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POLYNOMIAL SPLINE COLLOCATION METHODS FOR THE NONLINEAR BASSET EQUATION

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Abstract—We discuss the application of a class of spline collocation methods to a first-order Volterra integro-differential equation arising in fluid dynamics. Since this equation possesses a weakly singular kernel its solutions, even for analytic data, are not smooth on the entire interval of integration. It is shown that while the use of uniform grids leads to poor convergence rates, high-order convergence is restored on suitably graded grids.

1. INTRODUCTION

The mathematical modelling of the diffusion of discrete particle in a turbulent fluid leads to Volterra integro-differential equations (VIDEs) of the type

$$y'(t) = f(t, y(t)) + \int_0^t (t-s)^{-\alpha} k(t, s, y'(s))\, ds, \quad t \in I := [0, T], \tag{1}$$

with initial condition $y(0) = y_0$, and with $0 < \alpha < 1$ (typically, $\alpha = 1/2$). Here, $f\colon I \times \mathbb{R} \to \mathbb{R}$ and $k\colon S \times \mathbb{R} \to \mathbb{R}$ (where $S := \{(t, s)\colon 0 \leqslant s \leqslant t \leqslant T\}$) are given (smooth) functions [cf. 1–7]. For historical reasons, equation (1) is often referred to as the Basset equation.

Observe that, in contrast to "standard" VIDEs, the kernel function k in (1) depends on the derivative y' instead of the solution y itself; moreover, the integrand contains the weakly singular (i.e. integrable) factor $(t-s)^{-\alpha}$. As will be seen below this factor complicates the numerical treatment of (1) since solutions of (1) will, in general, be such that $y \in C^1(I)$ but $y''(t) \approx t^{-\alpha}$ as $t \to 0$: this singular behavior has the effect that numerical approximations to y generated by adaptations of numerical methods for VIDEs with regular kernel functions (and smooth solutions) will show a poor convergence behavior [8, 9]. Thus, new approaches tailored to the nonsmooth structure of y have to be found.

Using modifications of fractional linear multistep methods originally introduced by Lubich [10] for second-kind Volterra integral equations with weakly singular kernels, Hairer and Maass [7] derived an efficient class of high-order methods on uniform grids. (The product-integration methods in [5] and [6] yield high-order convergence on uniform grids only under the assumption that the exact solution of (1) exhibit continuous differentiability of sufficiently high order on the (closed) interval I.)

In this paper we study a class of collocation methods (and their discretizations) for (1), using suitably graded grids. The underlying ideas were introduced in [8] and [11]; however, here we also analyze the attainable order of convergence at the grid points.

As for the numerical treatment of standard VIDEs, the reader is referred [9, 12–14], [15] VIDEs with regular kernels) and [8, 13, 16, 17] (VIDEs with weakly singular kernels); most of these references contain information about other relevant papers.

2. SPLINE COLLOCATION METHODS ON GRADED GRIDS

As mentioned in Section 1, the analytical solution of the VIDE (1) does not inherit the assumed smoothness (i.e. the differentiability properties) of the given functions $f(t, y)$ and $k(t, s, z)$ (where $k(t, s, z) \not\equiv 0$). More precisely, it can be shown using the techniques in [10] (see also [8, 13, 18] for linear VIDEs with weakly singular kernels) that if f and k have continuous derivatives of order

m then there exists a function $Y = Y(t, v)$ possessing continuous derivatives of order $m + 1$, such that the solution of (1) can be written as

$$y(t) = Y(t, t^{2-\alpha}), \tag{2}$$

at least for $t \in [0, c]$, for some $c > 0$. This implies that near $t = 0$ its second derivative behaves like $y''(t) \approx t^{-\alpha}$.

There are essentially two ways of obtaining numerical methods whose approximations to $y(t)$ exhibit an acceptable order of convergence:

(i) The construction of new ("nonclassical") methods which reflect the specific singular behavior of $y(t)$ near $t = 0$. Methods of this type (using *uniform* grids) are the nonpolynomial spline collocation methods suggested in [18] and, above all, the fractional linear multistep methods of [7] (here, the construction of the special initial weights can be shown to correspond to an application of nonpolynomial spline collocation).

