Pointwise convergence rate for nonlinear conservation laws

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Abstract. We introduce a new method to obtain pointwise error estimates for vanishing viscosity and finite difference approximations of scalar conservation laws with piecewise smooth solutions. This method can deal with finitely many shocks with possible collisions. The key ingredient in our approach is an interpolation inequality between the L^1 and Lip^+-bounds, which enables us to convert a global result into a (non-optimal) local estimate. A bootstrap argument yields optimal pointwise error bound for both the vanishing viscosity and finite difference approximations.

1. Introduction

We study solutions to the single hyperbolic conservation laws with small viscosity of the form

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon, \qquad x \in \mathbf{R},\, t > 0, \epsilon > 0 \tag{1}$$

subject to the initial condition

$$u_0^\epsilon(x) = u_0(x). \tag{2}$$

We are interested in the relation between its solution, u^ϵ, and the solution u of the corresponding conservation laws without viscosity

$$u_t + f(u)_x = 0, \qquad x \in \mathbf{R},\, t > 0. \tag{3}$$

The initial condition for (3) is given by

$$u(x, 0) = u_0(x). \tag{4}$$

We will investigate in this paper the *pointwise* error estimates between u and u^ϵ, when u has finitely many shocks.

It is well-known that $u^\epsilon(\cdot, t)$ converges strongly in L^1 to $u(\cdot, t)$, where $u(\cdot, t)$ is the unique, so-called entropy solution of (3)-(4). It is shown in [8] that if the flux f is strictly convex,

$$f'' \geq \beta > 0, \tag{5}$$

then the L^1 convergence rate in this case is upper bounded by

$$\|u^\epsilon(\cdot,t)-u(\cdot,t)\|_{L^1}\leq \text{const.}\epsilon. \tag{6}$$

It is understood that the L^1 error estimate is a global one, while in many practical cases we are interested in the *local* behavior of $u(x,t)$. Consequently, when the error is measured by the L^1-norm, there is a loss of information due to the poor resolution of shock waves in $u(x,t)$. In this work, we will provide the *optimal* pointwise convergence rate for the viscosity approximation. The previous results for the optimal order one convergence rates, in both L^1 and L^∞ spaces, are all based on a matching method and traveling wave solutions, see e.g. [1, 2, 8]. In this work, however, we will not use the traveling wave solutions; instead our arguments are based on energy-like estimates. The proof of our results is based upon two ingredients: (1): Lip^+-boundedness along [4] which enables us to "convert" a global result into a local estimate, (2): A weighted quantity of the error satisfying a transport inequality such that the maximum principal applies.

Unlike previous work on pointwise estimates [1, 2], this framework can deal with finitely many shocks with *possible collisions*. The extensions to the general case can be found in [6]. Moreover, although we only consider the Lax-Friedrichs scheme the idea can be used to obtain the same results for monotone schemes [7].

The paper is organized as follows. In §2 we consider the viscosity methods when there are finitely many shocks. In §3 we discuss the extensions to the Lax-Friedrichs scheme.

2. Viscosity methods

To begin with, we let $\|\bullet\|_{Lip^+}$ denote the Lip^+-seminorm

$$\|w\|_{Lip^+}:=\operatorname*{ess\,sup}_{x\neq y}\left[\frac{w(x)-w(y)}{x-y}\right]^+,$$

where $[w]^+=H(w)w$, with $H(\bullet)$ the Heaviside function.

To convert global L^1-error bounds (for Lip^+ bounded solutions) into local pointwise error estimates, the following lemma is at the heart of matter.

Lemma 2.1. *Assume that $v\in L^1\bigcap Lip^+(I)$, and $w\in C^1_{loc}(x-\delta,x+\delta)$ for an interior x such that $(x-\delta,x+\delta)\in I$. Then the following estimate holds:*

$$|v(x)-w(x)|\leq Const\cdot\left[\frac{1}{\delta}\|v-w\|_{L^1}+\delta\Big\{\|v\|_{Lip^+(x-\delta,x+\delta)}+|w|_{C^1_{loc}(x-\delta,x+\delta)}\Big\}\right].$$

