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ON THE REGULARITY OF APPROXIMATE SOLUTIONS TO CONSERVATION LAWS WITH PIECEWISE SMOOTH SOLUTIONS*

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Abstract. In this paper we address the questions of the convergence rate for approximate solutions to conservation laws with piecewise smooth solutions in a weighted $W^{1,1}$ space. Convergence rate for the *derivative* of the approximate solutions is established under the assumption that a weak *pointwise-error* estimate is given. In other words, we are able to convert weak pointwise-error estimates to optimal error bounds in a weighted $W^{1,1}$ space.

For convex conservation laws, the assumption of a weak pointwise-error estimate is verified by Tadmor [*SIAM J. Numer. Anal.*, 28 (1991), pp. 891–906]. Therefore, one immediate application of our $W^{1,1}$- convergence theory is that for convex conservation laws we indeed have $W^{1,1}$-error bounds for the approximate solutions to conservation laws. Furthermore, the $\mathcal{O}(\epsilon)$-pointwise-error estimates of Tadmor and Tang [*SIAM J. Numer. Anal.*, 36 (1999), pp. 1739–1758] are recovered by the use of the $W^{1,1}$-convergence result.

Key words. conservation laws, error estimates, viscosity approximation, optimal convergence rate

AMS subject classifications. 35L65, 65M10, 65M15

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1. Introduction.

We study the convergence of vanishing viscosity solutions governed by the single conservation law

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon, \qquad x \in \mathbf{R},\ t > 0, \epsilon > 0 \tag{1.1}$$

and subject to the initial condition prescribed at $t = 0$,

$$u^\epsilon(x, 0) = u_0(x). \tag{1.2}$$

We are interested in the convergence rate of u_x^ϵ towards the derivative of the inviscid solution, u_x, of the corresponding inviscid conservation law

$$u_t + f(u)_x = 0, \qquad x \in \mathbf{R},\ t > 0, \tag{1.3}$$

which is subject to the same initial condition

$$u(x, 0) = u_0(x). \tag{1.4}$$

There has been an enormous amount of papers related to the error estimates for the viscosity or more general approximations to scalar conservation laws. The methods of analysis include matching method and traveling wave solutions (see, e.g., Goodman and Xin [5]); matching the Green function of the linearized problem (see, e.g., Liu [13]); weak $W^{-1,1}$ convergence theory (see, e.g., Tadmor [21]); the Kruzkov-functional method (see, e.g., Kuznetsov [9]); and energy-like methods (see, e.g., Tadmor and Tang [22]). The results on error estimates include

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- for BV entropy solutions to (1.3), an $\mathcal{O}(\sqrt{\epsilon})$ convergence rate in L^1 obtained by Kuznetsov [9], Lucier [14], Sanders [18], Cockburn–Gremaud–Yang [3], etc. (in the BV-solution space, it is shown by Sabac [17] and Tang and Teng [27] that the L^1-convergence rate of order $\mathcal{O}(\sqrt{\epsilon})$ is optimal);
- for BV entropy solutions, an $\mathcal{O}(\epsilon)$ convergence rate in $W^{-1,1}$ obtained by Tadmor [21], Nessyahu and Tadmor [15], Nessyahu–Tadmor–Tassa [16], Liu and Warnecke [11], Liu, Wang, and Warnecke [12] etc.;
- for piecewise smooth solutions for (1.3), an $\mathcal{O}(\epsilon)$ convergence rate in L^1 obtained by Bakhvalov [1], Harabetian [6], Teng and Zhang [30], Fan [2], Tang and Teng [28], Teng [29], etc.;
- for piecewise smooth solutions, an $\mathcal{O}(\epsilon)$ convergence rate in the smooth region of the entropy solution obtained by Goodman and Xin [5], Engquist and Sjögreen [4], Tadmor and Tang [22], [25], etc.

The results listed above are concerned with the convergence of the approximate solution itself; essentially nothing is obtained for its derivative. In this work, we will investigate the convergence of the first derivative of the approximate solutions. We will assume that the inviscid solution u has finitely many discontinuities, which is the generic situation, [19], [26]. By properly choosing a weighted function, we will obtain an $\mathcal{O}(\epsilon)$-bound for $u^\epsilon - u$ in a weighted $W^{1,1}$ space. More precisely, we will show that the following estimate holds:

$$\int_{\mathbf{R}} \rho(x,t)\Big(|u^\epsilon_x - u_x| + |u^\epsilon - u|\Big)dx \le C\epsilon\,,$$

where ρ is a *distance* function to the singular support of $u(x,t)$. In case that there is only one shock discontinuity $x = X(t)$, the above result implies that

$$\|u^\epsilon_x(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\setminus[X(t)-h,X(t)+h])} \le C(h)\epsilon$$

for any given $h > 0$. If there are finitely many shock curves $S(t) = \{(x,t)|x = X_k(t)\}_{k=1}^K$, then we have

$$\|u^\epsilon_x(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\setminus\cup_k[X_k(t)-h,X_k(t)+h])} \le C(h)\epsilon.$$

