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RIEMANN PROBLEM FOR A COMBUSTION MODEL SYSTEM: THE Z-N-D SOLUTIONS

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Abstract The Riemann problem for a combustion model system with special kind of viscosity and chemical reaction is considered and the existence of the Riemann problem is proved. The limit of the Riemann solution as vanished viscosity is also investigated.
Key Words Riemann problem; combustion model; vanished viscosity.
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1. Introduction

If fulid flow is accompanied by chemical reaction, then very complicated wave motion phenomena occur. Chapman and Jouguet used a simple and typical model which showed various waves of combustion: strong detonation wave, weak detonation wave, strong deflagration wave, weak deflagration wave, and their critical states, the so-called Chapman-Jouguet detonation wave and deflagration wave (see, *e.g.*, [1–2]). Afterwards, many authors have done various works about the structure of these waves and their formative conditions using different kinds of models. More research works have been done in the laboratories and by numerical experiments (see, *e.g.*, [3–4]).

It is an interesting problem how a mathematical model can be applied to these phenomena and one may investigate them by the theory of differential equations. Some authors investigated the travelling wave solutions with some Riemann initial value problems but up to now these investigations are not so deep as that for shock waves (see, *e.g.*, [5–6]).

A system of combustion model has been introduced in [7]. The governing equations are

$$\frac{\partial}{\partial t}(u+qZ)+\frac{\partial}{\partial x}(f(u))=\varepsilon t\frac{\partial^2 u}{\partial x^2} \tag{1a}$$

$$\frac{\partial Z}{\partial t}=-\frac{K}{t}\phi(u)Z \tag{1b}$$

where u is a lumped variable representing some features of density, velocity, and the temperature while Z represents the fraction of unburnt gas. The constants $q > 0, \varepsilon > 0$ and $K > 0$ represent the binding energy, viscosity and the rate of chemical reaction, respectively; and $f(u)$ is a convex strongly nonlinear function satisfying

$$f'(u) > 0, \quad f''(u) > 0 \quad (u \le 0), \quad f''(u) \ge \delta > 0 \quad (u > 0) \tag{2}$$

Function ϕ is defined as

$$\phi(u) = \begin{cases} 0, & u \le 0 \\ 1, & u > 0 \end{cases} \tag{3}$$

The present work is to consider Eqs (1) with the initial conditions,

$$(u(x,0); Z(x,0)) = \begin{cases} (u_L; 0), & x \le 0 \\ (u_R; 1), & x > 0 \end{cases} \tag{4}$$

where $u_L > 0 > u_R, \varepsilon, q, K > 0$. Then the problem considered in Teng and Ying [13] is a special case ($\varepsilon = 0$) of (1) and (4). The so-called Z-N-D solution refers to the solutions of Eqs (1) and (4) with finite rate of chemical reaction and vanished viscosity, namely, in Eqs (1) let K be fixed and $\varepsilon \to 0+$.

2. Some Lemmas

Let $\xi = x/t$, the Eqs. (1) and (4) become

$$\varepsilon u'' = (f'(u) - \xi)u' - q\xi Z' \tag{5a}$$

$$\xi Z' = K\phi(u)Z \tag{5b}$$

$$(u; Z)(-\infty) = (u_L; 0), \quad (u, Z)(+\infty) = (u_R; 1) \tag{5c}$$

with $\xi \in \boldsymbol{R}$. Here $u = u(\xi)$ and $Z = Z(\xi)$.

In this section, C always denotes some positive constants which depend only on u_L, u_R, q, K and δ but not on ε; while $C(\varepsilon)$ denotes those constants dependent on ε. For case of notation, we will denote u_ε by u in all of proofs. Moreover, we assume throughout the paper that $\varepsilon \le \varepsilon^*$. Here ε^* is a fixed constant.

Lemma 1 (see [7]). *If $\varepsilon \le \varepsilon^*$, then the Eqs. (5) possesses a solution $(u_\varepsilon, Z_\varepsilon)$ with $u_\varepsilon \in C^1(\boldsymbol{R})$ and $Z_\varepsilon \in C(\boldsymbol{R})$. Moreover, the solution $(u_\varepsilon, Z_\varepsilon)$ has only two possible cases, namely,*

(i) (Case I) *$u_\varepsilon(\xi)$ decreases monotonically on $\boldsymbol{R}$ with the unique zero-point $\eta > 0$ (see Fig. 1);*

