Regularity and Global Structure of Solutions to Hamilton-Jacobi Equations I. Convex Hamiltonian

Yinchuan Zhao* Tao Tang† Jinghua Wang‡

September 18, 2007

Abstract

This paper is concerned with the Hamilton-Jacobi (HJ) equations of multidimensional space variables with convex Hamiltonian. Using Hopf's formula (I), we will study the differentiability of the HJ solutions. For any given point, we give a sufficient and necessary condition under which the solutions are C^k smooth in some neighborhood of the point. We also study the characteristics of the HJ equations. It is shown that there are only two kinds of characteristics, one never touches the point of singularity, and the other touches the point of singularity in a finite time. The sufficient and necessary condition under which the characteristic never touches the point of singularity is given. Based on these results, we study the global structure of the set of singularity points for the HJ solutions. It is shown that there exists a one-to-one correspondence between the path connected components of the set of singularity points and the path connected components of a set on which the initial function does not attain its minimum. A path connected component of the set of singularity points never terminates at a finite time. Our results are independent of the particular forms of the equations as long as the Hamiltonian is convex.

1 Introduction

Consider the Cauchy problem for the following Hamilton-Jacobi (HJ) equation

*LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China. Email: zyc747250@hotmail.com.

†Department of Mathematics, Hong Kong Baptist University, Kowloon Tong Hong Kong & Institute of Computational Mathematics, Chinese Academy of Sciences, Beijing, China. Email: ttang@math.hkbu.edu.hk.

‡Institute of Systems Sciences, Academy of Mathematics and System Sciences, The Chinese Academy of Sciences, Beijing 100080, P. R. China. Email: jwang@amss.ac.cn.

$$
\begin{cases} u_t + H(Du) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{1.1}
$$

where the Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$ is C^k smooth with $k \geq 2$, and uniformly convex (with constant $\alpha > 0$):

$$
\sum_{i,j=1}^{n} H_{p_i p_j}(p)\xi_i \xi_j \geq \alpha \left|\xi\right|^2, \quad \forall\; p,\; \xi \in \mathbb{R}^n; \tag{1.2}
$$

$g : \mathbb{R}^n \to \mathbb{R}$ is C^k smooth with $k \geq 2$ and satisfies

$$
\sup_{y \in \mathbb{R}^n} |Dg(y)| < \infty. \tag{1.3}
$$

It is known that the solutions to (1.1) are given by the Hopf's formula (I)[1], see, e.g., [2, 4, 5, 6],

$$
u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}, \tag{1.4}
$$

where L is the Legendre transform of H, i.e.,

$$
L(p) = \sup_{q \in \mathbb{R}^n} \{p \cdot q - H(q)\} \quad (p \in \mathbb{R}^n). \tag{1.5}
$$

It can be verified that

$$
L(DH(p)) = p \cdot DH(p) - H(p). \tag{1.6}
$$

Note that the mapping $DH : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and onto (since H is uniform convex). Another equivalent formula for the solutions is given by Kruzkov [7]:

$$
u(x,t) = \min_{p \in \mathbb{R}^n} F(x,t,p), \tag{1.7}
$$

where

$$
F(x,t,p) = tL(DH(p)) + g(x - tDH(p)), \tag{1.8}
$$

and the initial function g is assumed to be lower semi-continuous and satisfies

$$
|g(x)| \leq M(|x| + 1). \tag{1.9}
$$

The regularity of the solutions to the HJ equations has attracted attention of many authors, see, e.g., [1, 3, 5, 6, 7, 13, 14, 15] and references there in. It is known [2] that the solutions $u(x,t)$ defined by the Hopf's formula (I) are viscosity solutions of (1.1), which are differentiable a.e. in $\mathbb{R}^n \times (0, \infty)$. However, in general $u(x,t)$ is not of class C^1 in the sense that their gradient may have discontinuities.

[1] It is called the "Lax formula" in [10].

Let U be the set of all points (x,t) such that $F(x,t,\bullet)$ has a unique non-degenerate minimizer. Then U is open on which the solutions are C^k smooth. We study the properties of characteristics, which are also interesting in their own sake and have other applications. Given $y_0 \in \mathbb{R}^n$, let

$$C = \{(x,t)|\ x = y_0 + DH(Dg(y_0))t,\ t > 0\}. \tag{1.10}$$

A characteristic segment

$$\bar{C} = C \bigcap \{0 < t \le T_1\}$$

is said to be valid if $Dg(y_0)$ is a minimizer for $F(x,t,\bullet)$ for each $(x,t) \in \bar{C}$. In the case that $\max T_1 < \infty$, we prove there exists a point $(x_s(y_0), t_s(y_0))$, where

$$t_s(y_0) = \max T_1,\ x_s(y_0) = y_0 + t_s(y_0)DH(Dg(y_0))), \tag{1.11}$$

such that $Dg(y_0)$ is a unique degenerate minimizer or one of the minimizers for $F(x_s(y_0), t_s(y_0), \bullet)$, while $Dg(y_0)$ will no longer be a minimizer for $F(x,t,\bullet)$ for $(x,t) \in C, t > t_s(y_0)$ and $Dg(y_0)$ is a unique non-degenerate minimizer for $F(x,t,\bullet)$ for $(x,t) \in C,\ t < t_s(y_0)$. We define $(x_s(y_0), t_s(y_0))$ as a singularity point. Let S be the set of singularity points.