In particular, if the size of the smoothness neighborhood for w can be chosen so that

$$\delta\sim\|v-w\|^{1/2}_{L^1(I)}\cdot\Big(\|v\|_{Lip^+}+|w|_{C^1_{loc}}\Big)^{-1/2}\leq\frac{1}{2}|I|$$

then the following estimate holds:

$$|v(x)-w(x)|\leq Const\cdot\|v-w\|^{1/2}_{L^1(I)}\cdot\Big[\|v\|_{Lip^+}+|w|_{C^1_{loc(x-\delta,x+\delta)}}\Big]^{1/2}. \tag{7}$$

Thus, (7) tells us that if the global L^1-error $\|v - w\|_{L^1}$ is small, then the pointwise error $|v(x) - w(x)|$ is also small whenever w_x is bounded. This does not require the C^1-boundedness of v; the weaker one-sided Lip^+ bound will suffice. The detailed proof of the above lemma can be found in [6].

In this section, we first assume that the entropy solution of (3)-(4) has only one shock discontinuity. The shock curve $x = X(t)$ satisfies the Rankine-Hugoniot and the Lax conditions:

$$X' = \frac{[f(u(X,t)]}{[u(X,t)]}, \tag{8}$$

$$f'(u(X(t)-,t)) > X'(t) > f'(u(X(t)+,t)). \tag{9}$$

Owing to the convexity of the flux f, the viscosity solutions of (1) satisfy a Lip^+-stability condition, similar to the familiar Oleinik's E-condition, which asserts an a priori upper bound for the Lip^+-seminorm of the viscosity solution

$$\|u^\epsilon(\cdot,t)\|_{Lip^+} \le \frac{1}{\|u_0\|^{-1}_{Lip^+} + \beta t}, \tag{10}$$

where u^ϵ is the solution of (1)-(2), β is the convexity constant of the flux f given by (5). The above result suggests that if the initial data do not contain non-Lipschitzian increasing discontinuities then the viscosity solution of (1) will keep the same property. The same is true for entropy solution of (3)-(4). Equipped with (7), together with the global error bound (6) and the Lip^+-boundedness (10), we obtain the following pointwise error bound:

$$|u^\epsilon(x,t) - u(x,t)| \le C\sqrt{\epsilon}, \quad \text{for} \quad dist(x, S(t)) \ge \sqrt{\epsilon}. \tag{11}$$

The basic idea of the pointwise error estimate in this section is as follows:

- **Step** #1: Set

  $$E(x,t) := (u^\epsilon(x,t) - u(x,t))\rho(x,t), \tag{12}$$

  where ρ is a suitably defined *distance* function to the shock curve $x = X(t)$. We will also choose a suitable domain of smoothness, D, such that the following differential equation holds:

  $$E_t + h(x,t)E_x - \epsilon E_{xx} = p(x,t)E + q(x,t)\epsilon, \quad (x,t) \in D. \tag{13}$$

  Here h, p and q are smooth functions in D.
- **Step** #2: The functions p and q in (13) can be (uniformly) upper bounded and bounded, respectively:

  $$p(x,t) \le Const., \qquad |q(x,t)| \le Const., \qquad \text{for all} \quad (x,t) \in D. \tag{14}$$

- **Step** #3: Let ∂D denote the usual boundary for this domain of smoothness, it will be shown that

  $$\max_{(x,t)\in\partial D} |E(x,t)| \le C\epsilon. \tag{15}$$

The inequality (15), together with the maximum principal for (13)-(14), yield $|E(x,t)| \leq C\epsilon$, for all $(x,t) \in D$, which in turn implies the pointwise estimate $|u^\epsilon(x,t) - u(x,t)| \leq C\epsilon$, for (x,t) away from the shock curve $x = X(t)$.

In Step #1 mentioned above, the function E is a weighted error function which is continuous for $(x,t) \in \mathbf{R} \times (0,T]$. The key point in this step is to introduce the *distance* function ρ, which satisfies $\rho \to 0$ as $dist(x, S(t)) \to 0$ and $\rho \sim O(1)$ when $dist(x, S(t)) \sim O(1)$. The proof for Step #2 is based upon the interpolation between the *global* L^1-error estimate and the Lip^+-stability that leads to a *local* pointwise estimate. The proper use of the Lax entropy condition (9) is also crucial in this step. The third step is dependent on the choice of the weighted distance function, ρ.