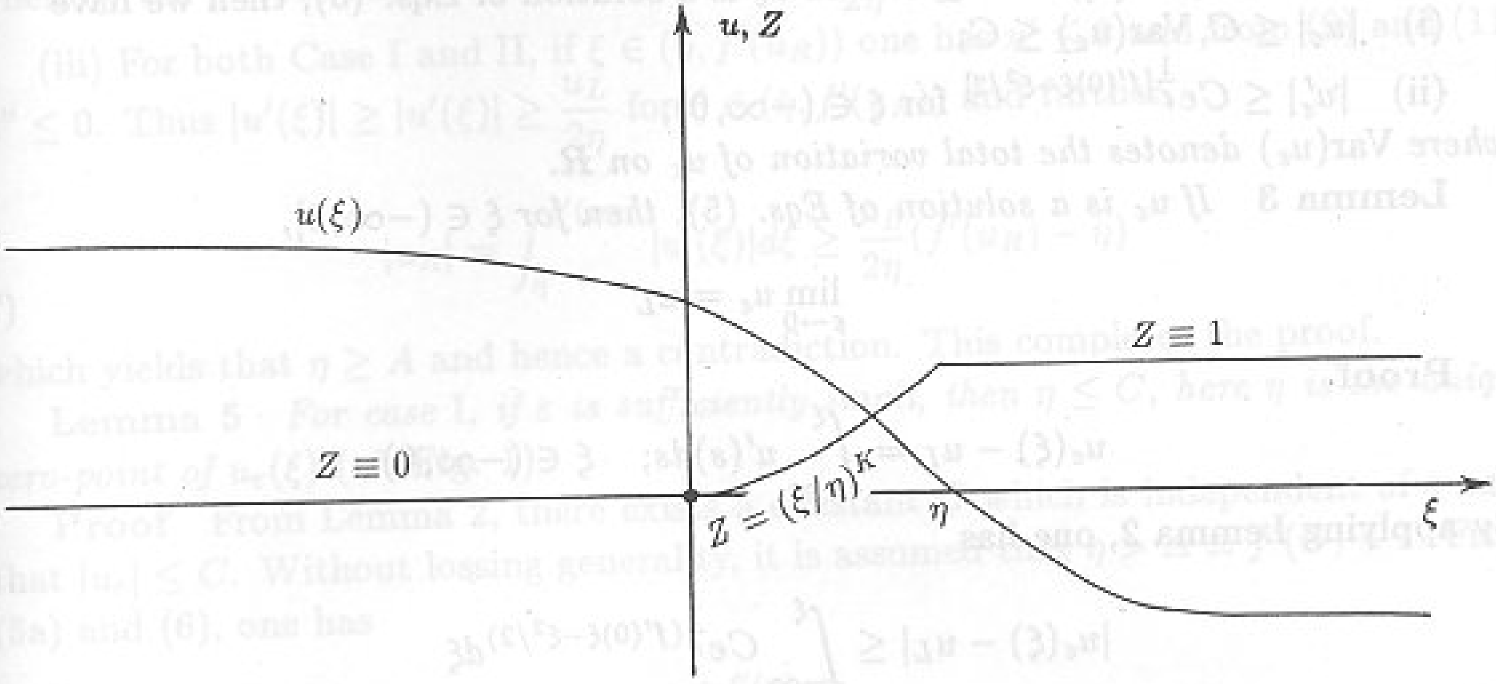

Fig.1 The solution structure.

(ii) (Case II) *there exists a (unique) maximum value point* $\eta_1 > 0$ *such that* $u_\varepsilon(\eta_1) = \max\{u_\varepsilon(\xi) : \xi \in \boldsymbol{R}\}$ *and* u_ε *increases monotonically on* $(-\infty, \eta_1)$ *and decreases monotonically on* (η_1, ∞) (see Fig. 2).

