}^2, \quad t > 0, \tag{1.1}$$

$$u(x,y,0) = u_0(x,y), \tag{1.2}$$

with $u_0 \in \mathrm{BV}(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$, $f, g \in C^1(\mathbf{R})$, and ϱ satisfies the Lipschitz condition, with a Lipschitz constant L; $\varrho(0) = 0$. For ease of exposition, we assume in this paper that $\varrho = \varrho(u)$, but our results can be easily extended to the more general case $\varrho = \varrho(x,y,t,u)$. If

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the source term $\varrho \equiv 0$, then (1.1) reduces to the homogeneous conservation laws of the form

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}=0, \quad (x,y)\in \mathbf{R}^2, \quad t>0. \tag{1.3}$$

Most problems of technological interest are nonhomogeneous, or multi–dimensional, or both. The source term in the nonhomogeneous problem is due to physical effects or to geometrical effects. For example, the one–dimensional equations for an inviscid, non–heat–conducting, radially symmetric flow can be written in the form (see, [15])

$$\mathbf{U}_t+\mathbf{F}(\mathbf{U})_r=-\mathbf{W}(\mathbf{U}),$$

with

$$\begin{aligned}\mathbf{U}&=[\rho,m,e]^T, \mathbf{F}(\mathbf{U})=[m,m^2/\rho+p,m(e+p)/\rho]^T\\ \mathbf{W}(\mathbf{U})&=(\alpha-1)[m/r,m^2/(\rho r),m(e+p)/(\rho r)]^T,\end{aligned}$$

where ρ is the density, $m=\rho u$ is the momentum, u is the velocity, p is the pressure, e is the energy per unit volume, t is time, r is the space co–ordinate of symmetry, α is a constant which is equal to 2 for cylindrical symmetry and 3 for spherical symmetry. Other examples include the elasticity model describing shearing and flux–turning effects, one–dimensional gas flow in a variable area duct and compressible reacting gas flow (see, e.g. [13, 20]).

The nonhomogeneous conservation laws haven been investigated theoretically by several authors (see, e.g. [9, 11]). In the scalar case, Kruzkov [9] proved the existence and uniqueness of the solution. Numerical schemes based on time–splitting method [3, 15] and on the solution of a generalized Riemann problem with stationary data [6, 19] have been proposed. It is noted that the solution of the (scalar) nonhomogeneous equations does not possess TVD (Total Variation Decreasing) property due to the effect of the source term, but locally TVB (Total Variation Bounded) schemes can be constructed, see [1]. Theoretical analysis for some finite difference methods to (1.1)-(1.2) with stiff source terms can be found in [2, 14].

A finite difference scheme for the problem (1.3) is said to be conservative if it is of the form

$$\begin{aligned}U^{n+1}_{j,k} &= H(U^n_{j-r,k-s},\cdots,U^n_{j+r,k+s})\\ &:= U^n_{j,k}-\lambda_x\left(\bar f^n_{j+1/2,k}-\bar f^n_{j-1/2,k}\right)-\lambda_y\left(\bar g^n_{j,k+1/2}-\bar g^n_{j,k-1/2}\right),\end{aligned} \tag{1.4}$$

where r and s are two fixed positive integers, λ_x and λ_y are fixed constants defined by

$$\lambda_x=\Delta t/\Delta x, \qquad \lambda_y=\Delta t/\Delta y,$$

Δt, Δx and Δy are mesh sizes in time and space directions [1]. The numerical fluxes $\bar f$ and $\bar g$

[1]In practice, the step size in time direction may not be a constant, i.e. $\Delta t=\Delta t^n$. In this case, λ_x and λ_y are also not constant. However, our main results in this paper can be established for the general case with some trivial modifications in the proofs.

are functions of neighboring values of U:

$$
\left\{
\begin{array}{l}
\bar{f}^n_{j+1/2,k} = \bar{f}\left(U^n_{j-r+1,k-s}, \cdots, U^n_{j+r,k+s}\right), \\
\bar{g}^n_{j,k+1/2} = \bar{g}\left(U^n_{j-r,k-s+1}, \cdots, U^n_{j+r,k+s}\right).
\end{array}
\right.
\tag{1.5}
$$

We assume $\bar{f}$ and $\bar{g}$ are consistent with f and g,

$$
\bar{f}(u, \cdots, u) = f(u), \qquad \bar{g}(u, \cdots, u) = g(u),
\tag{1.6}
$$

and are Lipschitz continuous in all arguments (with a Lipschitz constant $\bar{L}$). The scheme (1.4) for (1.3) is *monotone* if H is a monotone nondecreasing function of all arguments.

The monotone schemes for the homogeneous conservation laws can be easily extended to the nonhomogeneous problem (1.1). To compute weak solutions of problem (1.1)–(1.2), we use the following finite difference schemes

$$
\left\{
\begin{array}{l}
U^{n+1}_{j,k} = H\left(U^n_{j-r,k-s}, \cdots, U^n_{j+r,k-s}\right) + \Delta t \varrho\left(U^n_{j,k}\right), \quad n \geq 0, \\
U^0_{j,k} = \frac{1}{\Delta x \Delta y} \int_{y_{k-1/2}}^{y_{k+1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x,y)\,dxdy,
\end{array}
\right.
\tag{1.7}
$$

where $x_{j+1/2} = (j+1/2)\Delta x, y_{k+1/2} = (k+1/2)\Delta y$. The scheme (1.7) for (1.1)-(1.2) is *monotone* if H is a monotone nondecreasing function of all arguments.