(ii) The specific implementation, reflecting the behavior of $y(t)$ near $t = 0$, of certain methods introduced originally for VIDEs with smooth solutions. This implementation is based on certain *nonuniform* (graded) grids depending on the value of α.

In the present paper we analyze the second of these two approaches. The methods (collocation and Runge–Kutta-type methods) generate as approximations to the solution y of (1) elements of the polynomial spline space

$$S_m^{(0)}(Z_N) := \{u \in C(I): u|_{\sigma_m} := u_n \in \pi_m, 0 \leqslant n \leqslant N - 1\}, \tag{3}$$

associated with a given partition

$$\Pi_N: 0 = t_0^{(N)} < t_1^{(N)} < \cdots < t_N^{(N)} = T, \quad N \geqslant 1$$

of the interval $I = [0, T]$. Here, π_m is the set of (real) polynomials of degree not exceeding m (with $m \geqslant 1$), and we have set $\sigma_n := [t_n^{(N)}, t_{n+1}^{(N)}]$ $(n = 0, \ldots, N - 1)$, $Z_N := \{t_n^{(N)}: 1 \leqslant n \leqslant N - 1\}$ (the set of interior grid points). For ease of notation we shall subsequently omit the subscript N in $t_n^{(N)}$ etc.; note, however, that quantities like $h_n := t_{n+1} - t_n$,

$$h := \max\{h_n: 0 \leqslant n \leqslant N - 1\}, \quad h' := \min\{h_n: 0 \leqslant n \leqslant N - 1\} \tag{4}$$

depend on N (i.e. on the particular partition under consideration). The quantity h is often called the diameter of the grid Π_N.

The desired approximation to y is the element $u \in S_m^{(0)}(Z_N)$ satisfying

$$u'(t) = f(t, u(t)) + \int_0^t (t - s)^{-\alpha} k(t, s, u'(s))\,\mathrm{d}s, \quad t \in X(N), \tag{5}$$

with

$$X(N) := \bigcup_{n=0}^{N-1} X_n$$

and

$$X_n := \{t_{n,j} := t_n + c_j h_n: 0 \leqslant c_1 < \cdots < c_m \leqslant 1\}.$$

In other words, the collocation solution for (1) is the element in the polynomial spline space $S_m^{(0)}(Z_N)$ which solves (1) on the set of collocation points $X(N)$. This set is characterized, for a given partition Π_N, by the collocation parameters $\{c_j\}$.

Using standard contraction mapping arguments it is straightforward to show that, under suitable assumptions on $f(t, y)$ and $k(t, s, z)$ (e.g. Lipschitz conditions with respect to y and z), the collocation equation (5) and the initial condition $u(0) = y_0$ define a unique collocation solution $u \in S_m^{(0)}(Z_N)$ whenever the grid diameter h is sufficiently small.

In the following convergence analysis two types of grids will occur:

(i) Quasi-uniform grids: here, the quotient h/h' (cf. definition 4) is bounded by some finite constant $\gamma \geqslant 1$ for all values of N. This implies that

$$h \leqslant \gamma T N^{-1} = O(N^{-1}). \tag{6}$$

For $\gamma = 1$ we have, of course, uniform grids, with $h = h' = TN^{-1}$.

(ii) Graded grids: the grid points are given by

$$t_n = (n/N)^r T, \quad n = 0, \ldots, N, \tag{7}$$

where the grading exponent r satisfies $r \geqslant 1$. It is easily seen that grids of this type are not quasi-uniform if $r > 1$; however, we have

$$h \leqslant rTN^{-1} = O(N^{-1}) \tag{8}$$

for all $r \geqslant 1$. For $r = 1$ we obtain the uniform grid.

Theorem 2.1

Let $f(t, y)$ and $k(t, s, z)$ in (1) be m times continuously differentiable and such that the initial-value problem (1) has a unique solution y on I.