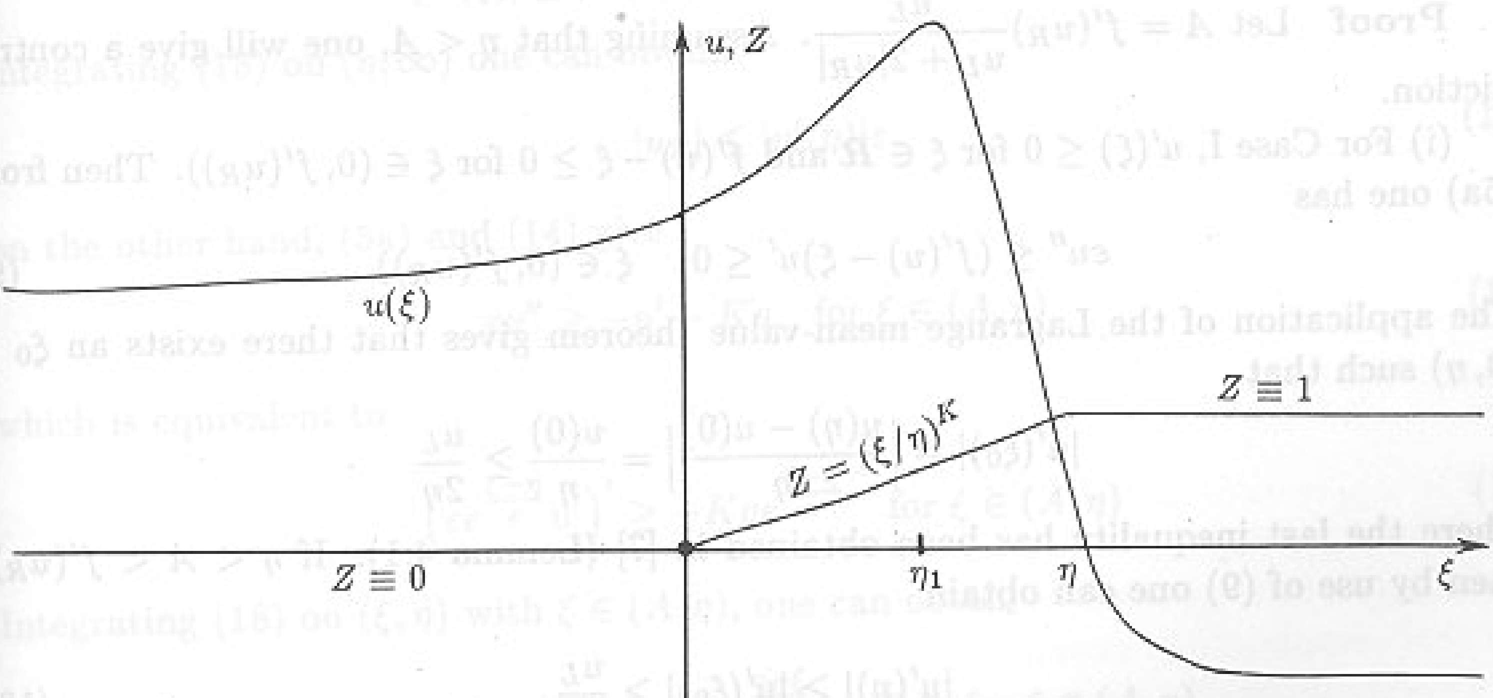

Fig.2 The solution structure.

In both cases, Z_ε *can be expressed as*

$$Z_\varepsilon(\xi) = \begin{cases} 0, & \xi \le 0 \\ (\xi/\eta)^K, & 0 < \xi < \eta \\ 1, & \xi \ge \eta \end{cases} \tag{6}$$

Lemma 2 (see [7]). If $\varepsilon \le \varepsilon^*$ and u_ε is a solution of Eqs. (5), then we have
(i) $|u_\varepsilon| \le C, \mathrm{Var}(u_\varepsilon) \le C$;
(ii) $|u'_\varepsilon| \le Ce^{\frac{1}{\varepsilon}[f'(0)\xi-\xi^2/2]}$ for $\xi \in (-\infty, 0)$,
where $\mathrm{Var}(u_\varepsilon)$ *denotes the total variation of* u_ε *on* $\boldsymbol{R}$.

Lemma 3 *If* u_ε *is a solution of Eqs.* (5), *then for* $\xi \in (-\infty, 0]$,

$$\lim_{\varepsilon\to 0} u_\varepsilon = u_L \tag{7}$$

Proof

$$u_\varepsilon(\xi) - u_L = \int_{-\infty}^{\xi} u'(s)ds; \quad \xi \in (-\infty, 0)$$

By applying Lemma 2, one has

$$\begin{aligned}|u_\varepsilon(\xi) - u_L| &\le \int_{-\infty}^{\xi} Ce^{\frac{1}{\varepsilon}(f'(0)\xi-\xi^2/2)}d\xi \\ &\le \int_{-\infty}^{\xi} Ce^{\frac{1}{\varepsilon}f'(0)\xi}d\xi \le \frac{C}{f'(0)}\varepsilon\end{aligned} \tag{8}$$

Hence the inequality (8) leads to (7) and this completes the proof.

Lemma 4 *For both Cases I and II, if* ε *is sufficiently small, then* $\eta \ge f'(u_R)\dfrac{u_L}{u_L+2|u_R|}$, *here* η *is the (unique) zero-point of* u_ε.

Proof Let $A = f'(u_R)\dfrac{u_L}{u_L+2|u_R|}$. Assuming that $\eta < A$, one will give a contradiction.

(i) For Case I, $u'(\xi) \le 0$ for $\xi \in \boldsymbol{R}$ and $f'(u) - \xi \ge 0$ for $\xi \in (0, f'(u_R))$. Then from (5a) one has

$$\varepsilon u'' \le (f'(u) - \xi)u' \le 0, \quad \xi \in (0, f'(u_R)) \tag{9}$$

The application of the Lagrange mean-value theorem gives that there exists an $\xi_0 \in (0, \eta)$ such that

$$|u'(\xi_0)| = \left|\frac{u(\eta) - u(0)}{\eta}\right| = \frac{u(0)}{\eta} \ge \frac{u_L}{2\eta}$$

where the last inequality has been obtained in [7] (Lemma 4.1). If $\eta < A < f'(u_R)$, then by use of (9) one can obtain

$$|u'(\eta)| \ge |u'(\xi_0)| \ge \frac{u_L}{2\eta} \tag{10}$$

(ii) For Case II, $u'(\xi) \le 0$ for $\xi \in (\eta_1, \infty)$ and $f'(u) - \xi \ge 0$ for $\xi \in (\eta_1, f'(u_R))$. Then one can deduce that

$$u''(\xi) \le 0 \quad \text{for } \xi \in (\eta_1, f'(u_R)) \tag{11}$$

Similarly, there exists an $\xi_1 \in (\eta_1, \eta)$ such that

$$|u'(\xi_1)| = \left|\frac{u(\eta) - u(\eta_1)}{\eta - \eta_1}\right| \ge \frac{u_L}{\eta} \ge \frac{u_L}{2\eta} \tag{12}$$

Then by use of (11) one has $|u'(\eta)| \geq |u'(\xi_1)| \geq \frac{u_L}{2\eta}$.