Let $\delta = \max(\Delta x, \Delta y, \Delta t)$ and

$$
\Omega_{j,k} = [x_{j-1/2}, x_{j+1/2}) \times [y_{k-1/2}, y_{k+1/2}).
$$

Extend the lattice function $U^m_{j,k}$ to continuous values of x, y, t by setting

$$
u_\delta(x,y,t) = U^m_{j,k}, \qquad (x,y,t) \in \Omega_{j,k} \times [m\Delta t, (m+1)\Delta t).
\tag{1.8}
$$

That is, u_δ is a piecewise constant function with $u_\delta(x_j, y_k, t_m) = U^m_{j,k}$. For any $u \in L^1(\mathbf{R}^2)$, we define operator T_δ as

$$
T_\delta(u)(x,y) = \frac{1}{\Delta x \Delta y} \int_{\Omega_{j,k}} u(\zeta, \eta)\,d\zeta d\eta, \quad (x,y) \in \Omega_{j,k}.
\tag{1.9}
$$

In this paper, we shall prove two main results. First, we shall obtain the convergence rate of the monotone scheme (1.7).

Theorem 1 *Assume that* $u_0 \in \mathrm{BV}(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2), f, g \in C^1(\mathbf{R}), \varrho \in Lip(\mathbf{R})$ *and* $\varrho(0) = 0$. *Let* (1.7) *be a monotone scheme for* (1.1)–(1.2), *which has Lipschitz continuous numerical fluxes* $\bar{f}$ *and* $\bar{g}$. *Let* u_δ *be given by* (1.7)–(1.8). *If* δ *is sufficiently small, then the following estimate holds*

$$
\|u(\cdot,\cdot,t) - u_\delta(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \leq C\sqrt{\delta}, \quad t \in [0,T],
\tag{1.10}
$$

where u *is the unique entropy solution of* (1.1)–(1.2), *and* C *is a constant independent of* δ.

Next, under a weak assumption on the initial function u_0, i.e. without the assumption $u_0 \in \mathrm{BV}(\mathbf{R}^2)$, we shall establish the following convergence results.

Theorem 2 *Assume that* $u_0 \in L^\infty(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$. *If* $f, g, \varrho, \bar{f}, \bar{g}, u_\delta$ *satisfy the conditions stated in Theorem* 1, *then as* $\delta \to 0$, u_δ *converges to the unique solution of* (1.1)–(1.2), u, *in* $L^1(\mathbf{R}^2)$ *uniformly for bounded* $t \geq 0$. *More precisely,*

$$\lim_{\delta\to 0} \sup_{0\leq t\leq T} \int_{\mathbf{R}^2} |u(x,y,t) - u_\delta(x,y,t)| dxdy = 0, \tag{1.11}$$

for each $T > 0$.

Remark. For the homogeneous equation (1.3), Kuznetsov [10] obtained the results stated in Theorem 1 and Crandall and Majda [4] obtained the results stated in Theorem 2. This work is to extend their results to the nonhomogeneous equation (1.1).

2 Preliminaries

A bounded measurable function, u, is a weak solution of (1.1) and (1.2) if for all $\phi \in C^1(\mathbf{R}^2 \times [0,T])$ with compact support in $\mathbf{R}^2 \times [0,T)$,

$$\int_0^T \int_{\mathbf{R}^2} (u\phi_t + f(u)\phi_x + g(u)\phi_y)\, dxdydt + \int_{\mathbf{R}^2} u_0(x,y)\phi(x,y,0)dxdy = -\int_0^T \int_{\mathbf{R}^2} g\phi dxdydt.$$

Weak solutions are not uniquely determined by their initial data and additional principles, entropy conditions, are needed to select the appropriate physical solution. A weak solution, u, of (1.1) and (1.2) is an entropy solution if for all nonnegative $\phi \in C_0^1(\mathbf{R}^2 \times (0,T))$ and any $c \in \mathbf{R}$,

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}^2} \Big(|u-c|\phi_t + \mathrm{sign}(u-c)(f(u)-f(c))\phi_x \\ &\qquad + \mathrm{sign}(u-c)(g(u)-g(c))\phi_y + \mathrm{sign}(u-c)g\phi \Big) dxdydt \geq 0. \end{aligned} \tag{2.1}$$

The bounded–variation seminorm of a function $w \in \mathrm{BV}(\mathbf{R}^2)$ is defined by

$$\begin{aligned} &|w(\cdot,\cdot)|_{\mathrm{BV}(\mathbf{R}^2)} \\ = \;& \sup_{0<|h|<\infty} \frac{1}{|h|} \|w(\cdot+h,\cdot) - w(\cdot,\cdot)\|_{L^1(\mathbf{R}^2)} + \sup_{0<|h|<\infty} \frac{1}{|h|} \|w(\cdot,\cdot+h) - w(\cdot,\cdot)\|_{L^1(\mathbf{R}^2)}. \end{aligned}$$

Following [4, 9, 10, 16], we can establish the following nonlinear stability results.

Lemma 2.1 *If* u_0, f, g *and* ϱ *satisfy the conditions stated in Theorem 1, then there exists a unique weak solution,* $u(x,y,t) \equiv S(t)u_0$, *belonging to* $L^\infty(\mathbf{R}^2 \times [0,T]) \cap \mathrm{BV}(\mathbf{R}^2 \times [0,T]) \cap$