We first consider the pointwise error estimate in the region $x > X(t)$. Let $e(x,t) := u^\epsilon - u$ and set the weighted error

$$E(x,t) = e(x,t)\,\phi(x - X(t))\,.$$

Here, $\phi(x - X(t))$ is a weighted distance to the shock set. The function $\phi(x) \in C^2([0,\infty))$ satisfies

$$\phi(x) \sim \begin{cases} x^\alpha, & \text{if} \quad 0 \leq x \ll 1 \\ 1, & \text{if} \quad x \gg 1, \end{cases} \tag{16}$$

with $\alpha \geq 1$ to be determined later. More precisely, the function ϕ satisfies

$$\phi(0) = 0, \qquad \phi'(x) > 0, \quad \phi(x) \leq x^\alpha, \quad \text{for} \quad x > 0; \tag{17}$$

$$x\phi'(x) \leq \alpha\phi(x), \quad \text{for} \quad x \geq 0; \tag{18}$$

$$|\phi^{(k)}(x)| \leq Const, \quad x \geq 0, \tag{19}$$

e.g., $\phi(x) = (1 - e^{-x})^\alpha$. Roughly speaking, the weighted function behaves like $\phi(x) \sim \min(|x|^\alpha, 1)$. Direct calculations using the definition of E give us

$$\begin{aligned} E_t + f'(u^\epsilon)E_x - \epsilon E_{xx} &= \underbrace{\Big(e_t + f'(u^\epsilon)e_x - \epsilon e_{xx}\Big)\phi}_{I_1} \\ &+ \underbrace{\Big(-X'(t) + f'(u^\epsilon)\Big)\phi' e}_{I_2} - 2\epsilon\phi' e_x - \epsilon\phi'' e. \end{aligned} \tag{20}$$

For ease of notation, ϕ denotes $\phi(x - X(t))$ in the remaining of this section. It follows from the viscosity equation (1) and the limit equation (3) that

$$\begin{aligned} I_1 &= \Big(-f'(u^\epsilon)u_x + f'(u)u_x + \epsilon u_{xx}\Big)\phi \\ &= -\phi f''(\bullet)(u^\epsilon - u)u_x + \epsilon\phi u_{xx} \\ &= -f''(\bullet)u_x\, E + \epsilon\phi u_{xx} \end{aligned} \tag{21}$$

where (and below) $\bullet$ denotes some intermediate value between $-\|u_0\|_\infty$ and $\|u_0\|_\infty$. Let $u_\pm(t) = u(X(t) \pm 0, t)$ and let

$$I_3(t) = -X'(t) + f'(u_+).$$

Observing that $u - u_+ = u_x(\zeta_1)(x - X(t))$, where ζ_1 is an intermediate value between x and $X(t)$, we obtain

$$\begin{aligned} I_2 &= \Big(I_3 - f'(u_+) + f'(u) - f'(u) + f'(u^\epsilon)\Big)\phi' e \\ &= \phi' e I_3 + \phi' e f''(\bullet)(u - u_+) + \phi' f''(\bullet) e^2 \\ &= \Big(I_3 + f''(\bullet) e\Big)\frac{\phi'}{\phi} E + f''(\bullet) u_x(\zeta_1)\frac{(x - X(t))\phi'}{\phi} E, \qquad (22) \end{aligned}$$

where in the last step we have used the fact $E = e\phi$. It is noted that $e_x = (E_x - \phi' e)/\phi$. This, together with (20)-(22), yield the first desired result, (13):

$$E_t + h(x,t)E_x - \epsilon E_{xx} = p(x,t)E + q(x,t)\,\epsilon\,,$$

where the coefficient of the convection term is given by

$$h(x,t) = f'(u^\epsilon) + 2\epsilon\frac{\phi'}{\phi}\,, \qquad (23)$$

and the functions $p := p_1 + p_2$ and q are given by

$$p_1(x,t) := I_3\frac{\phi'}{\phi} + f''(\bullet) e\frac{\phi'}{\phi} + 2\epsilon\left(\frac{\phi'}{\phi}\right)^2; \qquad (24)$$

$$p_2(x,t) := -f''(\bullet)\, u_x + f''(\bullet) u_x(\zeta_1)\frac{(x - X(t))\phi'}{\phi}; \qquad (25)$$

$$q(x,t) := \phi u_{xx} - \phi''\, e\,. \qquad (26)$$

We have then finished the Step #1.