We will introduce a singularity mapping based on the properties of the characteristics. A singularity mapping is defined as

$$\mathscr{S}(y) = (x_s(y), t_s(y)) \tag{1.12}$$

from $\mathbb{R}^n$ to $\mathbb{R}^n \times (0,\infty)$. It will be shown that $t_s(y_0)$ is finite if and only if

$$g(y_0) > \inf_{y \in \mathbb{R}^n} g(y). \tag{1.13}$$

Thus the domain of $\mathscr{S}$ is

$$\widetilde{\mathbb{R}}^n = \{y \in \mathbb{R}^n|\ g(y) > \inf_{x \in \mathbb{R}^n} g(x)\}. \tag{1.14}$$

Furthermore, it will be proved that the singularity mapping is continuous. Thus the singularity mapping is continuous from $\widetilde{\mathbb{R}}^n \subset \mathbb{R}^n$ to $\mathbb{R}^n \times (0,\infty)$ and

$$S = \{(x_s(y), t_s(y))|\ y \in \widetilde{\mathbb{R}}^n\}.$$

The second part of the paper is concerned with the differentiability of the HJ solutions. We prove that $u(x,t)$ is non-differentiable at (x_0,t_0) if $F(x_0,t_0,\bullet)$ has more than one minimizer, and (x_0,t_0) is a cluster point of non-differentiable points of the solution $u(x,t)$ if $F(x_0,t_0,\bullet)$ has a unique degenerate minimizer. Thus an equivalent definition of the singularity point can be stated as follows: We call a point a singularity point if it is a non-differentiable point of the solution $u(x,t)$ or a cluster point of non-differentiable points of $u(x,t)$. We will show that the solution of (1.1) is smooth in some neighborhood of a point (x_0,t_0) if and only if there exists a unique non-degenerate minimizer for $F(x_0,t_0,\bullet)$.

We are interested in the global structures of S. It will be shown that the set of singularity points consists of several path connected components. We prove that there exists a one-to-one correspondence between the path connected components of $\widetilde{\mathbb{R}}^n$ and the path connected component S_i of the set of singularity points and each path connected component of the set of singularity points never vanishes as t increases. Moreover, our results are independent of the particular forms of the equations as long as the Hamiltonian is convex. It is pointed out that this work is a kind of extension of our earlier work on the solution structures of the entropy solutions to scalar conservation laws [8, 12].

2 Hopf's formula (I) and characteristics

In this section we will give several lemmas and propositions on characteristics, which play important role in studying the regularity and global structure of the HJ solutions. The fact that solutions $u(x,t)$ of (1.1) are given by Hopf's formula (I) shows that the minimizers of $F(x,t,\bullet)$ belong to a subset of the set

$$\{Dg(y)|y \in \mathbb{R}^n\},$$

where y is a point from which the characteristic C:

$$C: \quad \{(x,t)\,|x = y + tDH(Dg(y)), t > 0\} \tag{2.1}$$

emanates and passes through $(x,,t)$. We will show that $Dg(y)$ is a unique non-degenerate minimizer of $F(x,t,\bullet)$ for $(x,t) \in C$, $t > 0$ if and only if

$$g(y) = \inf_{y \in \mathbb{R}^n} g(y). \tag{2.2}$$

In the case that

$$g(y) > \inf_{y \in \mathbb{R}^n} g(y),$$

there exists $(x_s(y), t_s(y))$ belonging to C such that $Dg(y)$ is a minimizer of $F(x,t,\bullet)$ for $(x,t) \in C$, $t \leq t_s(y)$ and $Dg(y)$ is not minimizer of $F(x,t,\bullet)$ for $(x,t) \in C$, $t > t_s(y)$.

We record here the following relations that will be needed later

$$D_pF(x,t,p) = tD^2H(p) \cdot (p - Dg(x - tDH(p))). \tag{2.3}$$

If $D_pF(x,t,p) = 0$, then

$$D_xF(x,t,p) = p, \quad D_tF(x,t,p) = -H(p), \tag{2.4}$$

$$D_p^2F(x,t,p) = tD^2H(p)[I_n + t \cdot D^2g(x - tDH(p) \cdot D^2H(p)]. \tag{2.5}$$

Definition 2.1 Let p_0 be a minimizer for $F(x_0,t_0,\bullet)$. Then p_0 is called non-degenerate (degenerate) if $|D_p^2F(x_0,t_0,p_0)| \neq 0$ $(= 0)$.

Lemma 2.1 *Let*

$$U = \{(x,t) : \exists \text{ unique non-degenerate minimizer for } F(x,t,\bullet)\}. \tag{2.6}$$

Then U is an open subset of $\mathbb{R}^n \times (0,\infty)$, and $u(x,t)$ is C^k smooth on U.

Proof: The proof is a straightforward generalization of Lemma 1.1 in [11] for scalar conservation laws in one space dimension. ■

Lemma 2.2 *Suppose $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$, $p_0 = p(x_0,t_0)$ is a minimizer of $F(x_0,t_0,\bullet)$, l is an open straight line segment connecting (x_0,t_0) and $(y_0,0)$, where*

$$y_0 = y(x_0,t_0) = x_0 - t_0 DH(p_0).$$

If $(x_1,t_1) \in l$, then there is a unique point

$$p(x_1,t_1) = p(x_0,t_0) = p_0$$

which minimizes $F(x_1,t_1,\bullet)$. Furthermore,

$$u(x_0,t_0) = u(x_1,t_1) + (t_0 - t_1) L\left(\frac{x_0 - y_0}{t_0}\right). \tag{2.7}$$