In this work, the above estimates are established under the assumption that a weak pointwise-error estimate holds away from the singular support of the entropy solution $u(x,t)$. In other words, we are able to convert weak pointwise-error estimates to optimal error bounds in a weighted $W^{1,1}$ space. It is noticed that such a weak pointwise-error estimate is verified for convex conservation laws by Tadmor [21]. Furthermore, our $W^{1,1}$-estimate recovers the $\mathcal{O}(\epsilon)$-pointwise error bound obtained by Tadmor and Tang [22], i.e.,

$$|(u^\epsilon - u)(x,t)| \le C(h)\epsilon\,, \qquad dist(x,S(t)) \ge h,$$

for any given $h > 0$, where $S(t) = \{(x,t)|x = X_k(t)\}_{k=1}^K$. Since our result is obtained in a completely different way, this work gives an independent check of the result in [22].

There are also many studies on the convergence and stability of the viscosity approximations to the *system* of conservation laws, see, e.g., [5], [7], [8], [13], [20], [31]. Kreiss and Kreiss [7] and Kreiss, Kreiss, and Lorenz [8] provide an energy-method approach (which is different from the ones used in this work and [22]) for the question of interior regularity. In [13], Liu used a pointwise-error estimate to study the asymptotic stability of a viscous shock profile. The convergence of viscous solutions to inviscid solutions with one shock is investigated by Goodman and Xin [5] and Yu [31]. In the scalar case, we wish to obtain a clear picture about the pointwise convergence rate, not only for one shock but also for finitely many shock discontinuities. Moreover, we wish to provide some general methods which can be applied to other types of approximate schemes; see [22], [23].

We close this section by emphasizing two important points of this work: (1) Unlike most of previous work on error estimates, the present paper gives the error bounds for the *derivative* of the approximate solutions. This result, together with previous results by Tadmor, etc. for pointwise-error estimates, gives a clear picture about the pointwise convergence rates of approximate solutions to scalar conservation laws. (2) The present results suggest that if a weak pointwise-error estimate can be established for the *nonconvex* conservation laws, then the optimal pointwise-error estimate can be obtained for the *nonconvex* case. So far, almost no pointwise-error estimates for numerical approximations have been obtained for nonconvex conservation laws.

2. Preliminaries. For ease of exposition we shall make the following assumptions in this section:

- (A1) the initial data u_0 is piecewisely C^3-smooth and is compactly supported;
- (A2) we assume that there exists a smooth curve, $x = X(t)$, such that $u(x,t)$ is smooth at any point away from $x = X(t)$;
- (A3) there exists a constant $0 < \gamma \leq 1$ and a constant $C_T > 0$ such that

$$|u^\epsilon(x,t) - u(x,t)| \leq C_T \epsilon^\gamma \qquad \text{for} \quad |x - X(t)| > C_T \epsilon^\gamma\,. \tag{2.1}$$

Remark 1. We make the following remarks and observations:

- (a) assumption (A1) implies that $u_{xxx}(\bullet, t) \in L^1(\mathbf{R})$ which will be used in the proofs of the next section (see the estimate (3.8)); also, if we are interested in pointwise-error estimates in a finite domain (not too far from the shock curve), then it is reasonable to assume that u_0 is compactly supported;
- (b) although we consider only the case with one shock, the extension to finitely many shocks/rarefaction waves can be carried out by following [22]; see section 4 for more detail discussion;
- (c) assumption (A3) is satisfied for convex conservation laws with Lip^+-bounded initial data, with $\gamma = 1/3$ (see Tadmor [21]); it can be improved to $\gamma = 1/2$ as discussed in Tadmor and Tang [22]; however, it is still unclear if (A3) holds for nonconvex conservation laws;
- (d) there is no convexity assumption for the flux function f in this section. $\square$

At a point on the shock curve $x = X(t)$, we have the Rankine–Hugoniot condition

$$X'(t) = \frac{f(u(X(t)+,0)) - f(u(X(t)-,0))}{u(X(t)+,0) - u(X(t)-,0)}. \tag{2.2}$$

Also the Lax geometrical entropy condition is satisfied [10]:

$$f'(u(X(t)-,t)) \geq X'(t) \geq f'(u(X(t)+,t)). \tag{2.3}$$

The above properties will be used in the analysis in the following section. Following [22], we introduce a function $\phi(x) \in C^2(\mathbf{R})$ which satisfies