Next we move to Step #2, verifying the boundedness of the coefficients p and q inside a suitable domain. We now choose a proper domain of smoothness, D, inside the region $x > X(t)$. Let

$$D := \Big\{(x,t) \;\Big|\; x \geq X(t) + \epsilon^{1/2}\,,\, 0 \leq t \leq T\Big\}. \qquad (27)$$

Using Lax geometrical entropy condition (9), $u_+(t) \leq u_-(t)$, and the convexity of f, it follows that I_3 is nonpositive

$$\begin{aligned} I_3(t) &= -X'(t) + f'(u_+) = \int_0^1 \Big[f'(u_+) - f'(\theta u_+ + (1-\theta)u_-)\Big] d\theta \\ &= \int_0^1 f''(\bullet)(1-\theta) d\theta\, (u_+ - u_-) \leq 0. \end{aligned}$$

For $(x,t) \in D, x > X(t) + \sqrt{\epsilon}$, and hence by the property (18) of the weighted distance function ϕ we have

$$0 \leq \frac{\phi'}{\phi} \leq \frac{C}{x - X(t)} \leq C\epsilon^{-1/2}, \quad \text{for} \quad (x,t) \in D.$$

The last two upper bounds, together with (11), lead to the following estimate for p_1

$$p_1 \leq 0 + C\epsilon^{1/2}\epsilon^{-1/2} + C\epsilon\epsilon^{-1} \leq C, \quad \text{for} \quad (x,t) \in D\,. \qquad (28)$$

By the property of ϕ, $(x - X(t))\phi'(x - X(t))/\phi(x - X(t)) \leq Const$ and the regularity of u, $|u_x| \leq Const$ for all $(x,t) \in D$, we obtain that p_2 is also upper bounded. Again, due to the C^2-smoothness assumption on u, q is bounded in the domain of smoothness, D. This completes Step #2.

Finally, we need to verify Step #3, upper bounding E on ∂D. We first check that the maximum value for E on the left boundary is bounded by $O(\epsilon)$. On the left boundary, we have $x - X(t) = \epsilon^{1/2}$; hence by $|\phi| \leq x^\alpha$, and by $|e(x,t)| = O(\epsilon^{1/2})$, we have

$$|E(x,t)| \leq \epsilon^{\alpha/2}|e(x,t)| \leq C\epsilon^{\alpha/2}\,\epsilon^{1/2}\,.$$

Choosing $\alpha = 1$, we have $E(x,t) = O(\epsilon)$ on the left boundary of the domain D. On the right and the bottom of D, $E(x,t)$ vanishes. This completes Step #3. Hence, the maximum principal gives

$$|E(x,t)| \leq C\epsilon, \qquad \text{for} \quad (x,t) \in D.$$

This implies that the *weighted* error $u^\epsilon(x,t) - u(x,t)\,\phi(x - X(t))$ is bounded by $O(\epsilon)$, in particular for (x,t) bounded away from the shock curve $x = X(t)$ we have $O(\epsilon)$ *pointwise error* bound. Similarly, we can show that the same is true when (x,t) is on the left side of the shock. The argument can be extended when there are finitely many shocks.

We summarize what we have shown by stating the following:

Theorem 2.2. *Let $u^\epsilon(x,t)$ be the viscosity solutions of (1)-(2). Let $u(x,t)$ be the entropy solution of (3)-(4), and assume it has finitely many shock discontinuities, then the following error estimates hold:*

- *For a weighted distance function ϕ, $\phi(x) \sim \min(|x|, 1)$,*

$$|(u^\epsilon - u)(x,t)|\phi\left(|x - X(t)|\right) = O(\epsilon). \tag{29}$$

- *In particular, if (x,t) is bounded away from the singular support of u, then*

$$|(u^\epsilon - u)(x,t)| \leq C(h)\epsilon, \qquad for \quad dist(x, S(t)) \geq h > 0. \tag{30}$$

- *Since the weighted function $\phi(x) \sim |x|$, it follows from (29) that*

$$|(u^\epsilon - u)(x,t)| \sim \epsilon\, dist(x, S(t))^{-1}. \tag{31}$$

This implies that the thickness of the shock layer is of order $O(\epsilon)$.