$L^1(\mathbf{R}^2 \times [0,T]) \cap C([0,T], L^1(\mathbf{R}^2))$ *and satisfying the following entropy condition: for any nonnegative function* $\phi \in C_0^1(\mathbf{R}^2 \times \overline{\mathbf{R}^+})$, *any* $c \in \mathbf{R}$, *and any* $\tau_1, \tau_2 \in [0,T]$ *such that* $\tau_1 \le \tau_2$,

$$
\begin{aligned}
&-\int_{\tau_1}^{\tau_2}\int_{\mathbf{R}^2}\Big(|u-c|\phi_\tau + \mathrm{sign}(u-c)(f(u)-f(c))\phi_x + \mathrm{sign}(u-c)(g(u)-g(c)) \\
&+\mathrm{sign}(u-c)\varrho(u)\phi\Big)\,dxdyd\tau + \int_{\mathbf{R}^2}\Big(|u-c|\phi\Big)\Big|_{\tau=\tau_1}^{\tau=\tau_2} dx \le 0,
\end{aligned}
\tag{2.2}
$$

where $u = u(x,y,\tau), \phi = \phi(x,y,\tau)$, *and* $h(x,y,\tau)\Big|_{\tau=\tau_1}^{\tau=\tau_2} = h(x,y,\tau_2) - h(x,y,\tau_1)$. *Moreover, the function* $S(t)u_0$ *satisfies the following properties*

$$
\begin{aligned}
&\|S(t)u_0\|_{L^\infty(\mathbf{R}^2)} \le e^{Lt}\|u_0\|_{L^\infty(\mathbf{R}^2)}; \\
&|S(t)u_0|_{\mathrm{BV}(\mathbf{R}^2)} \le e^{Lt}|u_0|_{\mathrm{BV}(\mathbf{R}^2)}; \\
&\|S(t)u_0\|_{L^1(\mathbf{R}^2)} \le e^{Lt}\|u_0\|_{L^1(\mathbf{R}^2)}; \\
&\|S(\tau_2)u_0 - S(\tau_1)u_0\|_{L^1(\mathbf{R}^2)} \le e^{LT}\Big(C_1|u_0|_{\mathrm{BV}(\mathbf{R}^2)} + L\|u_0\|_{L^1(\mathbf{R}^2)}\Big)|\tau_2-\tau_1|,
\end{aligned}
$$

where L *is the Lipschitz constant of* ϱ, *and*

$$
C_1 = \max_{|u|\le e^{LT}\|u_0\|_{L^\infty(\mathbf{R}^2)}} |f'(u)| + \max_{|u|\le e^{LT}\|u_0\|_{L^\infty(\mathbf{R}^2)}} |g'(u)|.
$$

Lemma 2.2 *Assume* u_0, f, g *and* ϱ *satisfy the conditions stated in Theorem 1. Let* U^n *be computed by* (1.7). *If* H *is a monotone nondecreasing function of all arguments, then* u_δ *defined by* (1.8) *satisfies the following properties:*

$$
\begin{aligned}
&\|u_\delta(\cdot,\cdot,t)\|_{L^\infty(\mathbf{R}^2)} \le e^{LT}\|u_0\|_{L^\infty(\mathbf{R}^2)}; \\
&|u_\delta(\cdot,\cdot,t)|_{\mathrm{BV}(\mathbf{R}^2)} \le e^{LT}|u_0|_{\mathrm{BV}(\mathbf{R}^2)}; \\
&\|u_\delta(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \le e^{LT}\|u_0\|_{L^1(\mathbf{R}^2)}; \\
&\|u_\delta(\cdot,\cdot,\tau_2) - u_\delta(\cdot,\cdot,\tau_1)\|_{L^1(\mathbf{R}^2)} \\
&\qquad \le e^{LT}\Big(C_2|u_0|_{\mathrm{BV}(\mathbf{R}^2)} + L\|u_0\|_{L^1(\mathbf{R}^2)}\Big)(|\tau_2-\tau_1| + \Delta t)
\end{aligned}
\tag{2.3}
$$

for all $t, \tau_1, \tau_2 \in [0,T]$, *where* $C_2 = (2r+1)(2s+1)\bar{L}$. *Moreover, if* $u_\delta(x,y,0)$ *and* $v_\delta(x,y,0)$ *belong to* $L^1(\mathbf{R}^2)$, *then*

$$
\|u_\delta(\cdot,\cdot,t) - v_\delta(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \le e^{LT}\|u_\delta(\cdot,\cdot,0) - v_\delta(\cdot,\cdot,0)\|_{L^1(\mathbf{R}^2)} \tag{2.4}
$$

for $t \in [0,T]$.

We let $\theta(\xi)$ designate a function which is infinitely differentiable on $(-\infty,\infty)$ such that $\theta(\xi) \ge 0, \theta(\xi) \equiv 0$ for $|\xi| \ge 1$, and

$$
\mathrm{sign}(\xi)\theta'(\xi) \le 0, \qquad \int_{-\infty}^{\infty}\theta(\xi)d\xi = 1. \tag{2.5}
$$

The first inequality in (2.5) indicates that $\theta(\xi)$ is a nondecreasing function for $\xi \leq 0$ and is a nonincreasing function for $\xi > 0$. For any number $\epsilon > 0$ we set

$$\theta^\epsilon(\xi) = \frac{1}{\epsilon}\theta\left(\frac{\xi}{\epsilon}\right), \qquad \tilde{\omega}(x,y,t) = \theta^\epsilon(x)\theta^\epsilon(y)\theta^\epsilon(t). \tag{2.6}$$

It is obvious that $\theta^\epsilon \in C_0^\infty(\mathbf{R}), \tilde{\omega} \in C_0^\infty(\mathbf{R}^3), \mathrm{supp}\theta^\epsilon \subset [-\epsilon,\epsilon], \mathrm{supp}\tilde{\omega} \subset [-\epsilon,\epsilon]^3$, and

$$\|\theta^\epsilon\|_{L^1(\mathbf{R})} = 1, \qquad \|\tilde{\omega}\|_{L^1(\mathbf{R}^3)} = 1, \tag{2.7}$$

$$\|\tilde{\omega}_x\|_{L^1(\mathbf{R}^3)} = \|\tilde{\omega}_y\|_{L^1(\mathbf{R}^3)} = \|\tilde{\omega}_t\|_{L^1(\mathbf{R}^3)} = \frac{2}{\epsilon}\theta(0). \tag{2.8}$$