- (i) $\phi(x) \sim |x|^\alpha\,, \qquad |x| \ll 1,$
- (ii) $x\phi'(x) > 0, \qquad x \neq 0,$

- (iii) $\phi(x) \to 1, \qquad |x| \to \infty\,,$

where $\alpha \geq 1$ is a finite constant. The second requirement above implies that ϕ is monotonely decreasing for $x < 0$ and increasing for $x > 0$. More precisely, the distance function ϕ is required to satisfy

$$\left\{ \begin{array}{l} \phi(0) = 0, \qquad 0 < x\phi'(x) \leq \alpha\phi(x) \quad \text{for} \quad x \neq 0\,, \\ \\ |\phi(x)| \leq |x|^{\alpha}, \qquad |\phi^{(k)}(x)| \leq C, \quad k = 0, 1, 2, \end{array} \right. \tag{2.4}$$

where the constants C are independent of x. The functions satisfying the above requirement include $\phi(x) = (1 - e^{-x^2})^{\alpha/2}$.

Remark 2. Unlike in [22], the distance function ϕ is now extended to the whole $\mathbf{R}$. This enables us to make a uniform treatment to the (weighted) error function. ∎

In this work, we choose $\alpha = \gamma^{-1} + 1$, where γ is the constant given in assumption (A3). In other words,

$$\phi(x) \sim \left\{ \begin{array}{l} x^{1/\gamma+1}\,, \qquad |x| \ll 1\,, \\ \\ 1\,, \qquad |x| \gg 1\,. \end{array} \right.$$

As to be shown in section 4, we have $\gamma = \frac{1}{2}$ for convex conservation laws.

3. Main results. We define the error function between the viscosity solution and the entropy solution as $e(x,t) := u^{\epsilon}(x,t) - u(x,t)$. The main results of this paper are given in the following theorem.

THEOREM 3.1. *Assume that assumptions* (A1)–(A3) *are satisfied. Then for a weighted distance function ϕ, $\phi \sim \min(|x|^{1/\gamma+1}, 1)$, there exists a positive constant $C(T)$ independent of ϵ such that*

$$\int_{\mathbf{R}} \phi(x - X(t))\Big(|e(x,t)| + |e_x(x,t)|\Big)dx \leq C(T)\epsilon \tag{3.1}$$

for $0 \leq t \leq T$. In particular, for any given $h > 0$, there exists a constant $C(T,h)$ independent of ϵ such that

$$\|u_x^{\epsilon}(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\setminus[X(t)-h, X(t)+h])} \leq C(T,h)\epsilon \quad \textit{for} \quad 0 \leq t \leq T. \tag{3.2}$$

An immediate application of the above theorem is to recover the (optimal) pointwise-error estimate of Tadmor and Tang [22].

COROLLARY 3.2. *Assume that assumptions* (A1)–(A3) *are satisfied. Then for a weighted distance function ϕ, $\phi \sim \min(|x|^{1/\gamma+1}, 1)$,*

$$|(u^{\epsilon} - u)(x,t)\,\phi(x - X(t))| = \mathcal{O}(\epsilon). \tag{3.3}$$

In particular, if (x,t) is away from the shock discontinuity $S(t) = \{(x,t)|\, x = X(t)\}$, then

$$|(u^{\epsilon} - u)(x,t)| \leq C(h)\epsilon, \qquad dist(x, S(t)) \geq h. \tag{3.4}$$

3.1. Some lemmas. In order to establish the results in Theorem 3.1, we need the following three lemmas which will lead to a Gronwall inequality for the left-hand side function of (3.1). Moreover, in the remainder of this section, we always denote $\phi = \phi(x - X(t)), \phi' = \phi'(x - X(t))$.

LEMMA 3.3. *For a weighted distance function ϕ, $\phi \sim \min(|x|^{1/\gamma+1}, 1)$, and for any $F \in L^1(\mathbf{R})$, we have*

$$\int_{\mathbf{R}} \Big(f'(u^\epsilon) - \dot{X}(t)\Big)\phi'|F|dx \le C\int_{\mathbf{R}} \phi|F|dx + C\epsilon\,. \tag{3.5}$$