3. The Lax-Friedrichs scheme

The Lax-Friedrichs (LxF) scheme is of the following form:

$$v_j^{n+1} = \frac{1}{2}\Big(v_{j+1}^n + v_{j-1}^n\Big) - \frac{\lambda}{2}\Big(f(v_{j+1}^n) - f(v_{j-1}^n)\Big). \tag{32}$$

It is well known that the truncation error of the LxF scheme is $O(\Delta t^2)$. This, and the assumption that u_{xx} is uniformly bounded in the region $x > X(t)$, imply that

$$U_j^{n+1} - \frac{1}{2}\Big(U_{j+1}^n + U_{j-1}^n\Big) + \frac{\lambda}{2}\Big(f(U_{j+1}^n) - f(U_{j-1}^n)\Big) = O(\Delta t^2), \tag{33}$$

where $U_j^n := u(x_j, t_n), x_j = j\Delta x, t_n = n\Delta t$. We require that $\lambda := \Delta t/\Delta x$ satisfies the standard CFL condition

$$\lambda \max_j |f'(v_j^n)| < 1. \tag{34}$$

For a fixed integer n, we define

$$J(n) = \min\Big\{ j \;\Big|\; x_j \geq X(t_n) + \Delta t^{1/4} \Big\}. \tag{35}$$

The following lemmas are useful in obtaining our error bounds. However, due to the limitation of space we will omit the detail proofs. They can be found in [7].

Lemma 3.1. *For any given $T > 0$ and given integer $m > 0$, there exists a positive constant $C(m,T) > 0$ such that*

$$|v_j^n - U_j^n| \leq C(m,T)\Delta t^{1/4}, \qquad j \geq J(n) - m, \quad n \leq T/\Delta t\,, \tag{36}$$

where $U_j^n := u(x_j, t_n)$, $J(n)$ is defined by (35).

The above lemma is established by using Lemma 2.1, the interpolation between L^1 and Lip^+ estimates. The uniform Lip^+-bounds are obtained by Nessyahu and Tadmor [5]. In the discrete case, the optimal L^1 error bounds for the case with finitely many shocks are not available. The best result was obtained by Kuznetsov [3]:

$$\|v^n - U^n\|_{L^1} := \sum_j \Delta x |v_j^n - U_j^n| \leq Const \cdot \Delta t^{1/2}. \tag{37}$$

With the above non-optimal L^1-error bound, the order of the pointwise error bound (36) is even less than 1/2. However, it will suffice to derive the optimal error bound by a bootstrap argument.

Lemma 3.2. *Let $e_j^n = v_j^n - U_j^n$ and*

$$\rho_j^n = \int_0^1 f'(\theta v_{j+1}^n + (1-\theta)v_{j-1}^n)d\theta\,.$$

In the smooth region of u, the following result holds:

$$e_j^{n+1} - \frac{1}{2}\Big(e_{j+1}^n + e_{j-1}^n\Big) + \frac{\lambda}{2}\rho_j^n\Big(e_{j+1}^n - e_{j-1}^n\Big) = \Delta t T_j^n + O(\Delta t^2)\,, \quad j \geq J(n)\,, \tag{38}$$

where the term $\mathbf{T}_j^n$ can be bounded by

$$\left|\mathbf{T}_j^n\right| \leq C\left(|e_{j+1}^n| + |e_{j-1}^n|\right)\,. \tag{39}$$

The above lemma is established by using Taylor expansions. The main part of our error analysis is to estimate

$$E_j^{n+1} - \frac{1}{2}\Big(E_{j+1}^n + E_{j-1}^n\Big) + \frac{\lambda}{2}\rho_j^n\Big(E_{j+1}^n - E_{j-1}^n\Big), \quad j \ge J(n), \tag{40}$$

where $E_j^n = \phi(x_j - X(t_n)) \cdot e_j^n$, ϕ is given in §2 (with $\alpha = 3$). Let $\phi_j^n = \phi(x_j - X(t_n))$ The following facts will be used frequently

$$\begin{cases} \phi_{j\pm1}^n = \phi_j^n \pm \Delta x (\phi')_j^n + O(\Delta x^2)\,, \\ \phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n = (\phi'')_j^n \Delta x^2 + O(\Delta x^3)\,. \end{cases} \tag{41}$$