(a) If $\{\Pi_N\}$ is a quasi-uniform grid sequence, then the collocation solution $u \in S_m^{(0)}(Z_N)$ defined by (5) satisfies

$$\| y^{(k)} - u^{(k)} \|_\infty = O(N^{-(1-\alpha)}), \quad k \in \{0, 1\} \tag{9}$$

as $N \to \infty$. This holds for any set $\{c_j\}$ of collocation parameters and any value of $m \geqslant 1$.

(b) If $\{\Pi_N\}$ is the sequence of graded grids given by (7) and with grading exponent r chosen as

$$r = \mu/(1-\alpha), \quad 1 - \alpha \leqslant \mu \leqslant m, \tag{10}$$

then the collocation solution $u \in S_m^{(0)}(Z_N)$ determined by (5) satisfies

$$\| y^{(k)} - u^{(k)} \|_\infty = O(N^{-\mu}), \quad k \in \{0, 1\}, \tag{11}$$

as $N \to \infty$. □

The proof of Theorem 2.1 is based on techniques introduced in [8], using the result (2) on the form of the analytical solution of (1) and the theory of weakly singular discrete Gronwall inequalities [19, 20]. We omit the details.

It is well known [12, 13] that the global convergence rate of collocation approximations to solutions of VIDEs with regular kernels (and smooth solutions) is given by $O(N^{-m})$. Part (b) of Theorem 2.1 shows that this rate can be attained also for VIDEs with weakly singular kernels, provided we employ graded grids of the form (7) with grading exponent $r = m/(1-\alpha)$. The motivation for using graded grids in the approximation of nonsmooth functions like $f(t) = t^\beta$ ($\beta > 0$) on [0, 1] by spline functions was given by Rice [21] (see also [22, pp. 44–47] or [23, pp. 254–257]). Collocation and related methods for weakly singular Fredholm integral equations of the second kind, using graded grids, were studied, for example, by Schneider [24] and Vainikko and Uba [25] (compare also the survey paper [26]). The convergence properties of spline collocation methods on graded grids for Volterra integral and integro-differential equations with weakly singular kernels were analyzed in [11] and [8], respectively.

We now turn our attention to the order of convergence of $y(t_n) - u(t_n)$ where $t_n \in \bar{Z}_N := Z_N \cup \{T\}$ is a grid point; i.e. we shall analyze the discrete (or local) convergence properties of the collocation solution $u \in S_m^{(0)}(Z_N)$. This is of interest since in many applications one is interested in generating approximations to $y(t)$ at the right endpoint $t = T$ of I which exhibit a high order of convergence at that point. To our knowledge, results of this kind for VIDEs with weakly singular kernel, e.g. (1), have not yet been obtained, not even in the linear case.

Theorem 2.2

Let $f(t, y)$ and $k(t, s, z)$ in (1) be $m + 1$ times continuously differentiable and such that (1) has a unique solution on I. Assume that $u \in S_m^{(0)}(Z_N)$ is the collocation solution defined by (5), with collocation parameters $\{c_j\}$ such that

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_j)\, ds = 0. \tag{12}$$

(a) The use of quasi-uniform grids Π_N yields

$$\max_{(n)} |y(t_n) - u(t_n)| = O(N^{-2(1-\alpha)}), \tag{13}$$

as $N \to \infty$.

(b) If, however, one employs graded grids (7) with grading exponent satisfying

$$r = \mu/(1-\alpha), \quad \text{with} \quad \mu \geqslant (m+2-\alpha)/2, \tag{14}$$

$$\max_{(n)} |y(t_n) - u(t_n)| = O(N^{-(m+1-\alpha)}), \tag{15}$$

as $N \to \infty$. □

Before turning to the proof of Theorem 2.2 we add some remarks.

(i) For VIDEs with smooth kernels and smooth solutions, a much higher rate of convergence than (15) can be obtained on uniform grids: if, for example, the collocation parameters $\{c_j\}$ are the Gauss points for (0, 1) (i.e. the zeros of the shifted Legendre polynomial $P_m(2s-1)$), then

$$\max_{(n)} |y(t_n) - u(t_n)| = O(N^{-2m})$$

(cf. [9, 12, 13]).