Proof. We split the left-hand side of (3.5) into two parts: $I_1 + I_2$, where

$$\begin{aligned} I_1 &= \int_{|x-X(t)|\ge\epsilon^\gamma} \Big(f'(u^\epsilon) - \dot{X}(t)\Big)\phi'(x - X(t))|F|dx, \\ I_2 &= \int_{|x-X(t)|\le\epsilon^\gamma} \Big(f'(u^\epsilon) - \dot{X}(t)\Big)\phi'(x - X(t))|F|dx\,, \end{aligned}$$

where γ is the constant given in (A3). It follows from (2.4) that $|\phi'(x)| \le C|x|^{1/\gamma}$. This result gives that

$$I_2 \le C\epsilon. \tag{3.6}$$

Now for $x - X(t) \ge \epsilon^\gamma$, we use the facts that $\phi' \ge 0, \dot{X}(t) > f'(u_+)$, where $u_+ = u(X(t)+0,t)$, to obtain

$$\begin{aligned} &\Big(f'(u^\epsilon) - \dot{X}(t)\Big)\phi' \\ &\le \Big(f'(u^\epsilon) - f'(u)\Big)\phi' + \Big(f'(u) - f'(u_+)\Big)\phi' \\ &\le Cf''(\bullet)\epsilon^\gamma\phi' + f''(\bullet)u_x(\bullet)(x - X(t))\phi' \qquad \text{(using (A3))} \\ &\le Cf''(\bullet)(x - X(t))\phi' + f''(\bullet)u_x(\bullet)(x - X(t))\phi' \\ &\le C(x - X(t))\phi' \\ &\le C\phi\,. \end{aligned}$$

Similarly, by noting that $\phi' \le 0$ for $x \le X(t)$ we can also prove that

$$\Big(f'(u^\epsilon) - \dot{X}(t)\Big)\phi' \le C\phi \qquad \text{for} \quad x - X(t) \le -\epsilon^\gamma\,.$$

The above results lead to $I_1 \le C\|\phi F\|_{L^1(\mathbf{R})}$. This, together with (3.6), yields the inequality (3.5). The proof of this lemma is complete. □

LEMMA 3.4. *For a weighted distance function ϕ, $\phi \sim \min(|x|^{1/\gamma+1}, 1)$, there exists a constant C independent of ϵ such that*

$$\int_{\mathbf{R}} \phi\,\mathrm{sgn}(e_x)\partial_t e_x dx \le \int_{\mathbf{R}} \phi' f'(u^\epsilon)|e_x|dx + C\int_{\mathbf{R}} \phi|e_x|dx + C\int_{\mathbf{R}} \phi|e|dx + C\epsilon\,. \tag{3.7}$$

Proof. Direct calculation from the viscosity equation (1.1) gives

$$\partial_t e_x + (f(u^\epsilon) - f(u))_{xx} = \epsilon(e_x)_{xx} + \epsilon u_{xxx}\,.$$

Applying the above result gives

$$\int_{\mathbf{R}} \phi\,\mathrm{sgn}(e_x)\partial_t e_x dx = J_1 + J_2 + J_3,$$

where

$$J_1 = -\int_{\mathbf{R}} \phi \operatorname{sgn}(e_x)(f(u^\epsilon) - f(u))_{xx} dx\,,$$
$$J_2 = \epsilon \int_{\mathbf{R}} \phi \operatorname{sgn}(e_x)(e_x)_{xx} dx\,,$$
$$J_3 = \epsilon \int_{\mathbf{R}} \phi \operatorname{sgn}(e_x) u_{xxx} dx\,.$$

It is easy to verify that

$$J_3 = \mathcal{O}(\epsilon). \tag{3.8}$$

We now estimate J_2 by integration by parts. It is noted that $e_x(\cdot, t) \in C((X(t), \infty))$. As in Lax [10], we divide the interval $[X(t), \infty)$ into intervals, $[X(t), \infty) = \cup_m I_m(t)$, $I_m(t) = [p_m(t), p_{m+1}(t))$, with $p_0(t) = X(t)$ and e_x changing signs across points p_m, $m \geq 1$. Assuming that

$$(-1)^s = \operatorname{sgn}(e_x) \Big|_{x \in I_0(t)}\,,$$

we have

$$\phi \operatorname{sgn}(e_x) e_{xx} \Big|_{x=p_0(t)} = 0\,,$$
$$(-1)^{s+m} e_{xx} \Big|_{x=p_{m+1}(t)} \leq 0\,, \qquad (-1)^{s+m} e_{xx} \Big|_{x=p_m(t)} \geq 0 \quad \text{for} \quad m \geq 1\,. \tag{3.9}$$