(iii) For both Case I and II, if $\xi \in (\eta, f'(u_R))$ one has $u' \leq 0$ and from (9) and (11) $u'' \leq 0$. Thus $|u'(\xi)| \geq |u'(\xi)| \geq \frac{u_L}{2\eta}$ for $\xi \in (\eta, f'(u_R))$ and further,

$$|u_R| = \int_\eta^{f'(u_R)} |u'(\xi)| d\xi \geq \frac{u_L}{2\eta}(f'(u_R) - \eta)$$

which yields that $\eta \geq A$ and hence a contradiction. This completes the proof.

Lemma 5 *For case* I, *if ε is sufficiently small, then $\eta \leq C$, here η is the unique zero-point of $u_\varepsilon(\xi)$* (cf. Fig. 1).

Proof From Lemma 2, there exists a constant C which is independent of ε such that $|u_\varepsilon| \leq C$. Without lossing generality, it is assumed that $\eta > A \equiv f'(C) + 1$. From (5a) and (6), one has

$$|u'(\xi)| = |u'(\eta)| e^{\int_\eta^\xi \frac{f'(u) - s}{\varepsilon} ds} \quad \text{for } \xi \in (\eta, \infty) \tag{13}$$

If $\xi > A$, then

$$f'(u) - \xi \leq -1 \quad \text{for } \xi \in (\eta, \infty) \tag{14}$$

Thus, (14) gives

$$|u'(\xi)| \leq |u'(\eta)| e^{\frac{1}{\varepsilon}(\eta - \xi)} \quad \text{for } \xi \in (\eta, \infty) \tag{15}$$

Integrating (15) on (η, ∞) one can obtain

$$|u_R| \leq |u'(\eta)| \varepsilon \tag{16}$$

on the other hand, (5a) and (14) give

$$\varepsilon u'' \geq -u' - Kq \quad \text{for } \xi \in (A, \eta) \tag{17}$$

which is equivalent to

$$\left(\varepsilon e^{\frac{\xi - \eta}{\varepsilon}} u'\right)' \geq -Kq e^{\frac{\xi - \eta}{\varepsilon}} \quad \text{for } \xi \in (A, \eta) \tag{18}$$

Integrating (18) on (ξ, η) with $\xi \in (A, \eta)$, one can obtain

$$u'(\xi) \leq (u'(\eta) + Kq) e^{\frac{\eta - \xi}{\varepsilon}} - Kq \quad \text{for } \xi \in (A, \eta)$$

Integrating (18) on (ξ, η), one can obtain

$$u'(\xi) \leq (u'(\eta) + Kq) e^{\frac{\xi - \eta}{\varepsilon}} - Kq, \quad \xi \in (A, \eta) \tag{19}$$

By use of (16) and for sufficiently small ε, ons has

$$u'(\eta) + Kq \leq -\frac{|u_R|}{\varepsilon} + Kq \leq 0 \tag{20}$$

Thus (19) gives

$$u'(\xi) \le Kq, \quad \xi \in (A, \eta) \tag{21}$$

Integrating (21) on (A, η), one has

$$-C \le -u(A) = \int_A^\eta u'(\xi)d\xi \le -Kq(\eta - A)$$

which gives

$$\eta \le A + \frac{C}{Kq} \tag{22}$$

This completes the proof.

Lemma 6 *If ε is sufficiently small, then $u'_\varepsilon(0) > 0$.*

Proof Let $A = \min\{\eta, f'(u_R)\}$. From (5a) and (6) one has for $\xi \in (0, A)$

$$u'(\xi) = u'(0)e^{\int_0^\xi \frac{f'(u)-s}{\varepsilon}ds} - \int_0^\xi \frac{qK}{\varepsilon}\left(\frac{s}{\eta}\right)^K e^{\int_0^\xi \frac{f'(u)-x}{\varepsilon}dx}ds \tag{23}$$

For $\xi \in (0, A), \xi \le f'(u_R) \le f'(u)$. If $u'(0) \le 0$ (cf. Fig. 1), then from (23) one has

$$u'(\xi) \le -\int_0^\xi \frac{qK}{\varepsilon}\left(\frac{s}{\eta}\right)^K e^{\int_0^\xi \frac{f'(u)-x}{\varepsilon}dx}ds$$

$$\le -\int_0^\xi \frac{qK}{\varepsilon}\left(\frac{s}{\eta}\right)^K ds = -\frac{qK}{\varepsilon}\frac{1}{K+1}\frac{\xi^{K+1}}{\eta^K}, \quad \xi \in (0, A) \tag{24}$$

Thus,

$$u_R - u_L \le u(A) - u(0) = \int_0^A u'(\xi)d\xi$$

$$\le -\frac{qK}{\varepsilon}\frac{1}{(K+1)(K+2)}\frac{A^{K+2}}{\eta^K} \tag{25}$$

By use of Lemmas 4 and 5, one has

$$u_R - u_L \le -\frac{qK}{\varepsilon}C \tag{26}$$

where C is a positive constant independent of ε. Let $\varepsilon \to 0+$, (26) gives a contradiction and this completes the proof.