Proof: Suppose p_1 is a minimizer for $F(x_1,t_1,\bullet)$. Set $y_1 = x_1 - t_1 DH(p_1)$. It follows from the Hopf's formula (I) that

$$\begin{aligned} u(x_1,t_1) &= t_1 L(\frac{x_1 - y_1}{t_1}) + g(y_1) \\ &\le t_1 L(\frac{x_1 - y_0}{t_1}) + g(y_0). \end{aligned} \tag{2.8}$$

Using the fact that L is strictly convex, we have from (2.8) that

$$\begin{aligned} u(x_0,t_0) &= t_0 L(\frac{x_0 - y_0}{t_0}) + g(y_0) \\ &= (t_0 - t_1) L(\frac{x_0 - y_0}{t_0}) + t_1 L(\frac{x_0 - y_0}{t_0}) + g(y_0) \\ &= (t_0 - t_1) L(\frac{x_0 - x_1}{t_0 - t_1}) + t_1 L(\frac{x_1 - y_0}{t_1}) + g(y_0) \\ &\ge (t_0 - t_1) L(\frac{x_0 - x_1}{t_0 - t_1}) + t_1 L(\frac{x_1 - y_1}{t_1}) + g(y_1) \\ &= t_0 \left[\frac{t_0 - t_1}{t_0} L(\frac{x_0 - x_1}{t_0 - t_1}) + \frac{t_1}{t_0} L(\frac{x_1 - y_1}{t_1}) \right] + g(y_1) \\ &\ge t_0 L(\frac{x_0 - y_1}{t_0}) + g(y_1). \end{aligned} \tag{2.9}$$

Since

$$u(x_0,t_0) = \min_{y \in \mathbb{R}^n} \left\{ t_0 L(\frac{x_0 - y}{t_0}) + g(y) \right\},$$

we have

$$u(x_0,t_0)=t_0L(\frac{x_0-y_1}{t_0})+g(y_1).$$

Consequently,

$$\frac{t_0-t_1}{t_0}L(\frac{x_0-x_1}{t_0-t_1})+\frac{t_1}{t_0}L(\frac{x_1-y_1}{t_1})=L(\frac{x_0-y_1}{t_0}), \tag{2.10}$$

where we have used (2.9). Note that

$$\begin{aligned} p_0=p_1 \Longleftrightarrow DH(p_0)=DH(p_1) \Longleftrightarrow \frac{x_0-y_0}{t_0}=\frac{x_1-y_1}{t_1} \\ \Longleftrightarrow \frac{t_0-t_1}{t_0}L(\frac{x_0-x_1}{t_0-t_1})+\frac{t_1}{t_0}L(\frac{x_1-y_1}{t_1})=L(\frac{x_0-y_1}{t_0}). \end{aligned} \tag{2.11}$$

This, together with (2.10), implies that $p_0=p_1$. ■

We now discuss the relationship between the critical point of $F(x,t,\bullet)$ and the characteristic. Suppose p_0 is a critical point of $F(x_0,t_0,\bullet)$, i.e.,

$$D_pF(x_0,t_0,p_0)=0.$$

Then it follows from (2.3) that $p_0=Dg(x_0-t_0DH(p_0))$. Let

$$y_0=x_0-t_0DH(p_0). \tag{2.12}$$

The characteristic

$$x=y_0+tDH(Dg(y_0)) \tag{2.13}$$

will pass through (x_0,t_0) with the speed $DH(Dg(y_0))=DH(p_0)$.

On the other hand, consider a characteristic

$$C: \quad x=y+tDH(Dg(y)), \quad t>0. \tag{2.14}$$

Then $D_pF(x,t,Dg(y))=0$ for $(x,t)\in C$ due to the fact that

$$Dg(y)=Dg(x-tDH(Dg(y)))$$

and (2.3). This implies that $Dg(y)$ is a critical point of $F(x,t,\bullet)$.

It is natural to ask if $Dg(y)$ is a minimizer of $F(x,t,\bullet)$ for $(x,t)\in C$. The following lemma gives an answer.

Lemma 2.3 *Let $y_0\in\mathbb{R}^n$ and assume the corresponding characteristic C is given by* (1.10)*. If $g\in C^k$ is bounded, then precisely one of the following statements must hold:*

- `either` *$Dg(y_0)$ is the unique non-degenerate minimizer of $F(x,t,\bullet)$ for each $(x,t)\in C$;*

- `or` *there exits a point* $(x_s(y_0), t_s(y_0)) \in C$ *such that* $Dg(y_0)$ *is either the unique degenerate minimizer of* $F(x_s(y_0), t_s(y_0), \bullet)$ *or one of more than one minimizers for* $F(x_s(y_0), t_s(y_0), \bullet)$*. Furthermore* $Dg(y_0)$ *is a unique non-degenerate minimizer for* $F(x, t, \bullet)$ *for each*

$$(x, t) \in C^- := C \cap \{(x, t) : t_s(y_0) > t > 0\},$$

while for

$$(x, t) \in C^+ := C \cap \{(x, t) : t > t_s(y_0)\},$$

$Dg(y_0)$ *is no longer the minimizer of* $F(x, t, \bullet)$*.*

Proof: We first show that $Dg(y_0)$ is no longer a minimizer for $F(x, t, \bullet)$ for $(x, t) \in C^+$ if there exist more than one minimizer for $F(x_s(y_0), t_s(y_0), \bullet)$. Otherwise, there exists a point $(\tilde{x}, \tilde{t}) \in C^+$ such that $Dg(y_0)$ is a minimizer for $F(\tilde{x}, \tilde{t}, \bullet)$. Consequently, $Dg(y_0)$ is the unique minimizer for $F(x_s(y_0), t_s(y_0), \bullet)$ according to Lemma 2.2, which is a contradiction since there are more than one minimizers for $F(x_s(y_0), t_s(y_0), \bullet)$.