It follows from the above results that

$$\begin{aligned}
\int_{X(t)}^{\infty} \phi \operatorname{sgn}(e_x)(e_x)_{xx} dx &= \sum_{m=0}^{\infty} (-1)^{s+m} \int_{p_m(t)}^{p_{m+1}(t)} \phi (e_x)_{xx} dx \\
&= \sum_m (-1)^{m+s} \phi e_{xx} \Big|_{p_m(t)}^{p_{m+1}(t)} - \sum_m (-1)^{m+s} \int_{p_m(t)}^{p_{m+1}(t)} \phi' e_{xx} dx \\
&\leq 0 - \sum_m (-1)^{m+s} \int_{p_m(t)}^{p_{m+1}(t)} \phi' e_{xx} dx\,,
\end{aligned} \tag{3.10}$$

where in the last step we have used (3.9). Note that $\phi' = 0$ when $x = p_0(t) = X(t)$ and $e_x = 0$ when $x = p_m(t), m \geq 1$. Using integration by parts for (3.10) leads to

$$\begin{aligned}
\int_{X(t)}^{\infty} \phi \operatorname{sgn}(e_x)(e_x)_{xx} dx &\leq \sum_m (-1)^{m+s} \int_{p_m(t)}^{p_{m+1}(t)} \phi'' e_x dx \\
&= \int_{X(t)}^{\infty} \phi'' |e_x| dx \leq C \|u_0\|_{BV(\mathbf{R})} = \mathcal{O}(1),
\end{aligned}$$

where in the second to last step we have used the facts that u^ϵ and u are BV-bounded by $||u_0||_{BV(\mathbf{R})}$. Similarly, we can show that

$$\int_{-\infty}^{X(t)} \phi \operatorname{sgn}(e_x)(e_x)_{xx} dx \leq \mathcal{O}(1).$$

The above two results yield

$$J_2 \le \mathcal{O}(\epsilon). \tag{3.11}$$

Finally, we need to estimate J_1. Observe

$$\begin{aligned}
&-\int_{X(t)}^{\infty} \phi \,\mathrm{sgn}(e_x) \Big(f(u^\epsilon) - f(u)\Big)_{xx} dx \\
&= -\sum_m (-1)^{m+s} \phi\Big(f(u^\epsilon) - f(u)\Big)_x \Big|_{p_m(t)}^{p_{m+1}(t)} \\
&\quad + \sum_m (-1)^{m+s} \int_{p_m(t)}^{p_{m+1}(t)} \Big(f(u^\epsilon) - f(u)\Big)_x \phi' dx\,.
\end{aligned} \tag{3.12}$$

Using the following observation

$$\Big(f(u^\epsilon) - f(u)\Big)_x = \Big(f'(u^\epsilon) - f'(u)\Big) u_x\,, \qquad \text{when } x = p_m(t), m \ge 1\,,$$

we obtain from (3.12) that

$$\begin{aligned}
&-\int_{X(t)}^{\infty} \phi \,\mathrm{sgn}(e_x) \Big(f(u^\epsilon) - f(u)\Big)_{xx} dx \\
&= -\sum_m (-1)^{m+s} \phi\Big(f'(u^\epsilon) - f'(u)\Big) u_x \Big|_{p_m}^{p_{m+1}} \\
&\quad + \sum_m (-1)^{m+s} \int_{p_m}^{p_{m+1}} \phi' \Big[f'(u^\epsilon) e_x + (f'(u^\epsilon) - f'(u)) u_x\Big] dx \\
&= \underbrace{-\sum_m (-1)^{m+s} \int_{p_m}^{p_{m+1}} \Big(\phi(f'(u^\epsilon) - f'(u)) u_x\Big)_x dx}_{K_1} + \int_{X(t)}^{\infty} \phi' f'(u^\epsilon) |e_x| dx \\
&\quad \underbrace{+ \sum_m (-1)^{m+s} \int_{p_m}^{p_{m+1}} \phi' (f'(u^\epsilon) - f'(u)) u_x dx}_{K_2}\,.
\end{aligned} \tag{3.13}$$

Using product rule for the integrand of K_1 gives

$$\begin{aligned}
K_1 &= -K_2 - \sum_m (-1)^{m+s} \int_{p_m}^{p_{m+1}} \phi (f'(u^\epsilon) - f'(u))_x u_x dx \\
&\quad - \sum_m (-1)^{m+s} \int_{p_m}^{p_{m+1}} \phi (f'(u^\epsilon) - f'(u)) u_{xx} dx \\
&= -K_2 - \int_{X(t)}^{\infty} \phi \,\mathrm{sgn}(e_x) \Big[f''(u^\epsilon) e_x + (f''(u^\epsilon) - f''(u)) u_x\Big] u_x dx \\
&\quad - \int_{X(t)}^{\infty} \phi \mathrm{sgn}(e_x) (f'(u^\epsilon) - f'(u)) u_{xx} dx \\
&\le -K_2 + C \int_{X(t)}^{\infty} \phi |e_x| dx + C \int_{X(t)}^{\infty} \phi |e| dx\,.
\end{aligned} \tag{3.14}$$