The identities in (2.8) can be obtained by using the first inequality in (2.5). From Lemmas 2.1 and 2.2, we know that u, the entropy solution of (1.1) and (1.2), and u_δ, which is defined by (1.8) and is subject to the conditions stated in Lemma 2.2, belong to $L^\infty(\mathbf{R}^2 \times [0,T]) \cap \mathrm{BV}(\mathbf{R}^2 \times [0,T]) \cap L^1(\mathbf{R}^2 \times [0,T])$ and satisfy

$$\|u(\cdot + h_1, \cdot + h_2, \tau_2) - u(\cdot,\cdot,\tau_1)\|_{L^1(\mathbf{R}^2)} \leq C\max\{|h_1|,|h_2|,|\tau_2-\tau_1|\}; \tag{2.9}$$

$$\|u_\delta(\cdot + h_1, \cdot + h_2, \tau_2) - u_\delta(\cdot,\cdot,\tau_1)\|_{L^1(\mathbf{R}^2)} \leq C\max\{|h_1|,|h_2|,|\tau_2-\tau_1| + \Delta t\} \tag{2.10}$$

for any $h_1, h_2 \in \mathbf{R}; \tau_2, \tau_1 \in [0,T]$, where C is a constant independent of h_1, h_2, τ_2, τ_1 and Δt. For $p, q \in L^\infty(\mathbf{R}^2 \times [0,T]) \cap \mathrm{BV}(\mathbf{R}^2 \times [0,T]) \cap L^1(\mathbf{R}^2 \times [0,T])$, we define the functional Λ in the following form

$$\begin{aligned}\Lambda(p,q,t) = -\int_S\int_S \Big(&|p-q'|\omega_\tau + \mathrm{sign}(p-q')(f(p)-f(q'))\omega_x \\ &+\mathrm{sign}(p-q')(g(p)-g(q'))\omega_y\Big)dsds' + \int_S\int_{\mathbf{R}^2}\{|p-q'|\omega\}\Big|_{\tau=0}^{\tau=t} dxdyds',\end{aligned} \tag{2.11}$$

where $S = \mathbf{R}^2 \times [0,t], p = p(x,y,\tau), q' = q(x',y',\tau'), \omega(x,y,x',y',\tau,\tau') = \tilde{\omega}(x-x', y-y', \tau-\tau'), ds = dxdyd\tau$, and $ds' = dx'dy'd\tau'$. Following [10, 12, 16] we have the following generalized Kuznetsov's lemma.

Lemma 2.3 *Assume that the conditions stated in Theorem 1 hold. Let u be the entropy solution of* (1.1)–(1.2) *and u_δ be defined by* (1.8). *Then for any $t \in [0,T]$,*

$$\begin{aligned}&\|u(\cdot,\cdot,t) - u_\delta(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \\ \leq\ &\|u(\cdot,\cdot,0) - u_\delta(\cdot,\cdot,0)\|_{L^1(\mathbf{R}^2)} + \Lambda(u,u_\delta,t) + \Lambda(u_\delta,u,t) + C(\epsilon + \Delta t),\end{aligned} \tag{2.12}$$

where C is independent of ϵ and Δt.

Finally, we need the following discrete entropy inequality which will play a key role in the proof of Theorem 1.

Lemma 2.4 *If* (1.7) *is a monotone scheme for* (1.1)–(1.2), *then the following discrete entropy inequality holds*

$$
\begin{aligned}
(2.13) \qquad & T_t^+|U_{j,k}^n - c| + \lambda_x T_x^+ \bar{F}_{j-1/2,k}^n(c) + \lambda_y T_y^+ \bar{G}_{j,k-1/2}^n(c) \\
\leq \; & \mathrm{sign}(U_{j,k}^{n+1} - c)\Delta t \varrho(U_{j,k}^n),
\end{aligned}
$$

where the numerical entropy fluxes $\bar{F}$ and $\bar{G}$ are given by

(2.14)

$$
\begin{cases}
\bar{F}_{j+1/2,k}^n(c) = \bar{f}\left(U_{j-r+1,k-s}^n \vee c, \cdots, U_{j+r,k+s}^n \vee c\right) - \bar{f}\left(U_{j-r+1,k-s}^n \wedge c, \cdots, U_{j+r,k+s}^n \wedge c\right), \\
\bar{G}_{j+1/2,k}^n(c) = \bar{g}\left(U_{j-r,k-s+1}^n \vee c, \cdots, U_{j+r,k+s}^n \vee c\right) - \bar{g}\left(U_{j-r,k-s+1}^n \wedge c, \cdots, U_{j+r,k+s}^n \wedge c\right),
\end{cases}
$$

with $a \vee b = \max\{a, b\}, a \wedge b = \min\{a, b\}$, and the difference operators T are defined by

$$
T_t^+ V_{j,k}^n = V_{j,k}^{n+1} - V_{j,k}^n, \quad T_x^+ V_{j,k}^n = V_{j+1,k}^n - V_{j,k}^n, \quad T_y^+ V_{j,k}^n = V_{j,k+1}^n - V_{j,k}^n.
$$

Proof. Let $\bar{U}_{j,k}^n = H(U_{j-r,k-s}^n, \cdots, U_{j+r,k+s}^n)$. Since H is a monotone nondecreasing function for all arguments, we have (see, [4, 8])

$$
(2.15) \qquad |\bar{U}_{j,k}^n - c| - |U_{j,k}^n - c| + \lambda_x T_x^+ \bar{F}_{j-1/2,k}^n(c) + \lambda_y T_y^+ \bar{G}_{j,k-1/2}^n(c) \leq 0.
$$

Noting that for any $a, b \in \mathbf{R}$,

$$
|a - c| - |b - c| \leq \mathrm{sign}(a - c)(a - b),
$$

we have

$$
\begin{aligned}
|U_{j,k}^{n+1} - c| - |\bar{U}_{j,k}^n - c| &\leq \mathrm{sign}(U_{j,k}^{n+1} - c)(U_{j,k}^{n+1} - \bar{U}_{j,k}^n) \\
&= \mathrm{sign}(U_{j,k}^{n+1} - c)\Delta t \varrho(U_{j,k}^n).
\end{aligned}
$$

The above result, together with (2.15), leads to the desired inequality (2.13). Q.E.D.