If $Dg(y_0)$ is a unique degenerate minimizer for $F(x_s(y_0), t_s(y_0), \bullet)$, i.e.,

$$|D_p^2 F(x_s(y_0), t_s(y_0), Dg(y_0))| = 0. \tag{2.15}$$

Therefore, there exists a non-zero vector $\xi \in \mathbb{R}^n$ such that

$$\xi^T D_p^2 F(x_s(y_0), t_s(y_0), Dg(y_0))\xi = 0. \tag{2.16}$$

It follows from (2.4) and (2.5) that

$$t_s[\xi^T D^2 H(Dg(y_0))\xi + t_s \xi^T D^2 H(Dg(y_0)) D^2 g(y_0) D^2 H(Dg(y_0))\xi] = 0. \tag{2.17}$$

Consider the equation

$$\bar{g}(t) = \xi^T D^2 H(Dg(y_0))\xi + t\xi^T D^2 H(Dg(y_0)) D^2 g(y_0) D^2 H(Dg(y_0))\xi. \tag{2.18}$$

Thus

$$\xi D_p^2 F(x, t, Dg(y_0))\xi^T = t\bar{g}(t). \tag{2.19}$$

According to Lemma 2.2, $Dg(y_0)$ is a unique minimizer for $F(x, t, \bullet)$ for each $(x, t) \in C, t \leq t_s(y_0)$ since $Dg(y_0)$ is a minimizer for $F(x_s(y_0), t_s(y_0), \bullet)$. Then

$$\xi D_p^2 F(x, t, Dg(y_0))\xi^T \geq 0, \quad \forall\ (x, t) \in C,\ t \leq t_s(y_0). \tag{2.20}$$

On the other hand, (2.18) is a linear equation of t and has a unique root, $t = t_s(y_0)$. Thus

$$\bar{g}(t) < 0 \ \text{ for } t > t_s(y_0). \tag{2.21}$$

It follows from (2.19) and (2.21) that

$$\xi^T D_p^2 F(x, t, Dg(y_0))\xi < 0 \ \text{ for } (x, t) \in C, t > t_s(y_0), \tag{2.22}$$

which means that the matrix $D_p^2 F(x, t, Dg(y_0))$ is negative definite or non-definite. Consequently, $Dg(y_0)$ can not be a minimizer for $F(x, t, \bullet)$. This completes the proof of this lemma. ■

The analogue of Lemma 2.3 for convex scalar conservation laws was obtained by Li and Wang in [8].

Definition 2.2 *A characteristic segment*

$$\bar{C} = C \bigcap \{0 < t \leq T_1\}$$

is said to be valid if there exists $0 < T_1 \leq \infty$ *such that for each* $(x,t) \in \bar{C}$*,* $Dg(y_0)$ *is a minimizer for* $F(x,t,\bullet)$*,* C *is given by* (1.10)*.*

For each point $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$, there exists at least one valid characteristic segment $\bar{C} = C \bigcap \{0 < t \leq t_0\}$ passing through it. Lemma 2.3 can be used to judge if $F(x_0,t_0,\bullet)$ has a unique non-degenerate (or degenerate) minimizer or several minimizers by considering the relationship between C^- and $\bar{C}$. That is to say: if $C^- \subset\subset \bar{C}$, then $F(x_0,t_0,\bullet)$ has a unique degenerate minimizer or more than one minimizer; if $\bar{C} \subset\subset C^-$, then $F(x_0,t_0,\bullet)$ has a unique non-degenerate minimizer.

From the above lemma, for $y \in \mathbb{R}^n$ satisfying $t_s(y) < \infty$, consider C, a characteristic emanating from y of the form (2.1). We see that

$$\begin{cases} t_s(y) & = \max\{t \mid u(x,t) = F(x,t,Dg(y)),\ (x,t) \in C,\ t > 0\}, \\ x_s(y) & = y + t_s(y) DH(Dg(y)). \end{cases} \tag{2.23}$$

We define the point $(x_s(y), t_s(y))$ as singularity point of solution $u(x,t)$ and let S be the set of singularity points. In order to study the structure of the set of singularity points we introduce a singularity mapping $\mathscr{S}$ from some subset of $\mathbb{R}^n$ to $\mathbb{R}^n \times (0,\infty)$,

$$\mathscr{S}(y) = (x_s(y), t_s(y)). \tag{2.24}$$

In other words, $(x_s(y), t_s(y))$ is the point such that $F(x_s(y), t_s(y), \bullet)$ has a unique degenerate minimizer or more than one minimizer.

Lemma 2.4 $\mathscr{S}$ *defined by* (2.24) *is a continuous map.*

Proof: We need to prove that $t_s(y_n) \to t_s(y_0)$ if $y_n \to y_0$, where $y_n \in \mathbb{R}^n$. This will be done in two steps.