Combining the above results, (3.13) and (3.14), gives

$$-\int_{X(t)}^{\infty} \phi \,\mathrm{sgn}(e_x) \Big(f(u^\epsilon) - f(u)\Big)_{xx} dx$$
$$\le \int_{X(t)}^{\infty} \phi' f'(u^\epsilon)|e_x| dx + C \int_{X(t)}^{\infty} \phi |e_x| dx + C \int_{X(t)}^{\infty} \phi |e| dx \,.$$

Similarly, we can show that

$$-\int_{-\infty}^{X(t)} \phi \,\mathrm{sgn}(e_x) \Big(f(u^\epsilon) - f(u)\Big)_{xx} dx$$
$$\le \int_{-\infty}^{X(t)} \phi' f'(u^\epsilon)|e_x| dx + C \int_{-\infty}^{X(t)} \phi |e_x| dx + C \int_{-\infty}^{X(t)} \phi |e| dx \,.$$

The above two results lead to an estimate for J_1:

$$J_1 \le \int_{\mathbf{R}} \phi' f'(u^\epsilon)|e_x| dx + C \int_{\mathbf{R}} \phi |e_x| dx + C \int_{\mathbf{R}} \phi |e| dx \,. \tag{3.15}$$

The desired inequality (3.7) follows from the above estimates for J_1, J_2, and J_3. □

LEMMA 3.5. *For a weighted distance function ϕ, $\phi \sim \min(|x|^{1/\gamma+1}, 1)$, there exists a constant C independent of ϵ such that*

$$\frac{d}{dt} \int_{\mathbf{R}} \phi |e| dx \le C \int_{\mathbf{R}} \phi |e_x| dx + C \int_{\mathbf{R}} \phi |e| dx \; + C\epsilon \,. \tag{3.16}$$

Proof. It follows from the viscous equation (1.1) and the conservation law (1.3) that

$$e_t = -\Big(f(u^\epsilon) - f(u)\Big)_x + \epsilon e_{xx} + \epsilon u_{xx} \,.$$

Using the above equation gives

$$\begin{aligned} \int_{\mathbf{R}} \phi \mathrm{sgn}(e) e_t dx = &- \int_{\mathbf{R}} \phi \mathrm{sgn}(e) \Big(f(u^\epsilon) - f(u)\Big)_x dx \\ &+ \epsilon \int_{\mathbf{R}} \phi \mathrm{sgn}(e) e_{xx} dx + \epsilon \int_{\mathbf{R}} \phi \mathrm{sgn}(e) u_{xx} dx \,. \end{aligned} \tag{3.17}$$

It can be verified that the last term above is of order $\mathcal{O}(\epsilon)$, and the second to last term is bounded by

$$\epsilon \int_{\mathbf{R}} \phi \mathrm{sgn}(e) e_{xx} dx \le -\epsilon \int_{\mathbf{R}} \phi' \mathrm{sgn}(e) e_x dx = \mathcal{O}(\epsilon) \,,$$

where in the last step we have used the fact that u and u^ϵ are BV-bounded. Using the following observation

$$\begin{aligned} \Big(f(u^\epsilon) - f(u)\Big)_x &= f'(u^\epsilon) e_x + \Big(f'(u^\epsilon) - f'(u)\Big) u_x \\ &= f'(u^\epsilon) e_x + f''(\bullet)\, e\, u_x \,, \end{aligned}$$

and the fact that $u_x = \mathcal{O}(1)$ for (x, t) away from the shock curve, we can bound the first term on the right-hand side of (3.17):

$$-\int_{\mathbf{R}} \phi \mathrm{sgn}(e) \Big(f(u^\epsilon) - f(u)\Big)_x dx \le C \int_{\mathbf{R}} \phi |e_x| dx + C \int_{\mathbf{R}} \phi |e| dx \,.$$

Therefore, we have proved that

$$\int_{\mathbf{R}} \phi \operatorname{sgn}(e) e_t dx \le C \int_{\mathbf{R}} \phi |e_x| dx + C \int_{\mathbf{R}} \phi |e| dx + C\epsilon \,. \tag{3.18}$$

Using integration by parts we obtain

$$\begin{aligned} \int_{\mathbf{R}} -\dot{X}(t)\phi'|e|dx &= \dot{X}(t) \int_{\mathbf{R}} \phi \operatorname{sgn}(e) e_x dx \\ &\le C \int_{\mathbf{R}} \phi |e_x| dx \,. \end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19) we obtain the desired estimate (3.16). ∎