3 Proof of Theorem 1

In the following sections, C denotes a positive constant independent of $\epsilon, \Delta x, \Delta y$ and Δt. The following notations will be used frequently in this section:

$$
(3.1) \qquad \begin{cases}
u = u(x, y, \tau), \; u_\delta = u_\delta(x, y, \tau), \; u' = u(x', y', \tau'), \; u'_\delta = u_\delta(x', y', \tau'); \\
\omega = \omega(x, x', y, y', \tau, \tau') := \tilde{\omega}(x - x', y - y', \tau - \tau'); \\
S = \mathbf{R}^2 \times [0, t_n], \; t_n = n\Delta t, \; ds = dxdyd\tau, \; ds' = dx'dy'd\tau'.
\end{cases}
$$

We first prove the following lemmas.

Lemma 3.1 *If u is the entropy solution of* (1.1)–(1.2) *and u_δ is computed by* (1.7)*, then*

$$\Lambda(u_\delta, u, t_n) \leq \int_S \int_S \operatorname{sign}(u_\delta - u')\varrho(u_\delta)\omega ds ds' + C\left(\delta + \frac{\delta}{\epsilon}\right). \tag{3.2}$$

Proof. By the definition of $\Lambda(u_\delta, u, t_n)$, we have

$$\Lambda(u_\delta, u, t_n) = I_1 + I_2 + I_3,$$

where

$$\begin{cases} I_1 = -\int_S \int_S |u_\delta - u'|\omega_\tau ds ds' + \int_S \int_{\mathbf{R}^2} \left\{|u_\delta - u'|\omega\right\} \Big|_{\tau=0}^{\tau=t_n} dx dy ds', \\ I_2 = \int_S \int_S \Big(f(u_\delta \vee u') - f(u_\delta \wedge u')\Big)\omega_x ds ds', \\ I_3 = \int_S \int_S \Big(g(u_\delta \vee u') - g(u_\delta \wedge u')\Big)\omega_y ds ds'. \end{cases} \tag{3.3}$$

Since u_δ is a piecewise constant function, it can be shown that

$$I_1 = \int_S \int_{\mathbf{R}^2} \sum_{m=0}^{n-1} \Big(|u_\delta(x, y, t_{m+1}) - u'| - |u_\delta(x, y, t_m) - u'|\Big)\omega(x, x', y, y', t_{m+1}, \tau')dx dy ds'.$$

Using the Mean Value theorem, we obtain that

$$\begin{aligned} I_1 &= \sum_{m=0}^{n-1} \int_S \sum_{j,k} T_t^+ |U_{j,k}^m - u'| \int_{y_{k-1/2}}^{y_{k+1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} \omega(x, x', y, y', t_{m+1}, \tau')dx dy ds' \\ &= \sum_{m=0}^{n-1} \int_S \sum_{j,k} T_t^+ |U_{j,k}^m - u'|\omega(x_j^*, x', y_k^*, y', t_{m+1}, \tau')\Delta x \Delta y ds', \end{aligned} \tag{3.4}$$

where $x_j^* \in (x_{j-1/2}, x_{j+1/2}), y_k^* \in (y_{k-1/2}, y_{k+1/2})$. Similarly, we have

$$I_2 = \sum_{m=0}^{n-1} \int_S \sum_{j,k} T_x^+ \left(f(U_{j,k}^m \vee u') - f(U_{j,k}^m \wedge u')\right) \omega(x_{j+1/2}, x', y_k^{**}, y', t_m^*, \tau')\Delta y \Delta t ds', \tag{3.5}$$

$$I_3 = \sum_{m=0}^{n-1} \int_S \sum_{j,k} T_y^+ \left(g(U_{j,k}^m \vee u') - g(U_{j,k}^m \wedge u')\right) \omega(x_j^{**}, x', y_{k+1/2}, y', t_m^{**}, \tau')\Delta x \Delta t ds', \tag{3.6}$$

where $y_k^{**} \in (y_{k-1/2}, y_{k+1/2}), t_m^* \in (t_m, t_{m+1})$, $x_j^{**} \in (x_{j-1/2}, x_{j+1/2}), t_m^{**} \in (t_m, t_{m+1})$. Using Lemma 2.2 and the triangular inequality gives

$$\sum_{m=0}^{n-1} \|u_\delta(\cdot, \cdot, t_{m+1}) - u_\delta(\cdot, \cdot, t_m)\|_{L^1(\mathbf{R}^2)} \leq C;$$

$$\sum_{m=0}^{n-1} \sum_{j,k} \left|T_x^+ \left(f(U_{j,k}^m \vee u') - f(U_{j,k}^m \wedge u')\right) \Delta y \Delta t\right| \leq C;$$

$$\sum_{m=0}^{n-1} \sum_{j,k} \left|T_y^+ \left(g(U_{j,k}^m \vee u') - g(U_{j,k}^m \wedge u')\right) \Delta x \Delta t\right| \leq C.$$

Notice that if $\max\{|x_1-x_2|,|y_1-y_2|,|\tau_1-\tau_2|\}\le\delta$ then

$$\int_S |\omega(x_1,x',y_1,y',\tau_1,\tau')-\omega(x_2,x',y_2,y',\tau_2,\tau')|ds'\le\frac{C\delta}{\epsilon}. \tag{3.7}$$

The above four inequalities, together with (3.4)–(3.6), yield

$$\begin{aligned}&\Lambda(u_\delta,u,t_n)\\ \le\ &\sum_{m=0}^{n-1}\sum_{j,k}\int_S\Big(\Delta x\Delta y T_t^+|U_{j,k}^m-u'|+\Delta t\Delta y T_x^+\left(f(U_{j,k}^m\vee u')-f(U_{j,k}^m\wedge u')\right)\\ &\quad+\Delta t\Delta x T_y^+\left(g(U_{j,k}^m\vee u')-g(U_{j,k}^m\wedge u')\right)\Big)\omega(x_j,x',y_k,y',t_m,\tau')ds'+\frac{C\delta}{\epsilon}.\end{aligned} \tag{3.8}$$