`Step 1.` We claim

$$\limsup t_s(y_n) \leq t_s(y_0). \tag{2.25}$$

Otherwise there exists a subsequence $\{t_s(y_{n_k})\}$ of $\{t_s(y_n)\}$ such that $t_s(y_{n_k}) \to T_1 > t_s(y_0)$. Then according to the definition of $t_s(y_{n_k})$, for k big enough we have

$$u\Big(DH(Dg(y_{n_k}))T_1 + y_{n_k}, T_1\Big) = F\Big(DH(Dg(y_{n_k}))T_1 + y_{n_k}, T_1, Dg(y_{n_k})\Big). \tag{2.26}$$

Using the continuity property of $u(x,t)$ and $F(x,t,p)$, we obtain by letting $k \to \infty$ in (2.26) that

$$u\Big(DH(Dg(y_0))T_1 + y_0, T_1\Big) = F\Big(DH(Dg(y_0)T_1 + y_0, T_1, Dg(y_0)\Big), \tag{2.27}$$

which contradicts the definition of $t_s(y_0)$ and Lemma 2.3.

`Step 2.` We claim

$$\liminf t_s(y_n) \geq t_s(y_0). \tag{2.28}$$

Otherwise there exists some subsequence $\{t_s(y_{n_k})\}$ of $\{t_s(y_n)\}$ such that $t_s(y_{n_k}) \to T < t_s(y_0)$. Then there exists a neighborhood $U_{(x_1,T)}$ of (x_1, T), where

$$x_1 = y_0 + TDH(Dg(y_0)).$$

For each $(x,t) \in U_{(x_1,T)}$, there exists a unique non-degenerate minimizer for $F(x,t,\bullet)$. On the other hand, for sufficiently large k, we have

$$\Big(DH(Dg(y_{n_k}))t_s(y_{n_k}) + y_{n_k}, t_s(y_{n_k})\Big) \in U_{(x_1,T)}.$$

According to Lemma 2.3, there are more than one minimizers or a unique degenerate point for $F(y_{n_k} + t_s(y_{n_k})DH(Dg(y_{n_k})), t_s(y_{n_k}), \bullet)$, which is a contradiction. ■

Consider a characteristic given by *(1.10)*, we have shown that either $Dg(y_0)$ is a minimizer for $F(x,t,\bullet)$ for each $(x,t) \in C$ (in this case $t_s(y_0) = \infty$) or $Dg(y_0)$ is a minimizer for $F(x,t,\bullet)$ for $(x,t) \in C, t \leq t_s(y_0) < \infty$ while $Dg(y_0)$ will be no longer a minimizer for $F(x,t,\bullet)$ for $(x,t) \in C, t > t_s(y_0)$ in lemma 2.3. How to determine a point y_0 whether $t_s(y_0)$ is finite ? That is to say what the domain of definition of the singularity mapping $\mathscr{S}$ is. The following three propositions on the characteristic, which are also interesting in their own sake and have other applications, provide a criterion. The criterion is dependent on the initial data and independent of the particular forms of the equations as long as the Hamiltonian is convex.

Proposition 2.1 *Assume that the initial function $g(y)$ attain its minimum at y_0. Then $Dg(y_0)$ must be a unique non-degenerate minimizer for $F(x,t,\bullet)$ for $(x,t) \in C$, where C is defined by* (1.10).

Proof: We first show that for each $(x,t) \in C$, any other local minimum of $F(x,t,\bullet)$ is strictly greater than $F(x,t,Dg(y_0))$. Suppose $Dg(y_1)$ is another local minimizer for $F(x,t,\bullet)$. Thus $Dg(y_1) \neq 0$. Let

$$h(s) = L(DH(sDg(y_1))). \tag{2.29}$$

Direct computations yield

$$\begin{aligned} h(0) &= L(DH(0)) = L\Big(DH(Dg(y_0))\Big), \\ h(1) &= L\Big(DH(Dg(y_1))\Big), \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} h'(s) &= sDg(y_1)D^2H(sDg(y_1))Dg(y_1)^T \\ &\geq s\alpha|Dg(y_1)|^2 > 0, \end{aligned} \tag{2.31}$$

where we have used the fact that $Dg(y_0) = 0$ and $Dg(y_1) \neq 0$. Consequently, $h(s)$ is strictly increasing, which gives $h(1) > h(0)$, i.e.,

$$L\Big(DH(Dg(y_1))\Big) > L\Big(DH(Dg(y_0))\Big). \tag{2.32}$$

Since g attains its minimum at y_0, we have

$$\begin{aligned} & g\Big(x - tDH(Dg(y_1))\Big) - g\Big(x - tDH(Dg(y_0))\Big) \\ = \; & g\Big(x - tDH(Dg(y_1))\Big) - g(y_0) \geq 0. \end{aligned} \tag{2.33}$$

Therefore, we have

$$\begin{aligned} & F(x, t, Dg(y_1)) - F(x, t, Dg(y_0)) \\ = \; & t\Big[L(DH(Dg(y_1))) - L(DH(Dg(y_0)))\Big] \\ & + g\Big(x - tDH(Dg(y_1))\Big) - g\Big(x - tDH(Dg(y_0))\Big) > 0. \end{aligned} \tag{2.34}$$

The above result indicates that $Dg(y_0)$ is a unique non-degenerate minimizer for $F(x, t, \bullet)$. This completes the proof of the lemma. ■

Proposition 2.2 *Assume that $Dg(y) \to 0$ as $|y| \to \infty$, $g(y)$ does not attain its minimum at y_0 and $Dg(y_0) \neq 0$. Let C be given by (1.10). Then there exists $(\tilde{x}, \tilde{t}) \in C$ such that $Dg(y_0)$ is not the minimizer for $F(\tilde{x}, \tilde{t}, \bullet)$.*