3.2. Proof of Theorem 3.1. Having the above three lemmas, we are ready to prove Theorem 3.1. Observe that

$$\frac{d}{dt} \int_{\mathbf{R}} \phi |e_x| dx = \int_{\mathbf{R}} -\dot{X}(t)\phi'|e_x|dx + \int_{\mathbf{R}} \phi \partial_t |e_x| dx \,.$$

The above result, together with Lemma 3.4, yields

$$\frac{d}{dt} \int_{\mathbf{R}} \phi |e_x| dx \le \int_{\mathbf{R}} \phi' \Big(f'(u^\epsilon) - \dot{X}(t) \Big) |e_x| dx + C \int_{\mathbf{R}} \phi |e_x| dx + C \int_{\mathbf{R}} \phi |e| dx + C\epsilon \,.$$

Since $e_x \in L^1(\mathbf{R})$, we apply Lemma 3.3 to obtain

$$\frac{d}{dt} \int_{\mathbf{R}} \phi |e_x| dx \le C \int_{\mathbf{R}} \phi |e_x| dx + C \int_{\mathbf{R}} \phi |e| dx + C\epsilon \,.$$

The above result, together with Lemma 3.5, leads to

$$\frac{d}{dt} \int_{\mathbf{R}} \phi \Big(|e| + |e_x| \Big) dx \le C \int_{\mathbf{R}} \phi \Big(|e| + |e_x| \Big) dx + C\epsilon \,. \tag{3.20}$$

The estimate (3.1) in Theorem 3.1 follows immediately from the above Gronwall inequality. It follows from (2.4) that for any $h > 0$ there exists a constant $c(h) > 0$ such that

$$|\phi(x)| \ge c(h) \qquad \text{as} \quad |x| \ge h.$$

The estimate (3.2) follows from the above observation and (3.1). The proof of Theorem 3.1 is complete. ∎

3.3. Proof of Corollary 3.2. Consider the weighted error function $\phi(x - X(t))e(x,t)$ with (x,t) on the right-hand side of the shock curve, i.e., $x > X(t)$. In this case,

$$\phi(x - X(t))\, e(x,t) = - \int_x^\infty \phi e_x dx - \int_x^\infty \phi' e dx \,,$$

which leads to

$$|\phi(x - X(t)) e(x,t)| \le \int_{X(t)}^\infty \phi |e_x| dx + \int_{X(t)}^\infty \phi' |e| dx \,.$$

Using integration by parts gives

$$\int_{X(t)}^{\infty} \phi' |e| dx = -\int_{X(t)}^{\infty} \phi \mathrm{sgn}(e)\, e_x dx \leq \int_{X(t)}^{\infty} \phi |e_x| dx.$$

Combining the above two results we obtain

$$|\phi(x - X(t))e(x,t)| \leq 2 \int_{X(t)}^{\infty} \phi |e_x| dx \leq 2 \int_{\mathbf{R}} \phi |e_x| dx\,.$$

This, together with Theorem 3.1, yields

$$|\phi(x - X(t))e(x,t)| \leq C\epsilon\,, \qquad x > X(t)\,.$$

A similar result holds for (x,t) on the left-hand side of the shock curve. This completes the proof of Corollary 3.2. ∎

4. Application to convex conservation laws. Assume that the flux function f in (1.3) is convex and that there exists only one shock discontinuity for the entropy solution of (1.3)–(1.4). As in [21], we let $\| \bullet \|_{Lip^+}$ denote the Lip^+-seminorm

$$\|w\|_{Lip^+} := \text{ess}\sup_{x \neq y} \left[\frac{w(x) - w(y)}{x - y} \right]^+ ,$$

where $[w]^+ = H(w)w$, with $H(\bullet)$ the Heaviside function. Owing to the convexity of the flux f, the viscosity solutions of (1.1) satisfy a Lip^+-stability condition, similar to the familiar Oleinik's E-condition, which asserts an a priori upper bound for the Lip^+-seminorm of the viscosity solution

$$\frac{u^\epsilon(x,t) - u^\epsilon(y,t)}{x - y} \leq \|u^\epsilon(\cdot,t)\|_{Lip^+} \leq \frac{1}{\|u_0\|_{Lip^+}^{-1} + \beta t}\,, \tag{4.1}$$

where u^ϵ is the solution of (1.1)–(1.2), β is the convexity constant of the flux f, $f'' \geq \beta$; consult, e.g., [21]. The above result suggests that if the initial data do not contain nonLipschitz increasing discontinuities, then the viscosity solution of (1.1) will keep the same property. The same is true for entropy solution of (1.3)–(1.4).