(ii) For $m \geqslant 2$, the optimal (i.e. smallest) grading exponent for attaining the convergence rate of (15),

$$r = (m+2-\alpha)/(2(1-\alpha)), \tag{16}$$

is smaller than the grading exponent used in Theorem 2.1(b) to obtain global convergence of order m. This is of practical relevance since, for given and m, the initial step size $h_0 = TN^{-r}$ of the graded grid (7) will become very small as N is increased, thus representing a potential source of rounding errors in subsequent calculations. As an example, assume that we have $m = 3$ (cubic splines), $\alpha = 1/2$, and $T = 1$. If we choose r as in (10), with $\mu = m$, then $h_0 = N^{-6}$. However, if the value of (16) is chosen, then the initial step size is $h_0 = N^{-4.5}$.

(iii) Of the many sets $\{c_j\}$ satisfying the orthogonality condition (12) we mention the *Gauss points* [zeros of $P_m(2s-1)$], and the *Radau* points—given by either the zeros of $P_m(2s-1) + P_{m-1}(2s-1)$ [Radau (I)] or the zeros of $P_m(2s-1) - P_{m-1}(2s-1)$ [Radau (II)]. Condition (12) is also satisfied for the "Simpson points" $c_1 = 0$, $c_2 = 1/2$, $c_3 = 1$ (a special case of the Lobatto points). Observe that in this case (since $c_1 = 0$ and $c_m = 1$) the resulting collocation solution u is in the smoother space $S_m^{(1)}(Z_N) := S_m^{(0)}(Z_N) \cap C^1(I)$.

Proof of Theorem 2.2

We begin by writing the collocation equation (5) in "continuous" form,

$$u'(t) = f(t, u(t)) - \delta(t) + \int_0^t (t-s)^{-\alpha} k(t, s, u'(s))\, ds, \quad t \in I, \tag{17}$$

where the residual function $\delta(t)$ vanishes on the set $X(N)$ of collocation points. Since, by assumption, f and k have continuous derivatives of order $m+1 \geqslant 2$, we may apply Taylor's formula to the equation satisfied by the collocation error $e(t) := y(t) - u(t)$,

$$e'(t) = f(t, y(t)) - f(t, u(t)) + \delta(t) + \int_0^t (t-s)^{-\alpha}\{k(t, s, y'(s)) - k(t, s, u'(s))\}\, ds, \quad t \in I,$$

to obtain the following result.

Lemma 2.1

The collocation error induced by the collocation solution u is the solution of the initial-value problem

$$e'(t) = a(t)e(t) + \delta(t) + \int_0^t (t-s)^{-\alpha} H(t, s) e'(s)\, ds + (Qe)(t), \quad t \in I,$$

$$e(0) = 0, \tag{18}$$

with

$$a(t) := f_y(t, y(t)), \quad H(t,s) := k_z(t, s, y'(s)),$$

and

$$(Qe)(t) := -\tfrac{1}{2} f_{yy}(t, \cdot) e^2(t) - \frac{1}{2} \int_0^t (t-s)^{-\alpha} k_{zz}(t, s, \cdot)(e'(s))^2 \, ds.$$

In the last expression the unspecified arguments represent intermediate values arising in the respective remainder terms of Taylor's formula for $f(t, y(t) - e(t))$ and $k(t, s, y'(s) - e'(s))$.□

The following result concerning a representation of the solution of the VIDE (18) is one of the keys for establishing the discrete convergence result (15).