The following Taylor expansion

$$\phi_j^{n+1} = \phi_j^n - \Delta t X'(t_n)(\phi')_j^n + O(\Delta x^2)\,.$$

together with Lemma 3.2, lead to

$$\begin{aligned} & E_j^{n+1} - \frac{1}{2}\Big(E_{j+1}^n + E_{j-1}^n\Big) + \frac{\lambda}{2}\rho_j^n\Big(E_{j+1}^n - E_{j-1}^n\Big) \\ = \;& \Big(\phi_j^n - \Delta t X'(t_n)(\phi')_j^n + O(\Delta x^2)\Big)\Big(\frac{1}{2}(e_{j+1}^n + e_{j-1}^n) - \frac{\lambda}{2}\rho_j^n(e_{j+1}^n - e_{j-1}^n) \\ & \quad + \Delta t \mathbf{T}_j^n + O(\Delta t^2)\Big) - \frac{1}{2}\Big(E_{j+1}^n + E_{j-1}^n\Big) + \frac{\lambda}{2}\rho_j^n\Big(E_{j+1}^n - E_{j-1}^n\Big) \\ = \;& \frac{\phi_j^n - \phi_{j+1}^n}{2\phi_{j+1}^n}E_{j+1}^n + \frac{\phi_j^n - \phi_{j-1}^n}{2\phi_{j-1}^n}E_{j-1}^n - \frac{\lambda}{2}\phi_j^n\rho_j^n(e_{j+1}^n - e_{j-1}^n) \\ & \quad + \frac{\lambda}{2}\rho_j^n\Big(E_{j+1}^n - E_{j-1}^n\Big) - \Delta t X'(t_n)(\phi')_j^n\frac{1}{2}(e_{j+1}^n + e_{j-1}^n) \\ & \quad + \frac{\lambda}{2}\Delta t X'(t_n)(\phi')_j^n\rho_j^n(e_{j+1}^n - e_{j-1}^n) + \mathbf{T}_j^n\,O(\Delta t) + O(\Delta t^2)\,. \end{aligned} \tag{42}$$

It follows from (41) that

$$\begin{aligned} & -\frac{\lambda}{2}\phi_j^n\rho_j^n(e_{j+1}^n - e_{j-1}^n) + \frac{\lambda}{2}\rho_j^n\Big(E_{j+1}^n - E_{j-1}^n\Big) = -\frac{\lambda}{2}\phi_j^n\rho_j^n(e_{j+1}^n - e_{j-1}^n) \\ & \quad + \frac{\lambda}{2}\rho_j^n\Big[e_{j+1}^n(\phi_j^n + \Delta x(\phi')_j^n) - e_{j-1}^n(\phi_j^n - \Delta x(\phi')_j^n)\Big] + O(\Delta x^2) \\ = \;& \Delta t\,\rho_j^n(\phi')_j^n\frac{1}{2}(e_{j+1}^n + e_{j-1}^n) + O(\Delta t^2)\,. \end{aligned}$$

where in the last step we have used the fact $\lambda\Delta x = \Delta t$. This result, together with (42), lead to

$$\begin{aligned} & E_j^{n+1} - \frac{1}{2}\Big(E_{j+1}^n + E_{j-1}^n\Big) + \frac{\lambda}{2}\rho_j^n\Big(E_{j+1}^n - E_{j-1}^n\Big) \\ = \;& I_1 + J_1 + I_2 + J_2 + I_3 - J_3 + \mathbf{T}_j^n\,O(\Delta t) + O(\Delta t^2) \end{aligned} \tag{43}$$

for $j \geq J(n)$, where

$$I_1 = \frac{\phi_j^n - \phi_{j+1}^n}{2\phi_{j+1}^n}, \qquad J_1 = \frac{\phi_j^n - \phi_{j-1}^n}{2\phi_{j-1}^n};$$

$$I_2 = \frac{\Delta t}{2}\frac{(\phi')_j^n}{\phi_{j+1}^n}\Big(- X'(t_n) + \rho_j^n\Big), \quad J_2 = \frac{\Delta t}{2}\frac{(\phi')_j^n}{\phi_{j-1}^n}\Big(- X'(t_n) + \rho_j^n\Big);$$

$$I_3 = \frac{\lambda}{2}\Delta t X'(t_n)\frac{(\phi')_j^n}{\phi_{j+1}^n}\rho_j^n, \qquad J_3 = \frac{\lambda}{2}\Delta t X'(t_n)\frac{(\phi')_j^n}{\phi_{j-1}^n}\rho_j^n.$$