It is shown in [21] that with the Lip^+ initial data, the following pointwise-error bound holds:

$$|u^\epsilon(x,t) - u(x,t)| \leq C \sqrt[3]{\epsilon} \quad \text{for} \quad dist(x, S(t)) \geq \sqrt[3]{\epsilon}\,, \tag{4.2}$$

where $S(t) = \{(x,t) |\, x = X(t)\}$. This result can be further improved by using the results in [22] and [28]:

$$|u^\epsilon(x,t) - u(x,t)| \leq C \sqrt{\epsilon} \quad \text{for} \quad dist(x, S(t)) \geq \sqrt{\epsilon}\,. \tag{4.3}$$

In other words, assumption (A3) holds with $\gamma = 0.5$ for convex conservation laws with Lip^+ initial data.

4.1. One shock. It is noted that assumption (A2), i.e., the entropy solution has *only one* shock discontinuity, implies that u_0 must be Lip^+-stable. Therefore, the theories developed in the last section can be applied to convex conservation laws with one shock discontinuity. We summarize what we have shown by stating the following.

ASSERTION 1. *Let $u^\epsilon(x,t)$ be the viscosity solutions of* (1.1)–(1.2) *and $u(x,t)$ be the entropy solution of* (1.3)–(1.4)*. If the flux function f in* (1.3) *is convex and the entropy solution has* only one *shock discontinuity $S(t) = \{(x,t)|x = X(t)\}$, then the following error estimates hold:*

- *For a weighted distance function* ϕ, $\phi(x) \sim \min(|x|^3, 1)$,

$$\text{(a)} \quad \int_{\mathbf{R}} \phi(x - X(t))\Big(|(u^\epsilon - u)(x,t)| + |(u_x^\epsilon - u_x)(x,t)|\Big)dx \le C\epsilon\,,$$
$$\text{(b)} \quad |(u^\epsilon - u)(x,t)|\phi(x - X(t)) = \mathcal{O}(\epsilon)\,.$$

- *In particular, if* (x,t) *is away from the singular support, then for any given* $h > 0$

$$\text{(c)} \quad \|u_x^\epsilon(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\setminus[X(t)-h,X(t)+h])} \le C(h)\epsilon\,,$$
$$\text{(d)} \quad |(u^\epsilon - u)(x,t)| \le C(h)\epsilon\,, \qquad |x - X(t)| \ge h.$$

4.2. Finitely many shocks. In this general case, we define the weighted distance function as

$$\rho(x,t) = \prod_{k=1}^{K} \phi\left(x - X_k(t)\right). \tag{4.4}$$

We can apply the same techniques as used in section 3 for the weighted error functions $(u^\epsilon(x,t) - u(x,t))\rho(x,t)$ and $(u_x^\epsilon(x,t) - u_x(x,t))\rho(x,t)$. The results in Theorem 3.1 and Corollary 3.2 can be extended to these error functions. Following [22], we can extend Assertion 1 and conclude the following.

ASSERTION 2. *Let* $u^\epsilon(x,t)$ *be the viscosity solutions of* (1.1)–(1.2) *and* $u(x,t)$ *be the entropy solution of* (1.3)–(1.4). *If the flux function* f *in* (1.3) *is convex and the entropy solution has* finitely many *shock discontinuities* $S(t) = \{(x,t)|x = X_k(t)\}_{k=1}^K$, *then the following error estimates hold:*

- *For a weighted distance function* ϕ, $\phi(x) \sim \min(|x|^3, 1)$,

$$\text{(a)} \quad \int_{\mathbf{R}} \prod_{k=1}^{K} \phi\left(x - X_k(t)\right) \Big(|(u^\epsilon - u)(x,t)| + |(u_x^\epsilon - u_x)(x,t)|\Big)dx \le C\epsilon\,,$$

$$\text{(b)} \quad |(u^\epsilon - u)(x,t)| \prod_{k=1}^{K} \phi\left(x - X_k(t)\right) = \mathcal{O}(\epsilon)\,.$$

- *In particular, if* (x,t) *is away from the singular support, then for any* $h > 0$

$$\text{(c)} \quad \|u_x^\epsilon(\cdot,t) - u_x(\cdot,t)\|_{L^1(\mathbf{R}\setminus\cup_k[X_k(t)-h,X_k(t)+h])} \le C(h)\epsilon\,,$$
$$\text{(d)} \quad |(u^\epsilon - u)(x,t)| \le C(h)\epsilon\,, \qquad dist(x, S(t)) \ge h.$$

Finally, we point out that unlike previous work for studying the viscous conservation laws the approach used in Tadmor and Tang [22] does not follow the characteristics but instead makes use of the energy method. Therefore, their results for the viscosity approximations can be extended to finite difference methods [23], [24]. It is seen that the present work also makes use of the same energy method and it is expected that the results in this paper can be extended to other types of approximate solutions (such as the monotone difference methods with *smooth* numerical fluxs).

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