Proof: Choose $y_n \in \partial B(y_0, r_n)$ with $r_n \to \infty$ as $n \to \infty$. Then we have

$$Dg(y_n) \to 0, \quad \text{as } n \to \infty.$$

For $n \geq 0$, let

$$C_n = \{(x, t) : x = y_n + DH(Dg(y_n))t, t > 0\}.$$

Now we claim that there exists a point $y_n \in \partial B(y_0, r_n)$ for n sufficiently large such that the characteristics C_n intersects with the characteristic C. In fact, we only need to show that there exists a solution y_n for

$$y_n - y_0 = t\Big(DH(Dg(y_0)) - DH(Dg(y_n))\Big). \tag{2.35}$$

Let $f : \partial B(0, r_n) \to \partial B(0, r_n)$ be the mapping

$$\tilde{y}_n \to \frac{DH(Dg(y_0)) - DH(Dg(\tilde{y}_n + y_0))}{|DH(Dg(y_0)) - DH(Dg(\tilde{y}_n + y_0))|} r_n,$$

where $\tilde{y}_n = y_n - y_0$ and $y_n \in \partial B(y_0, r_n)$. It is obvious that f is continuous and $f(\partial B(0, r_n)) \neq \partial B(0, r_n)$ for sufficiently large n since

$$DH(Dg(y_n)) \to DH(0) \neq DH(Dg(y_0)), \quad \text{as } n \to \infty.$$

Then f has a fixed point according to an equivalent form of Brouwer's fixed point theorem. This implies that Eq. (2.35) has a solution.

Denote (x_n, t_n) with $x_n = y_n + t_n DH(Dg(y_n))$ being the intersection point of C_n and C. Then

$$y_0 + t_n DH(Dg(y_0)) = y_n + t_n DH(Dg(y_n)), \tag{2.36}$$

which gives

$$|y_n - y_0| = |DH(Dg(x_0)) - DH(Dg(y_n))| t_n. \tag{2.37}$$

Thus $t_n \to \infty$ as $n \to \infty$ by the fact that $|y_n - y_0| = r_n \to \infty$ and $|DH(Dg(x_0)) - DH(Dg(y_n))| \longrightarrow |DH(Dg(x_0)) - DH(0)| \neq 0$ as $n \to \infty$. We also have

$$\begin{aligned} & F(x_n, t_n, Dg(y_0)) - F(x_n, t_n, Dg(y_n)) \\ = \;& t_n \Big[L(DH(Dg(y_0))) - L(DH(Dg(y_n))) \Big] \\ & + g\Big(x_n - t_n DH(Dg(y_0))\Big) - g\Big(x_n - t_n DH(Dg(y_n))\Big) \\ = \;& t_n \Big[L(DH(Dg(y_0))) - L(DH(Dg(y_n))) \Big] + g(y_0) - g(y_n) \to \infty, \end{aligned}$$

where we have used the facts that $t_n \to \infty$ and

$$\begin{aligned} & L\Big(DH(Dg(y_0))\Big) - L\Big(DH(Dg(y_n))\Big) \\ \to \;& L\Big(DH(Dg(y_0))\Big) - L(DH(0)) > 0. \end{aligned} \tag{2.38}$$

The proof of (2.38) is as same as (2.32). This indicates that $Dg(y_0)$ is no longer the minimizer for $F(x_n, t_n, \bullet)$. ■

Remark 2.1 It is worth pointing out that the condition $Dg(y) \to 0$ as $|y| \to \infty$ is necessary in general. For example, it can be verified that there exists a global smooth solution of (1.1) if the initial function g is convex on $\mathbb{R}^n$.

Proposition 2.3 *Assume that $g(y)$ does not attain its minimum at y_0 and $Dg(y_0) = 0$. Let C be defined by* (1.10). *Then there exist $(\tilde{x}, \tilde{t}) \in C$ such that $Dg(y_0)$ is not the minimizer for $F(\tilde{x}, \tilde{t}, \bullet)$*

Proof: Note that

$$DL(DH(p)) = p, \quad \forall\, p \in \mathbb{R}^n. \tag{2.39}$$

Differentiating (2.39) with respect to p gives

$$D^2 L(DH(p)) D^2 H(p) = I_n, \tag{2.40}$$

where I_n is an identity matrix. Consequently,

$$D^2L(DH(p)) = (D^2H(p))^{-1} \le \frac{1}{\alpha} I_n, \tag{2.41}$$

where we have used the facts that $D^2H(p)$ is positive definite and $D^2H(p) \ge \alpha I_n$. Using (2.41) and $Dg(y_0) = 0$ gives

$$\begin{aligned} & L(DH(p)) - L(DH(Dg(y_0))) = DL\Big(DH(Dg(y_0))\Big) \cdot V(p; y_0) \\ & \quad + \int_0^1 \Big\{ sV(p; y_0)^T D^2L[sDH(p) + (1-s)DH(Dg(y_0))]V(p; y_0) \Big\} ds \\ \le & \ \frac{1}{2\alpha} |V(p; y_0)|^2, \end{aligned} \tag{2.42}$$

where for ease of notations, we let

$$V(x; y) = DH(x) - DH(Dg(y)). \tag{2.43}$$

For each point $(x, t) \in C$, i.e., $x = y_0 + tDH(Dg(y_0))$, we have

$$\begin{aligned} & F(x, t, Dg(y_0)) - F(x, t, p) \\ = & \ g(y_0) - g(y_0 - tV(p; y_0) - t\Big[L(DH(p)) - L(DH(Dg(y_0)))\Big] \\ \ge & \ g(y_0) - g(y_0 - tV(p; y_0) - \frac{t}{2\alpha} |V(p; y_0)|^2, \end{aligned} \tag{2.44}$$

where in the last step we have used (2.42). Let $m = \inf_{y \in \mathbb{R}^n} g(y)$. We have $g(y_0) - m > 0$ since $g(y)$ does not attain its minimum at point y_0. There are two cases to be considered.