This result implies that the structure of the solution of (5) contains only one possible case, namely, Case II as shown in Fig. 2 if ε is sufficiently small. Hence we will from now on concentrate on this case.

Lemma 7 *If ε is sufficiently small, then the first derivative of $u_\varepsilon(\xi), u'(\xi)$, is uniformly bounded on $(0, \eta_1)$, where η_1 is the (unique) maximum-value point of u_ε as shown in Fig.2.*

Proof By Lemma 4, if ε is sufficiently small, then there exists a constant β, such that $\eta \ge \beta$, here η is the unique zero-point of u_ε. Let $A \equiv \max\left\{\frac{1}{\beta}, \frac{qK^2+1}{\delta}\right\}$. It will be shown that if $u''(\xi) > 0$ for any $\xi \in (0, \eta_1)$ then $u'(\xi) \leq A$. If this were not true, then there exists an $a \in (0, \eta_1)$ such that $u''(a) > 0$ but $u'(a) > A$. In this case, one can show that $u''(\xi) > 0$ for all $\xi \in (a, \eta_1)$. The proof is given as follows. Assume that there exists $b \in (a, \eta_1)$ satisfying $u''(\xi) > 0$ for $\xi \in (a, b)$ but $u''(b) = 0$. Then by the Lagrange mean-value theorem, there exist c_1 and c_2 in (a, b) such that

$$qK\Big(\frac{b}{\eta}\Big)^K = qK\Big(\frac{a}{\eta}\Big)^k + qK^2 c_1^{K-1}\Big(\frac{1}{\eta}\Big)^K (b-a) \tag{27}$$

and

$$u'(b)h(b) = u'(a)h(a) + [u''(c_2)h(c_2) + u'(c_2)h'(c_2)](b-a) \tag{28}$$

where $h(\xi) \equiv f'(u) - \xi$. Since $u'' > 0$ and $u' > 0$ for $\xi \in (a, b)$, it can be deduced from (5a) and (6) that

$$h(\xi) > 0, \quad u'(\xi)h(\xi) \geq qK\Big(\frac{\xi}{\eta}\Big)^K \tag{29}$$

From (27), one has

$$\begin{aligned} qK\Big(\frac{b}{\eta}\Big)^K &\leq qK\Big(\frac{a}{\eta}\Big)^K + qK^2\frac{1}{\eta}(b-a) \\ &\leq qK\Big(\frac{a}{\eta}\Big)^K + qK^2\frac{1}{\beta}(b-a) \end{aligned} \tag{30}$$

Combining (28), (29) with (30), one has

$$\begin{aligned} u'(b)h(b) &> u'(a)h(a) + u'(c_2)h'(c_2)(b-a) \\ &> qK\Big(\frac{a}{\eta}\Big)^K + u'(c_2)[f''(u(c_2))u'(c_2) - 1](b-a) \\ &\geq qK\Big(\frac{a}{\eta}\Big)^K + A(\delta A - 1)(b-a) \\ &\geq qK\Big(\frac{a}{\eta}\Big)^K + \frac{1}{\beta}\Big(\delta\frac{qK^2+1}{\delta} - 1\Big)(b-a) \\ &\geq qK\Big(\frac{b}{\eta}\Big)^K \end{aligned} \tag{31}$$

From (5a) and (6), (31) implies that $u''(b) > 0$ and this shows that $u''(\xi) > 0$ for all $\xi \in (a, \eta_1)$. Then $u'(\eta_1) \geq u(a) > A$ but it contradicts that $u'(\eta_1) = 0$. This contradiction indicates that if $u''(\xi) > 0$ then $u'(\xi) \leq A$ for $\xi \in (0, \eta_1)$. Since $u' > 0$ on $(0, \eta_1)$ and $u''(0) > 0$, it can be obtained that $0 < u' \leq A$ on $(0, \eta_1)$.

3. The Properties of the Limit Solution

Since $\{\eta\}, \{\eta_1\}$ and $\{u_\varepsilon\}$ are uniformly bounded with ε (see Lemma 3.1 of [7] and Lemma 2 of the present paper) and $\mathrm{Var}(u_\varepsilon) \leq C$, there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$ for convenience) such that

$$\lim_{\varepsilon\to 0+} \eta = \eta^* \tag{32}$$

$$\lim_{\varepsilon\to 0+} \eta_1 = \eta_1^* \tag{33}$$

and

$$\lim_{\varepsilon\to 0+} u_\varepsilon(\xi) = u(\xi) \tag{34}$$

here η^* and η_1^* are real numbers and the limit function u is a real function defined on $\boldsymbol{R}$. It should be borne in mind that η and η_1 are dependent on ε. By Lemma 3, $u(\xi) \equiv u_L$ for $\xi \in (-\infty, 0]$. It is also easily to see that u monotonically increases on $(-\infty, \eta_1^*)$ and decreases on (η_1^*, ∞). Lemma 4 indicates that $\eta^* > 0$.