It follows from (43) that

$$\begin{aligned} E_j^{n+1} &= \Big(\frac{1}{2} - \frac{\lambda}{2}\rho_j^n + I_1 + I_2 + I_3\Big)E_{j+1}^n + \Big(\frac{1}{2} + \frac{\lambda}{2}\rho_j^n + J_1 \\ &\quad + J_2 - J_3\Big)E_{j-1}^n + \mathbf{T}_j^n\, O(\Delta t) + O(\Delta t^2), \end{aligned} \tag{44}$$

for $j \geq J(n)$. We can further show that for $j \geq J(n)$,

$$\begin{cases} I_1 + J_1 \leq C\Delta t^{3/2}, \quad I_2 \leq C\Delta t, \quad J_2 \leq C\Delta t, \quad I_3 - J_3 \leq C\Delta t^{3/2}, \\ \\ |I_k| \leq O(\Delta t^{3/4}), \quad |J_k| \leq O(\Delta t^{3/4}), \qquad k = 1, 2, 3. \end{cases} \tag{45}$$

It follows from (34) and the fact $f'' > 0$ that

$$\lambda \max_j |\rho_j^n| < 1. \tag{46}$$

This and the last two inequalities of (45) imply that the coefficients for $E_{j\pm 1}^n$ in (44) are *nonnegative*, provided that Δt is sufficiently small. This result implies that from (44) and the first three inequalities in (45) we have

$$\begin{aligned} |E_j^{n+1}| &\leq \Big(1 + C\Delta t\Big)\max_{j\geq J(n)-1}|E_j^n| + C\Delta t(|e_{j+1}^n| + |e_{j-1}^n|) + C\Delta t^2 \\ &\leq \Big(1 + C\Delta t\Big)\max_{j\geq J(n)-1}|E_j^n| + C\Delta t^2 \quad \text{for} \quad j \geq J(n). \end{aligned} \tag{47}$$

In other words, we have proved that

$$\max_{j\geq J(n)}|E_j^{n+1}| \leq \Big(1 + C\Delta t\Big)\max_{j\geq J(n)-1}|E_j^n| + C\Delta t^2. \tag{48}$$

The above inequality is not an exact Gronwall type inequality. By using the information on the numercal boundary $j = J(n)$, it can be improved to the standard Gronwall type inequality which yields

$$\max_{j\geq J(n), 0\leq n\leq N}|E_j^n| \leq C\Delta t. \tag{49}$$

This result implies that $v_j^n - u(x_j, t_n)$ can be bounded by $C\Delta t$, if (t_j, t_n) is on the right side of the shock and is of any $O(1)$ distance away from the shock curve. Similarly, we can show that the same is true if (x_j, t_n) is on the left side of the shock curve. The argument can be extended when there are finitely many shocks.

We summarize what we have shown by stating the following:

Theorem 3.3. *Let* $\{v_j^n\}$ *be the solution of the Lax-Friedrichs scheme (32). Let* $u(x,t)$ *be the entropy solution of (3)-(4), and assume it has finitely many shock discontinuities, then the following error estimate holds:*

- *For a weighted distance function* ϕ, $\phi(x) \sim \min(|x|^3, 1)$,
$$|v_j^n - u(x_j, t^n)|\,\phi(|x_j - X(t_n)|) = O(\Delta t)\,. \tag{50}$$
- *In particular, if* (x_j, t_n) *is bounded away from the singular support, then*
$$|v_j^n - u(x_j, t^n)| \leq C(h)\Delta t\,, \qquad dist(x_j, S(t_n)) \geq h > 0. \tag{51}$$

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