- *Case 1: there exists y_1 such that $m = g(y_1)$.* Then set

$$y_0 - tV(p; y_0) = y_1,$$

or equivalently,

$$\frac{y_0 - y_1}{t} = V(p; y_0), \tag{2.45}$$

where V is defined by (2.43). Select $t_n > 0$ with $t_n \to \infty$ and set $c_1 = |y_0 - y_1|$. Since the map DH is one-to-one and onto, we can find p_n satisfying

$$\frac{y_0 - y_1}{t_n} = V(p_n; y_0). \tag{2.46}$$

Consequently, $|V(p_n; y_0)| = c_1/t_n$, which gives that

$$\frac{t_n}{2\alpha} |V(p_n; y_0)|^2 = \frac{c_2}{t_n}, \tag{2.47}$$

where $c_2 = c_1^2/2\alpha$. For t_n sufficiently large, we have

$$\frac{c_2}{t_n} \le \frac{1}{2}\Big(g(y_0) - g(y_1)\Big), \tag{2.48}$$

Combining (2.42)-(2.48), we have, for sufficiently large n,

$$\begin{aligned} & F(x_n, t_n, Dg(y_0)) - F(x_n, t_n, p_n) \\ \geq\ & g(y_0) - g(y_1) - \frac{c_2}{t_n} \\ \geq\ & \frac{1}{2}\Big(g(y_0) - g(y_1)\Big) > 0, \end{aligned} \tag{2.49}$$

where $x_n = y_0 + t_n DH(Dg(y_0))$. This implies that for t_n sufficiently large, $Dg(y_0)$ is no longer a minimizer for $F(x_n, t_n, \bullet)$.

• *Case 2: there does not exist y_1 such that $m = g(y_1)$.* Then we can find a sequence $\{\bar{y}_n\}_{n\geq 1}$, such that $|\bar{y}_n| \to \infty$, and $g(\bar{y}_n) \to m$, as $n \to \infty$. Set

$$\frac{y_0 - \bar{y}_n}{t_n} = V(p_n; y_0), \quad t_n = |y_0 - \bar{y}_n|^3. \tag{2.50}$$

Then

$$\frac{1}{2\alpha} t_n |V(p_n; y_0)|^2 = \frac{1}{2\alpha} t_n^{-\frac{1}{3}}. \tag{2.51}$$

By the definition of $\bar{y}_n$, we have for $|\bar{y}_n|$ sufficiently large,

$$0 < g(\bar{y}_n) - m \leq \frac{g(y_0) - m}{3},$$

which gives

$$g(y_0) - g(\bar{y}_n) \geq \frac{2}{3}\Big(g(y_0) - m\Big). \tag{2.52}$$

Consequently,

$$\frac{1}{2\alpha} t_n^{-\frac{1}{3}} \leq \frac{1}{3}\Big(g(y_0) - m\Big). \tag{2.53}$$

Using (2.50)-(2.53), we have, for $x_n = y_0 + t_n DH(Dg(y_0))$,

$$\begin{aligned} & F(x_n, t_n, Dg(y_0)) - F(x_n, t_n, p_n) \\ \geq\ & g(y_0) - g(\bar{y}_n) - \frac{1}{2\alpha} t_n^{-\frac{1}{3}} \\ \geq\ & \frac{2}{3}[g(y_0) - m] - \frac{1}{3}[g(y_0) - m] \\ =\ & \frac{1}{3}[g(y_0) - m] > 0, \end{aligned} \tag{2.54}$$

which implies that

$$F(x_n, t_n, Dg(y_0)) - F(x_n, t_n, p_n) > 0. \tag{2.55}$$

The above result indicates that for t_n sufficiently large, $Dg(y_0)$ is no longer a minimizer for $F(x_n, t_n, \bullet)$. The proof is then complete. ■

From Lemmas 2.3-2.3, we see that $Dg(y) \to 0$ as $|y| \to \infty$, and that the domain of definition of the singularity mapping $\mathscr{S}$ is given by

$$\widetilde{\mathbb{R}}^n = \{y \in \mathbb{R}^n |\ g(y) > \inf_{x\in\mathbb{R}^n} g(x)\}. \tag{2.56}$$

Then the singularity mapping $\mathscr{S}$ is continuous from $\widetilde{\mathbb{R}}^n$ to $\mathbb{R}^n \times (0, \infty)$ and the set of singularity points can be written in the form

$$S = \{(x_s(y), t_s(y))| \; y \in \widetilde{\mathbb{R}}^n\} \tag{2.57}$$

where $(x_s(y), t_s(y))$ are defined by (2.23).

3 Regularity and global structure of solutions

This section is concerned with the regularity of the solutions and global structure of the set of singularity points S of the solutions in the upper half space $\mathbb{R}^n \times (0, \infty)$. We will show that S, as the complementary set of the set U in Lemma 2.1, is a closure of the set consisting of points at which the solution is non-differentiable. Then as a corollary we have the the result that the solution $u(x, t)$ is C^k smooth in some neighborhood of (x_0, t_0) if and only if there is a unique non-degenerate minimizer for $F(x_0, t_0, \bullet)$. The set of singularity points consists of several path connected components S_i. We will show that there exists one-to-one correspondence between the path connected components S_i of the set of singularity points and path connected components R_i of the subset of $\widetilde{\mathbb{R}}^n$ on which initial function does not attain its minimum. A singularity never terminates as t increases.