Lemma 2.2

The function $e(t)$ is a solution of the VIDE (18) on I if, and only if, it solves the equation

$$e(t) = \int_0^t r(t, s; \alpha)\delta(s) \, ds + \int_0^t r(t, s; \alpha)(Qe)(s) \, ds, \quad t \in I. \tag{19}$$

Here

$$r(t, s; \alpha) := 1 + \int_s^t R(v, s; \alpha) \, dv, \quad (t, s) \in S,$$

where $R(t, s; \alpha)$ denotes the resolvent kernel of the kernel

$$K(t, s; \alpha) := a(t) + (t-s)^{-\alpha} H(t, s) \tag{20}$$

of the linear integral equation

$$z(t) = \delta(t) + \int_0^t K(t, s; \alpha) z(s) \, ds. \quad \square \tag{21}$$

Proof. Setting

$$e(t) = \int_0^t e'(s) \, ds$$

(recall that $e(0) = 0$) we first rewrite equation (18) in the form

$$e'(t) = \delta(t) + \int_0^t \{a(t) + (t-s)^{-\alpha} H(t,s)\} e'(s) \, ds + (Qe)(t), \tag{22}$$

where the factor of $e'(s)$ under the integral sign equals $K(t, s; \alpha)$ defined in (20). Equation (22) may be viewed as a nonlinearly perturbed linear Volterra integral equation of the second kind for $e'(t)$, with perturbation term $(Qe)(t)$ given in Lemma 2.1.

It follows [27, pp. 191–193] that if $R(t, s; \alpha)$ is the resolvent kernel of $K(t, s; \alpha)$ in (21) then $e'(t)$ must also satisfy the equation

$$e'(t) = \delta(t) + (Qe)(t) + \int_0^t R(t, s; \alpha)\{\delta(s) + (Qe)(s)\} \, ds$$

(and vice versa). Integration of the above equation yields (19).

Let now $t = t_n$ be a grid point, and write (19) as

$$e(t_n) = S_0(t_n; \alpha) + S_1(t_n; \alpha), \tag{23a}$$

with

$$S_0(t_n; \alpha) := \int_0^{t_n} r(t_n, s; \alpha)\delta(s) \, ds \tag{23b}$$

and

$$S_1(t_n; \alpha) := \int_0^{t_n} r(t_n, s; \alpha)(Qe)(s) \, ds. \tag{23c}$$

Consider first (23c): it follows from the definition of $(Qe)(t)$ in Lemma 2.1 that

$$|(Qe)(t)| \leqslant \tfrac{1}{2}M_0 \| e \|_\infty^2 + \tfrac{1}{2}M_1 \| e' \|_\infty^2 \cdot T^{1-\alpha}/(1-\alpha), \quad t \in I,$$

where M_0 and M_1 denote suitable upper bounds for $|f_{yy}(t, y)|$ and $|k_{zz}(t, s, z)|$. Thus, the order of $\| Qe \|_\infty$ is governed by the order of global convergence of the collocation solution and its derivative; by Theorem 2.1 we have

$$\| e^{(k)} \|_\infty^2 = O(N^{-2\mu}) \quad \text{if} \quad r = \mu(1-\alpha) \quad (\text{with } 1-\alpha \leqslant \mu \leqslant m).$$

Hence

$$|S_1(t_n; \alpha)| \leqslant \| Qe \|_\infty \cdot \int_0^{t_n} |r(t_n, s; \alpha)| \, ds = O(N^{-2\mu}), \quad t_n \in \bar{Z}_N. \tag{24}$$

In order to estimate $S_0(t_n; \alpha)(n \geqslant 1)$ we write (23b) as

$$S_0(t_n; \alpha) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} g_n(s; \alpha)\delta(s) \, ds, \tag{25}$$

with

$$g_n(s; \alpha) := r(t_n, s; \alpha) = 1 + \int_s^{t_n} R(v, s; \alpha) \, dv$$

given in Lemma 2.2. Since $R(t, s; \alpha)$ is the resolvent kernel of the (weakly singular) kernel $K(t, s; \alpha)$ given by (20),

$$K(t, s; \alpha) = a(t) + (t-s)^{-\alpha} H(t, s), \quad \text{with} \quad H(t, s) \not\equiv 0,$$

it follows [13, pp. 15–16] that this resolvent must be of the form

$$R(t, s; \alpha) = (t-s)^{-\alpha} \cdot P(t, s; \alpha),$$

where $P(t, s; \alpha)$ is continuous for $(t, s) \in S$. This implies that $g_n(s; \alpha)$ is continuous for $0 \leqslant s \leqslant t_n$; however, its derivative, $dg_n(s; \alpha)/ds$, behaves like $(t_n - s)^{-\alpha}$ as s approaches t_n (while for $0 \leqslant s < t_n$, $g_n(s; \alpha)$ has continuous derivatives up to order m). In other words, using the terminology introduced in [21], the function $g_n(s; \alpha)$ is of type $(1-\alpha, m, \{t_n\})$. It is easily seen that the residual function $\delta(t)$ [cf. (17)] has the same properties.