Using (2.14) we can rewrite (3.8) in the form

$$\Lambda(u_\delta,u,t_n)\le I_4+I_5+I_6+\frac{C\delta}{\epsilon},$$

where

$$\begin{aligned}I_4:=&\sum_{m=0}^{n-1}\sum_{j,k}\int_S\Big(\Delta x\Delta y T_t^+|U_{j,k}^m-u'|+\Delta y\Delta t T_x^+\bar F_{j-1/2,k}^m(u')+\Delta x\Delta t\bar G_{j,k-1/2}^m(u')\Big)\\ &\quad\times\omega(x_j,x',y_k,y',t_m,\tau')ds'\end{aligned}$$

$$\begin{aligned}I_5:=&\sum_{m=0}^{n-1}\sum_{j,k}\int_S\Delta y\Delta t T_x^+\Big(f(U_{j,k}^m\vee u')-\bar f(U_{j-r,k-s}^m\vee u',\cdots,U_{j+r-1,k+s}^m\vee u')\\ &-f(U_{j,k}^m\wedge u')+\bar f(U_{j-r,k-s}^m\wedge u',\cdots,U_{j+r-1,k+s}^m\wedge u')\Big)\omega(x_j,x',y_k,y',t_m,\tau')ds'\end{aligned}$$

$$\begin{aligned}I_6:=&\sum_{m=0}^{n-1}\sum_{j,k}\int_S\Delta t\Delta x T_y^+\Big(g(U_{j,k}^m\vee u')-\bar g(U_{j-r,k-s}^m\vee u',\cdots,U_{j-r,k+s-1}^m\vee u')\\ &-g(U_{j,k}^m\wedge u')+\bar g(U_{j-r,k-s}^m\wedge u',\cdots,U_{j+r,k+s-1}^m\wedge u')\Big)\omega(x_j,x',y_k,y',t_m,\tau')ds'.\end{aligned}$$

Since the numerical flux $\bar f$ is Lipschitz continuous in all arguments and is consistent with f, we have

$$\begin{aligned}I_5\le&\sum_{m=0}^{n-1}\sum_{j,k}\Delta y\Delta t\int_S\Big|\bar f(U_{j,k}^m\vee u',\cdots,U_{j,k}^m\vee u')-\bar f(U_{j-r,k-s}^m\vee u',\cdots,U_{j+r-1,k+s}^m\vee u')\\ &-\bar f(U_{j,k}^m\wedge u',\cdots,U_{j,k}^m\wedge u')+\bar f(U_{j-r,k-s}^m\wedge u',\cdots,U_{j+r-1,k+s}^m\wedge u')\Big|\\ &\times\Big|T_x^+\omega(x_{j-1},x',y_k,y',t_m,\tau')\Big|ds'\\ \le\ &C\sum_{p=-r}^{r-1}\sum_{q=-s}^{s}\sum_{m=0}^{n-1}\|u_\delta(\cdot+p\Delta x,\cdot+q\Delta y,t_m)-u_\delta(\cdot,\cdot,t_m)\|_{L^1(\mathbf{R}^2)}\frac{\delta}{\epsilon}\\ \le\ &\frac{C\delta}{\epsilon},\end{aligned}$$

where in obtaining the last inequality we have used (2.10). Similarly, it can be shown that $I_6 \le C\delta/\epsilon$. The remaining work is to show that I_4 can be bounded by the right hand side of (3.2). Using Lemma 2.4 gives

$$I_4 \le \sum_{m=0}^{n-1} \sum_{j,k} \int_S \Delta x \Delta y \Delta t \,\mathrm{sign}\left(U_{j,k}^{m+1} - u'\right) \varrho\left(U_{j,k}^m\right) \omega(x_j, x', y_k, y', t_m, \tau') ds'. \tag{3.9}$$

Let

$$\begin{aligned}
I_7 &:= \int_S \int_S \mathrm{sign}\left(u_\delta(x, y, \tau + \Delta t) - u'\right) \varrho(u_\delta) \omega ds ds', \\
I_8 &:= \sum_{m=0}^{n-1} \sum_{j,k} \int_S \int_{t_m}^{t_{m+1}} \int_{y_{k-1/2}}^{y_{k+1/2}} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathrm{sign}(U_{j,k}^{m+1} - u') \varrho(U_{j,k}^m) \\
&\quad \times \Big(\omega(x_j, x', y_k, y', t_m, \tau') - \omega(x, x', y, y', \tau, \tau')\Big) dx dy d\tau ds'.
\end{aligned}$$

It follows from (3.9) that

$$I_4 \le I_7 + I_8.$$

Using the change of variable $\tau \to \tau - \Delta t$ gives

$$I_7 = I_9 + I_{10},$$

where

$$\begin{aligned}
I_9 &= \int_S \int_{\mathbf{R}^2} \int_{\Delta t}^{t_n + \Delta t} \mathrm{sign}(u_\delta - u') \varrho(u_\delta) \omega ds ds', \\
I_{10} &= \int_S \int_{\mathbf{R}^2} \int_{\Delta t}^{t_n + \Delta t} \mathrm{sign}(u_\delta - u') \Big(\varrho(u_\delta(x, y, \tau - \Delta t)) \omega(x, x', y, y', \tau - \Delta t, \tau') - \varrho(u_\delta)\omega\Big) ds ds'.
\end{aligned}$$

It can be shown that

$$I_9 \le \int_S \int_S \mathrm{sign}(u_\delta - u') \varrho(u_\delta) \omega ds ds' + C\Delta t.$$

Further, noting that ϱ is Lipschitz continuous, u_δ and ω satisfy (2.3) and (3.7), respectively, we can obtain

$$I_{10} \le C\left(\Delta t + \Delta t/\epsilon\right).$$

Moreover, using (3.7) yields $I_8 \le C\delta/\epsilon$. The upper bounds for I_8, I_9 and I_{10} imply that I_4 can be bounded by the right hand side of (3.2). This completes the proof of Lemma 3.1. Q.E.D.