The characteristic function $\chi_{(a,b)}(\xi)$ is defined as follows,

$$\chi_{(a,b)}(\xi) = \begin{cases} 1, & \xi \in (a,b) \\ 0. & \xi \notin (a,b) \end{cases} \tag{35}$$

It is clear that

$$\chi_{(0,\eta)}(\xi) \to \chi_{(0,\eta^*)}(\xi), \quad a.e. \quad \text{as } \varepsilon \to 0+ \tag{36}$$

Combining (5a) with (6) and integrating (5a) on $(0,\xi)$ one has

$$\begin{aligned} &\varepsilon(u_\varepsilon'(\xi) - u_\xi'(0)) \\ &= f(u_\varepsilon) - f(u_\varepsilon(0)) - \xi u_\varepsilon + \int_0^\xi u_\varepsilon(s)ds - \int_0^\xi Kq\Big(\frac{s}{\eta}\Big)^K \chi_{(0,\eta)}(s)ds \end{aligned} \tag{37}$$

Lemma 8 *The limit solution $u(\xi)$ defined by (34) satisfies*

$$\begin{cases} (f'(u) - \xi)\, u' = qK\Big(\dfrac{\xi}{\eta^*}\Big)^K, \quad \xi \in (0, \eta_1^*) \\ u(0) = u_L \end{cases} \tag{38}$$

Proof For $\xi \in (0, \eta_1^*)$, using Lemmas 3 and 7 and applying the Lebesgue's theorem to (37) one can obtain that

$$f(u) - f(u_L) - \xi u + \int_0^\xi u(s)ds - \int_0^\xi Kq\Big(\frac{s}{\eta^*}\Big)^K \chi_{(0,\eta^*)}(s)ds = 0 \tag{39}$$

Hence,

$$\begin{aligned} g(\xi) &\equiv f(u) - \xi u \\ &= f(u_L) - \int_0^\xi u(s)ds + \int_0^\xi Kq\Big(\frac{s}{\eta^*}\Big)^K ds, \ \xi \in (0, \eta_1^*) \end{aligned} \tag{40}$$

It can be obtained from Lemma 7 that $u \in C((0, \eta_1^*))$ and thus the right hand side of (40) is continuously differentiable on $(0, \eta_1^*)$. Then $g \in C^1(0, \eta_1^*)$ and satisfies

$$g'(\xi) = -u(\xi) + Kq\Big(\frac{\xi}{\eta^*}\Big)^K, \quad \xi \in (0, \eta_1^*) \tag{41}$$

For any given $\xi, a \in (0, \eta_1^*), \xi \neq a$, one has

$$\frac{u(\xi) - u(a)}{\xi - a} = \frac{\dfrac{g(\xi) - g(a)}{\xi - a} + u(\xi)}{\dfrac{f(u(\xi)) - f(u(a))}{u(\xi) - u(a)} - a} \tag{42}$$

It can be found that if $f'(u) - \xi \neq 0$ for $\xi \in (0, \eta_1^*)$, then in (42) let $a \to \xi$ one can obtain

$$u'(\xi) = \frac{g'(\xi) + u(\xi)}{f'(u) - \xi} \tag{43}$$

Let $h(\xi) \equiv f'(u) - \xi$. Then $h(0) = f'(u_L) > 0$. Assume that there exists an $c \in (0, \eta_1^*)$ satisfying $h(\xi) > 0, \xi \in (0, c)$ and $h(c) = 0$. By (41) and (43) one has $u'(c-) = +\infty$. On the other hand, it is easily to see that $h'(c-) \leq 0$, namely, $f''(u(c-))u'(c-) - 1 \leq 0$. Thus $u'(c-) \leq \dfrac{1}{f''(u(c-))} \leq \dfrac{1}{\delta}$ which gives an contradiction. This indicates that h is positive on $(0, \eta_1^*)$ and (41) and (43) lead to the conclusion of this Lemma.

The equations are now reduced to a system of second-order nonlinear ODEs and a singular perturbation is involved. The analytical and numerical approaches on singular perturbation problems are of theoretical and practical interests for many years (see, e.g. [8–10]). Mathematically, we may consider a system of ODEs where one or more of the highest derivatives appearing is multiplied by a small parameters ε. If we let ε approach 0, the order of the system reduces, and one can not in general expect all boundary conditions imposed on the original ODE system to be satisfiable. The perturbation is then "singular" when ε is not zero but is small, the solution is expected, under certain conditions, to exhibit narrow regions where of fast variation (so-called boundary or interior layers) which connect wider regions where it varies more slowly. Many interesting phenomena could be found by numerical experiments ([11–12]) for the limit process of the present problem and this gives the motivation of the theoretical work.