If we now replace each integral in (25) by the sum consisting of the interpolatory m-point quadrature formula based on the collocation parameters and the corresponding error term $E_{n,i}$, e.g.

$$\int_{t_i}^{t_{i+1}} g_n(s; \alpha)\delta(s) \, ds = \sum_{l=1}^{m} w_l g_n(t_{i,l})\delta(t_{i,l}) + E_{n,i},$$

we obtain, recalling that the residual vanishes at all collocation points,

$$S_0(t_n; \alpha) = \sum_{i=0}^{n-1} E_{n,i}, \quad n = 1, \ldots, N.$$

It follows from a result of Schneider [24, pp. 209–212] that, under the hypotheses (12) (choice of collocation parameters) and (13) (choice of grading exponent r) of Theorem 2.2, we have

$$\sum_{i=0}^{n-1} E_{n,i} = O(N^{-(m+1-\alpha)}), \quad n = 1, \ldots, N. \tag{26}$$

(For quasi-uniform grids we only have

$$\sum_{i=0}^{n-1} E_{n,i} = O(N^{-2(1-\alpha)}).)$$

Combining the results (24) and (26), and using the value $r = \mu/(1-\alpha)$, with $\mu \geqslant (m+2-\alpha)/2$, for the grading exponent, we find by (23a),

$$e(t_n) = O(N^{-2\mu}) + O(N^{-(m+1-\alpha)}) = O(N^{-(m+2-\alpha)}) + O(N^{-(m+1-\alpha)}) = O(N^{-(m+1-\alpha)}),$$

$n = 1, \ldots, N$, thus verifying (15). The discrete convergence order of (13) for quasi-uniform grids is established in the same way, using the remark following (26).

3. COMPUTATIONAL FORM OF THE COLLOCATION METHOD

In order to write the collocation (5) in a form which more clearly exhibits the recursive nature of the method, let $t = t_{n,j} \in X_n$ be a collocation point in the subinterval σ_n. Then (5) becomes

$$u_n'(t_{n,j}) = f(t_{n,j}, u_n(t_{n,j})) + h_n^{1-\alpha}\Phi_{n,n}^{(j)}[u_n'] + \sum_{i=0}^{n-1} h_i^{1-\alpha}\Phi_{n,i}^{(j)}[u_i'],$$

$$j = 1, \ldots, m \; (n = 0, \ldots, N-1), \tag{27}$$

where we have set

$$\Phi_{n,n}^{(j)}[u_n'] := \int_0^{c_j} (c_j - v)^{-\alpha} \cdot k(t_{n,j}, t_n + vh_n, u_n'(t_n + vh_n))\, dv \tag{28a}$$

and

$$\Phi_{n,i}^{(j)}[u_i'] := \int_0^1 [(t_{n,j} - t_i)/h_i - v)^{-\alpha} \cdot k(t_{n,j}, t_i + vh_i, u_i'(t_i + vh_i)]\, dv \tag{28b}$$

$(0 \leqslant i < n \leqslant N-1)$. In general, the integrals (28a) and (28b) cannot be found analytically. Thus the fully discretized (i.e. computational) form of (27) is obtained by choosing suitable quadrature approximations to these integrals. These approximations should of course be such that the resulting discrete collocation solution possesses the same convergence properties as the one defined by the exact collocation equation (27) (cf. Theorems 2.1 and 2.2). A natural way to approximate the integrals (28a) and (28b) is to use m-point product-integration formulas based on the collocation parameters $\{c_j\}$: here, $\Phi_{n,n}^{(j)}[u_n']$ is replaced by

$$\Psi_{n,n}^{(j)}[u_n'] := \sum_{s=1}^{m} w_{j,s}(\alpha) k[t_{n,j}, t_n + c_j c_s h_n, u_n'(t_n + c_j c_s h_n)] \tag{29a}$$