Lemma 3.2 *If u is the entropy solution of* (1.1)–(1.2) *and u_δ is computed by* (1.7), *then*

$$\int_S \int_S |u_\delta - u'| \omega ds ds' \le \int_0^{t_n} \|u(\cdot, \cdot, \tau) - u_\delta(\cdot, \cdot, \tau)\|_{L^1(\mathbf{R}^2)} d\tau + C\epsilon. \tag{3.10}$$

Proof. For a fixed $\tau \in [0, t_n]$, we consider the following inequalities:

$$
\begin{aligned}
(3.11) \qquad & \int_S \int_{\mathbf{R}^2} \Big(|u_\delta(x,y,\tau) - u(x',y',\tau')| - |u_\delta(x,y,\tau) - u(x,y,\tau)| \Big) \omega dx dy ds' \\
\leq \quad & \int_S \int_{\mathbf{R}^2} |u(x',y',\tau') - u(x,y,\tau)| \omega dx dy d\tau' \\
\leq \quad & \int_S \int_{\mathbf{R}^2} |u(x',y',\tau') - u(x,y,\tau')| \omega dx dy ds' + \int_S \int_{\mathbf{R}^2} |u(x,y,\tau') - u(x,y,\tau)| \omega dx dy ds' \\
=: \quad & I_{11} + I_{12}.
\end{aligned}
$$

Using change of variables $x - x' = h_1$ and $y - y' = h_2$ gives

$$
I_{11} = \int_S \int_{\mathbf{R}^2} |u(x',y',\tau') - u(x'+h_1, y'+h_2, \tau')| \omega(h_1, h_2, \tau - \tau') dh_1 dh_2 ds'.
$$

It follows from (2.9) that

$$
\begin{aligned}
I_{11} \quad & \leq \quad \int_0^{t_n} \int_{\mathbf{R}^2} \|u(\cdot,\cdot,\tau') - u(\cdot + h_1, \cdot + h_2, \tau')\|_{L^1(\mathbf{R}^2)} \omega(h_1, h_2, \tau - \tau') dh_1 dh_2 d\tau' \\
& \leq \quad C \int_0^{t_n} \int_{\mathbf{R}^2} \Big(|h_1| + |h_2| \Big) \omega(h_1, h_2, \tau - \tau') dh_1 dh_2 d\tau' \leq C\epsilon.
\end{aligned}
$$

Similarly, it can be shown that $I_{12} \leq C\epsilon$. Therefore, (3.11) gives

$$
\begin{aligned}
& \int_S \int_{\mathbf{R}^2} |u_\delta(x,y,\tau) - u(x',y',\tau')| \omega dx dy ds' \\
\leq \quad & \int_S \int_{\mathbf{R}^2} |u_\delta(x,y,\tau) - u(x,y,\tau)| \omega dx dy ds' + C\epsilon \\
\leq \quad & \int_0^{t_n} \|u_\delta(\cdot,\cdot,\tau) - u(\cdot,\cdot,\tau)\|_{L^1(\mathbf{R}^2)} d\tau \int_S \tilde{\omega}(x',y',\tau') ds' + C\epsilon \\
\leq \quad & \int_0^{t_n} \|u_\delta(\cdot,\cdot,\tau) - u(\cdot,\cdot,\tau)\|_{L^1(\mathbf{R}^2)} d\tau + C\epsilon,
\end{aligned}
$$

where in the last step we have used the second equation in (2.7). Q.E.D.

We now turn to the proof of Theorem 1. Letting $\tau_2 = t_n, \tau_1 = 0, c = u_\delta(x',y',\tau')$ and $\phi(x,y,\tau) = \omega(x,x',y,y',\tau,\tau')$ in (2.2), and integrating the resulting inequality with respect to x', y', τ' in $\mathbf{R}^2 \times [0, t_n]$ yield

$$
(3.12) \qquad \Lambda(u, u_\delta, t_n) \leq \int_S \int_S \operatorname{sign}(u - u'_\delta) \varrho(u) \omega ds ds' = -\int_S \int_S \operatorname{sign}(u_\delta - u') \varrho(u') \omega ds ds',
$$

where in the last step we have used the transformations $x \leftrightarrow x', y \leftrightarrow y'$ and $t \leftrightarrow t'$. An application of Lemma 2.3, together with (3.2) and (3.12), gives

$$
\begin{aligned}
(3.13) \qquad & \|u(\cdot,\cdot,t_n) - u_\delta(\cdot,\cdot,t_n)\|_{L^1(\mathbf{R}^2)} \leq \|u(\cdot,\cdot,0) - u_\delta(\cdot,\cdot,0)\|_{L^1(\mathbf{R}^2)} \\
& + C\left(\epsilon + \delta + \frac{\delta}{\epsilon}\right) + \int_S \int_S \operatorname{sign}(u_\delta - u')(\varrho(u_\delta) - \varrho(u')) \omega ds ds'.
\end{aligned}
$$

Lemma 3.2 and (3.13) lead to a Gronwall type inequality,

$$\begin{aligned}
&\|u(\cdot,\cdot,t_n) - u_\delta(\cdot,\cdot,t_n)\|_{L^1(\mathbf{R}^2)} \\
\leq\ & C\int_0^{t_n} \|u(\cdot,\cdot,\tau) - u_\delta(\cdot,\cdot,\tau)\| + \|u(\cdot,\cdot,0) - u_\delta(\cdot,\cdot,0)\|_{L^1(\mathbf{R}^2)} + C\left(\epsilon + \delta + \frac{\delta}{\epsilon}\right),
\end{aligned}$$

which gives

$$\|u(\cdot,\cdot,t_n) - u_\delta(\cdot,\cdot,t_n)\|_{L^1(\mathbf{R}^2)} \leq C\|u(\cdot,\cdot,0) - u_\delta(\cdot,\cdot,0)\|_{L^1(\mathbf{R}^2)} + C\left(\epsilon + \delta + \frac{\delta}{\epsilon}\right).$$

Setting $\epsilon = \sqrt{\delta}$ and noting that $\|u(\cdot,\cdot,0) - u_\delta(\cdot,\cdot,0)\|_{L^1(\mathbf{R}^2)} \leq C\delta$, we obtain Theorem 1.