Up to now it has been clear for the structure of the limit solution u on $(-\infty, \eta_1^*)$. We then turn to discuss the properties of u on $I \equiv (\eta_1^*, +\infty)$.

Let $h(\xi) \equiv f'(u) - \xi$. Since u decreases on I and $f'' > 0, h(\xi)$ decreases strictly on I and has at most one zero-point. It may be assumed that there exists an $\theta \in I$ such that

$$h(\xi) > 0, \quad \xi \in I_1 \equiv (\eta_1^*, \theta) \tag{44}$$

$$h(\xi) < 0, \quad \xi \in I_2 \equiv (\theta, +\infty) \tag{45}$$

Lemma 9 *If $\eta_1^* < \eta^*$, then $\eta_1^* = \theta$.*

Proof Assume that $\eta_1^* < \theta$, one will have a contradiction.

(i) It will be proved that $\lim\limits_{\varepsilon \to 0+} \varepsilon u'_\varepsilon = 0$ on I_1. If not, there would exist an $a \in I_1$, a constant $\beta > 0$ and a subsequence $\{\varepsilon_j\}$ such that $\varepsilon_j \to \infty$ as $j \to \infty$ and

$$|\varepsilon_j u_{\varepsilon_j}(a)| \geq \beta \tag{46}$$

It can be observed from (5a), (6) and (44) that if ε is sufficiently small, then $u''_\varepsilon \leq 0$ on $\left(a, \dfrac{a+\theta}{2}\right)$. Since $u'_\varepsilon \leq 0$ one has

$$|u'_\varepsilon(\xi)| \geq |u'_\varepsilon(a)|, \quad \xi \in \left(a, \frac{a+\theta}{2}\right) \tag{47}$$

By Lemma 2 there exists a constant C_1 such that $\mathrm{Var}(u_\varepsilon) \le C_1$ and then

$$C_1 \ge \int_a^{\frac{a+\theta}{2}} |u'_{\varepsilon_j}(\xi)| d\xi \ge |u'(a)| \frac{\theta - a}{2} \ge \frac{\beta}{2\varepsilon_j}(\theta - a) \tag{48}$$

Let $j \to \infty$ then (48) leads to a contradiction.

(ii) Eq. (1) is a system of conservative equation, it can be obtained by use of the theory on the conservative system that any discontinuous point ξ of u satisfies the so-called Rankine-Hugoniot condition, namely,

$$\xi = \frac{[f]}{[u]} = \frac{f(u^+) - f(u^-)}{u^+ - u^-} \tag{49}$$

where $u^\pm = u(\xi \pm 0)$. Since $f'' > 0$ and u decreases on I_1, then for any discontinuous point $\bar{\xi} \in I_1$ one has

$$\bar{\xi} > \min\{f'(u^+), f'(u^-)\} = f'(u^+) \tag{50}$$

On the other hand, (44) gives

$$\bar{\xi} \le \min\{f'(u^+), f'(u^-)\} = f'(u^+) \tag{51}$$

which contradicts (50) and this indicates that $u \in I_1$.

(iii) Since $h(\xi) > 0$ on I_1 and $\varepsilon u'_\varepsilon$ approaches 0 as $\varepsilon \to 0$, it can be obtained by using similar technique employed in the proof of Lemma 8 that $u \in C^1(I_1)$ and satisfies

$$h(\xi)u' = Kq\Big(\frac{\xi}{\eta^*}\Big)^K, \quad \xi \in (\eta_1^*, \min(\theta, \eta^*)) \tag{52}$$

Since η_1^* is less than either θ or $\eta^*, \eta_1^* < \min(\theta, \eta^*)$. On $I_1, u' \le 0$ but h and the right-side-hand of (52) are positive, thus give a contradiction.

Lemma 10 *If $\eta_1^* = \theta$, then $\eta_1^* = \eta^*$.*

Proof If $\eta_1^* = \theta$, then $h(\xi) < 0$ on $I \equiv (\eta_1^*, +\infty)$. By applying Lebesgue's theorem, one can show that $\lim_{\varepsilon \to 0} \varepsilon u'(\xi) = 0$ on I. By use of similar technique employed in the proof of Lemma 9, it can be obtained that $u \in C(I)$. Finally, it can be deduced that $u \in C^1((\eta_1^*, \eta^*) \cup (\eta^*, +\infty))$ and satisfies

$$h(\xi)u' = Kq\Big(\frac{\xi}{\eta^*}\Big)^K \chi_{(0,\eta^*)}(\xi) \tag{53}$$

Then by (45), one has $u' \equiv 0$ on $(\eta^*, +\infty)$. Since $u(+\infty) = u_R$ it can be obtained that $u(\xi) \equiv u_R$ for $\xi \in (\eta^*, +\infty)$. Then $u(\eta^*-) \ge 0$ and $u(\eta^*+)u_R < 0$, and thus η^* is a discontinuous point. This contradicts that $u \in C(I)$. Thus $\eta^* = \eta_1^*$.