(note that the kernel function $k(t, s, z)$ is, in general, given only for points (t, s) with $s \leqslant t$), while for the integrals $\Phi_{n,i}^{(j)}[u_i']$ in the lag term we have

$$\Psi_{n,i}^{(j)}[u_i'] := \sum_{s=1}^{m} w_{j,s}^{(n,i)}(\alpha) k(t_{n,j}, t_{i,s}, u_i'(t_{i,s})), \quad i < n. \tag{29b}$$

The quadrature weights are given by

$$w_{j,s}(\alpha) := c_j^{1-\alpha} \int_0^1 (1 - v)^{-\alpha} L_s(v)\, dv, \quad j, s = 1, \ldots, m,$$

and

$$w_{j,s}^{(n,i)}(\alpha) := \int_0^1 ((t_{n,j} - t_i)/h_i - v)^{-\alpha} L_s(v)\, dv, \quad j, s = 1, \ldots, m; i < n,$$

where

$$L_s(v) := \prod_{k \neq s}^{m} (v - c_k)/(c_s - c_k)$$

denotes the sth Lagrange fundamental polynomial with respect to the given collocation parameters.

Let the derivative of the discrete collocation solution be written in the form

$$u_n'(t_n + vh_n) = \sum_{s=1}^{m} L_s(v) Y_{n,s}, \quad t_n + vh_n \in \sigma_n, \tag{30}$$

with $Y_{n,s} := u_n'(t_{n,s})$ (where $t_{n,s} \in X_n$ is a collocation point). (Although the discrete collocation solution is an element in $S_m^{(0)}(Z_N)$ which, in general, differs from the exact collocation solution, we use the same notation.) It follows from (30) that on the subinterval σ_n the discrete collocation solution u is given by

$$u_n(t_n + vh_n) = y_n + h_n \sum_{s=1}^{m} a_s(v) Y_{n,s}, \tag{31}$$

where $y_n := u_n(t_n) (= u_{n-1}(t_n)$, with $u_0(t_0) = u(0) = y_0)$ and

$$a_s(v) := \int_0^v L_s(z)\, dz.$$

The values $Y_{n,1}, \ldots, Y_{n,m}$ are given by the solution of the system of nonlinear equations replacing the exact collocation equations (27),

$$Y_{n,j} = f\Big(t_{n,j}, y_n + h_n \sum_{s=1}^{m} a_s(c_j) Y_{n,s}\Big) + h_n^{1-\alpha} \Psi_{n,n}^{(j)}[u_n'] + \sum_{i=0}^{n-1} h_i^{1-\alpha} \Psi_{n,i}^{(j)}[u_i'], \quad j = 1, \ldots, m, \tag{32}$$

where the expressions $\Psi_{n,n}^{(j)}[u_n']$ and $\Psi_{n,i}^{(j)}[u_i'] (i < n)$ are defined in (29a) and (29b) respectively.

It can be shown (along the lines of the analogous proof for standard VIDEs with weakly singular kernels in [8]) that the global and discrete convergence results of Theorems 2.1 and 2.2 remain valid for the discrete collocation solution u generated by (31) and (32). This is a consequence of the fact that the order of the quadrature errors introduced when replacing the exact collocation equation (27) by (32) is given, under the assumptions (12) and (14) of Theorem 2.2, by (26). We leave the details of the proof to the reader.