4 Proof of Theorem 2

For general $u_0 \in L^1(\mathbf{R}^2)$ with $|u_0| \leq R$ *a.e.* we choose $u_0^{(m)} \in \mathrm{BV}(\mathbf{R}^2) \cap L^1(\mathbf{R}^2)$ with compact support such that

$$(4.1) \qquad |u_0^{(m)}| \leq R \quad a.e., \quad \text{and} \quad \lim_{m\to\infty} \|u_0^{(m)} - u_0\|_{L^1(\mathbf{R}^2)} = 0.$$

This can be done by any standard method. Denote the difference scheme solution corresponding to the initial–value $T_\delta(u_0^{(m)})$ (see, (1.9)) by $u_\delta^{(m)}$. By Theorem 1, there is a function $u^{(m)}$ such that

$$(4.2) \qquad \lim_{\delta\to 0} \sup_{0\leq t\leq T} \|u_\delta^{(m)}(\cdot,\cdot,t) - u^{(m)}(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} = 0.$$

Moreover, it follows from (2.4) that

$$(4.3) \qquad \sup_{0\leq t\leq T} \|u_\delta^{(m)}(\cdot,\cdot,t) - u_\delta^{(k)}(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \leq e^{LT}\|u_0^{(m)} - u_0^{(k)}\|_{L^1(\mathbf{R}^2)}.$$

Letting $\delta \to 0$, we obtain from (4.3) that

$$(4.4) \qquad \sup_{0\leq t\leq T} \|u^{(m)}(\cdot,\cdot,t) - u^{(k)}(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \leq e^{LT}\|u_0^{(m)} - u_0^{(k)}\|_{L^1(\mathbf{R}^2)},$$

which implies that $\{u^{(m)}\}$ is Cauchy in $L^1(\mathbf{R}^2)$ and hence converges uniformly to a limit $u \in L^1(\mathbf{R}^2)$. From (4.1) and (4.4) we also obtain

$$(4.5) \qquad \sup_{0\leq t\leq T} \|u^{(m)}(\cdot,\cdot,t) - u(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \leq e^{LT}\|u_0^{(m)} - u_0\|_{L^1(\mathbf{R}^2)}.$$

We now prove that u_δ converges to u in $L^1(\mathbf{R}^2)$ uniformly for $t \in [0,T]$. To this end, observe that

$$\begin{aligned}
&\|u_\delta(\cdot,\cdot,t) - u(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \leq \|u_\delta(\cdot,\cdot,t) - u_\delta^{(m)}(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \\
&\qquad + \|u_\delta^{(m)}(\cdot,\cdot,t) - u^{(m)}(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} + \|u^{(m)}(\cdot,\cdot,t) - u(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \\
\leq\ & e^{LT}\|u_0 - u_0^{(m)}\|_{L^1(\mathbf{R}^2)} + \|u_\delta^{(m)}(\cdot,\cdot,t) - u^{(m)}(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} + \|u^{(m)}(\cdot,\cdot,t) - u(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)}.
\end{aligned}$$

This, together with (4.5), gives

$$
\begin{aligned}
& \sup_{0\le t\le T} \|u_\delta(\cdot,\cdot,t) - u(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)} \\
\le\ & 2e^{LT}\|u_0^{(m)} - u_0\|_{L^1(\mathbf{R}^2)} + \sup_{0\le t\le T} \|u_\delta^{(m)}(\cdot,\cdot,t) - u^{(m)}(\cdot,\cdot,t)\|_{L^1(\mathbf{R}^2)}.
\end{aligned}
$$

The first term above can be made small by taking m large. The second term can be made small by taking δ small. Hence $u_\delta \to u$ in $L^1(\mathbf{R}^2)$ uniformly for $t \in [0,T]$. Moreover, it follows from (4.1) and (2.3) that $|u| \le R$, *a.e.* and

$$
\|u(\cdot,\cdot,t_2) - u(\cdot,\cdot,t_1)\|_{L^1(\mathbf{R}^2)} \le C\left(|t_2 - t_1| + \Delta t\right), \tag{4.6}
$$

for $t_1, t_2 \in [0,T]$. Using (2.13) we can also show that u satisfies (2.1). That is, u is an entropy solution of the conservation law (1.1). By (4.6) and $u(\cdot,\cdot,0) = u_0$, u assumes the initial value u_0 continuously in $L^1(\mathbf{R}^2)$. These results guarantee that u is the unique entropy solution of (1.1)–(1.2). This completes the proof of Theorem 2.

5 Concluding remarks

In this work we have studied the convergence and error bounds for monotone difference approximations to nonlinear scalar conservation laws with source terms in two space dimensions. As shown in [17], the order half convergence rate for the class of initial data considered in this paper is the optimal one. However, if the initial data is chosen such that the solution of a conservation law is piecewise smooth [18], then the order half convergence rate may be improved. For 1-D conservation laws without source terms, several authors have obtained order one convergence rate for difference methods or viscosity methods, see e.g. [5, 7, 18]. It remains to be seen if the order one rate holds for the two dimensional problems.

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