From Lemmas 9 and 10, it has been proved that $\eta^* = \eta_1^*$.

Lemma 11 *$u(\xi) \equiv u_R$ for $\xi \in (\eta^*, +\infty)$.*

Proof (i) It is possible that $\theta \ne \eta_1^*$, here θ is defined by (43) and (44). By similar procedure employed in the proof of Lemma 8, it can be deduced that $h(\xi)u' = 0$ for

$\xi \in I_1 \cup I_2$. Then $u' \equiv 0$ on I_1 and I_2. Hence $u \equiv c$ on I_1 and $u \equiv u_R$ on I_2, here c is a constant.

(ii) If θ is a discontinuous point of u, then $c \neq u_R$ and by (49) one has

$$\eta_1^* = \frac{f(u(\eta_1^*+)) - f(u(\eta_1^*-))}{u(\eta_1^*+) - u(\eta_1^*-)} > f'(u(\eta_1^*+)) = f'(c) \tag{54}$$

$$\theta = \frac{f(u_R) - f(c)}{u_R - c} < f'(c) \tag{55}$$

here one uses the fact that u decreases on $(\eta_1^*, +\infty)$. Then $\theta < \eta_1^*$ which gives a contradiction. Hence $c = u_R$ and $u(\xi) \equiv u_R$ for $\xi \in (\eta_1^*, +\infty)$.

4. The Main Theorem

Combined Lemmas 3, 8 with 11, the following main theorem can be obtained.

Theorem 1 *The similarity solutions, $(u_\varepsilon, Z_\varepsilon)$, of the Riemann problem (1) converges to a piecewise-smooth solution of the equations*

$$\frac{\partial}{\partial t}(u + qZ) + \frac{\partial}{\partial x}(f(u)) = 0 \tag{56a}$$

$$\frac{\partial Z}{\partial t} = -\frac{K}{t}\phi(u) Z \tag{56b}$$

$$(u(x,0), Z(x,0)) = \begin{cases} (u_L, & 0) \\ (u_R, & 1) \end{cases} \tag{5.6c}$$

as ε tends to 0. The limit solution $(u(x,t), Z(x,t)) = (u(\xi), Z(\xi))$, with $\xi = \frac{x}{t}$, consists of three pieces of smooth functions, i.e

$$(u(\xi), Z(\xi)) = \begin{cases} (u_L, 0), & \xi < 0 \\ \left(U(\xi), \left(\frac{\xi}{\eta^*}\right)^K\right), & 0 < \xi < \eta^* \\ (u_R, 1), & \xi > \eta^* \end{cases} \tag{57}$$

here η^* is a fixed constant and $U(\xi)$ is a monotonically increasing function defined by

$$\begin{cases} \frac{dU}{d\xi}(f'(U) - \xi) = Kq\left(\frac{\xi}{\eta^*}\right)^K, & \xi \in (0, \eta^*) \\ U(0) = u_L \end{cases} \tag{58}$$

References

[1] Williams F.A., Combustion Theory, *Addison-Wesley, Reading, Mass*, 1965.

[2] Courant R. & Friedrichs K.O., Supersonic Flow and Shock Waves, Interscience Publishes Inc., New York, 1948.

[3] Chorin A., Random choice methods with applications to reacting gas flow, *J. Comput. Phys.* **25**(1977), 253–272.

[4] Teng Z.H., Chorin A. & Liu T.P., Riemann problem for reacting gas, with application to transition, *SIAM J. Appl. Math.* **42**(1982), 964–981.

[5] Majda A., A qualitative model for dynamic combustion, *SIAM J. Appl. Math.* **41**(1981), 70–93.

[6] Ying L.A. & Teng Z.H., A hyperbolic model of combustion, Lecture Notes in Num. *Appl. Anal.* **5**(1982), 409–434.

[7] Fu E.Z, Tang T. and Teng Z.H., Riemann problem for a hyperbolic model of combustion (i) the existence and basic structure of the self-similar solutions, in Current Progress in Hyperbolic Systems: Riemann Problems and Computations, W.Lindquist ed. Contemperary Mathematics 100, AMS, 1988.

[8] O'Malley R.E., Introduction to Singular Perturbations, Academic Press, New York, 1974.

[9] Smith D.R., Singular Perturbation Theory, Cambridge University Press, Cambridge, 1985.

[10] Ascher U.M., Mattheij R.M.M and Russell R.D., Numerical Solution of Boundary Value Problems for Ordinary Differential Equations, Prentice-Hall, Englewood Cliffs, New Jersy, 1988.

[11] Li T., MSc Thesis, Department of Mathematics, Peking University, 1986.

[12] Fu E.Z., MSc Thesis, Department of Mathematics, Peking University, 1987.

[13] Teng Z.H. & Ying L.A., The Riemann problem for a hyperbolic combustion model system, *Chinese Ann. Math. Ser.* **A6**(1985), 13–22 (in Chinese).