4. NUMERICAL EXAMPLE

We illustrate the performance of the discretized collocation method (30)–(32) by applying it, with the value $m = 2$, i.e. $u \in S_2^{(0)}(Z_N)$, to the nonlinear Basset equation

$$y'(t) = -\frac{\cos(5t)}{1 + (y'(t))^2} + q(t) - \frac{1}{4} \int_0^t (t-s)^{-1/2} \cdot \frac{(y'(s))^2}{1+t^2}\, ds, \quad y(0) = 1.$$

Here, $q(t)$ has been chosen so that the equation has the solution $y(t) = 1 - t^{2-\alpha}$ [representative for the general form (2)], with $\alpha = 1/2$. The resulting systems of nonlinear equations (32) were solved by straightforward fixed-point iteration (using an error tolerance of 10^{-10} for the l_∞-norm of the difference of consecutive iterates; the calculations were performed in double precision on a VAX 8800). Tables 1 and 2 contain a selection of numerical results obtained by this method; they are representative for this and other collocation methods applied to various linear and nonlinear VIDEs of the form (1).

The grading exponents for the graded grids (7) are those given in Theorem 2.2 $(r = (m + 2 - \alpha)/(2(1 - \alpha)) = 3.5)$ and in Theorem 2.1 $(r = m/(1 - \alpha) = 4)$. In both cases the

Table 1. Collocation at Gauss points ($c_1 = (3 - \sqrt{3})/6$, $c_2 = (3 + \sqrt{3})/6$)

N	$r = 1$		$r = \frac{m+2-\alpha}{2(1-\alpha)}$		$r = \frac{m}{1-\alpha}$	
$(t_N = 1)$	$\lvert e(t_N)\rvert$	$\lvert e'(t_N)\rvert$	$\lvert e(t_N)\rvert$	$\lvert e'(t_N)\rvert$	$\lvert e(t_N)\rvert$	$\lvert e'(t_N)\rvert$
5	6.57D − 4	1.01D − 3	7.26D − 5	8.84D − 3	1.56D − 6	1.06D − 2
10	2.78D − 4	2.64D − 4	7.38D − 5	2.76D − 3	8.69D − 5	3.45D − 3
20	1.09D − 4	6.84D − 5	1.87D − 5	7.81D − 4	2.35D − 5	9.95D − 4
40	4.05D − 5	1.75D − 5	3.78D − 6	2.10D − 4	4.88D − 6	2.70D − 4

Table 2. Collocation at Radau (II) points ($c_1 = 1/3$, $c_2 = 1$)

N	$r = 1$		$r = \frac{m+2-\alpha}{2(1-\alpha)}$		$r = \frac{m}{1-\alpha}$	
$(t_N = 1)$	$\lvert e(t_N)\rvert$	$\lvert e'(t_N)\rvert$	$\lvert e(t_N)\rvert$	$\lvert e'(t_N)\rvert$	$\lvert e(t_N)\rvert$	$\lvert e'(t_N)\rvert$
5	3.31D − 3	3.48D − 4	4.19D − 3	1.79D − 3	5.43D − 3	2.30D − 3
10	1.03D − 3	6.49D − 5	7.19D − 4	3.62D − 4	9.40D − 4	4.76D − 4
20	3.32D − 4	1.10D − 5	1.21D − 4	6.88D − 5	1.58D − 4	9.12D − 5
40	1.11D − 4	1.52D − 6	2.04D − 5	1.27D − 5	2.67D − 5	1.69D − 5

predicted order of discrete convergence (cf. equation 15) is confirmed; for the "optimal" value of r $[r = (m + 2 - \alpha)/(2(1 - \alpha))]$ the errors $|e(t_N)|$ are somewhat smaller.

In both tables we have also listed the errors for the derivative u' at $t = t_N$. It can be shown [using the representation of $e'(t)$ preceding (23a)] that on graded grids (14) and under the orthogonality hypothesis (12) we have

$$\max_{(n)} |e'(t_n)| = \begin{cases} O(N^{-m}) & \text{if} \quad c_m < 1 \\ O(N^{-(m+1-\alpha)}) & \text{if} \quad c_m = 1 \end{cases}.$$

This explains why for the Radau (II) points both $|e(t_N)|$ and $|e'(t_N)|$ are of the same magnitude, while for the Gauss points the errors $|e'(t_N)|$ are substantially larger than $|e(t_N)|$.

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