Lemma 3.1 *If $F(x_0, t_0, \bullet)$ has a unique degenerate minimizer or more than one minimizer, then $u(x, t) = \min_{p\in\mathbb{R}^n} F(x, t, p)$ is not differentiable in any neighborhood $U_{(x_0,t_0)}$ of (x_0, t_0).*

Proof: Without loss of generality we assume that $Dg(y) \to 0$ as $|y| \to \infty$ since we are concerned with local properties of solutions $u(x, t)$.

The assertion that the solution $u(x, t)$ is not differentiable at (x_0, t_0) if there are more than one minimizer for $F(x_0, t_0, \bullet)$ can be deduced from Theorem 2.1 of Hoang [6].

For the case that there is a unique degenerate minimizer $p_0 = Dg(y_0)$ for $F(x_0, t_0, \bullet)$, we only need show that for any neighborhood $U_{(x_0,t_0)}$ of (x_0, t_0),

$$\begin{array}{l}\text{there exists a point } (x, t) \in U_{(x_0,t_0)} \text{ such that} \\ \text{there are more than one minimizer for } F(x, t, \bullet).\end{array} \tag{3.1}$$

Otherwise,

$$\begin{array}{l}\text{there exists a neighborhood } U_{(x_0,t_0)} \text{ of } (x_0, t_0) \text{ such that for each point} \\ (x, t) \in U_{(x_0,t_0)} \text{ there is a unique minimizer for } F(x, t, \bullet).\end{array} \tag{3.2}$$

It follows from Lemma 2.1 that $g(y_0) > \inf_{y\in\mathbb{R}^n} g(y)$. Then there exists a neighborhood $U_{y_0} \subset \mathbb{R}^n \times \{t = 0\}$ of y_0 such that $g(y) > \inf_{y\in\mathbb{R}^n} g(y)$ for each $y \in U_{y_0}$. By Lemmas 2.2 and 2.3, we conclude that $t_s(y)$ is finite for each $y \in U_{y_0}$. Then by Lemma 2.4 we have

$$\mathscr{S}(U_{y_0}) = \{(x_s(y), t_s(y))|y \in U_{y_0}\} \tag{3.3}$$

is a continuous manifold passing through (x_0, t_0).

It follows from (3.2) that the singularity mapping $\mathscr{S}$ from U_{y_0} to $\mathscr{S}(U_{y_0})$ is one-to-one if $\mathscr{S}(U_{y_0}) \subset U_{(x_0,t_0)}$ provided that U_{y_0} is small enough. Then $\mathscr{S}(U_{y_0})$ is a n-dimensional continuous hyper-surface. Moreover, there is a unique intersection point for each characteristic from U_{y_0} and the hypersurface, and (x_0, t_0) is an interior point of the hypersurface. Therefore, there exists a neighborhood of (x_0, t_0) as a subset of $U_{(x_0,t_0)}$ (denoted again by $U_{(x_0,t_0)}$) which can be divided by the hypersurface $\mathscr{S}(U_{y_0})$ into three parts. More precisely,

$$U_{(x_0,t_0)} = U^+_{(x_0,t_0)} \bigcup U^-_{(x_0,t_0)} \bigcup \mathscr{S}(U_{y_0}). \tag{3.4}$$

where

$$U^+_{(x_0,t_0)} = U_{(x_0,t_0)} \bigcap \{t > t_s(y)\}, \quad U^-_{(x_0,t_0)} = U_{(x_0,t_0)} \bigcap \{t < t_s(y)\}.$$

Consider a sequence $(\bar{x}_n, \bar{t}_n) \in U^+_{(x_0,t_0)}$, which is convergent to (x_0, t_0). Assume C^-_n is the valid segment of the characteristic emanating from y_n and passing through $(\bar{x}_n, \bar{t}_n)$, here the assumption (3.2) is used. Then C^-_n must meet $\mathscr{S}(U_{y_0})$ for n sufficiently large since there is a unique minimizer for $F(x_0, t_0, \bullet)$, which implies that $\bar{t}_n > t_s(y_n)$. This contradicts Lemma 2.3. Thus assertion (3.1) is true. ∎

Let

$$\begin{aligned} S^2 &= \{(x,t) \in \mathbb{R}^n \times (0,\infty) |\ F(x,t,\bullet) \text{ has more than one minimizer}\}, \\ S^1 &= \{(x,t) \in \mathbb{R}^n \times (0,\infty) |\ F(x,t,\bullet) \text{ has a unique degenerate minimizer}\}. \end{aligned} \tag{3.5}$$

It follows from the proof of Lemma 3.1 that (x, t) is a non-differentiable point of the solution $u(x, t)$ for $(x, t) \in S^2$ and (x, t) is a cluster point of non-differentiable points of the solution $u(x, t)$ for $(x, t) \in S^1$. This implies each point of S^1 is a cluster point of the elements of S^2. Furthermore, the set of singularity points S is a closure of S^2. Then another definition of a singularity point can be given: a point is called a singularity point if it is a non-differentiable point of the solution $u(x, t)$ or a cluster point of non-differentiable points of the solution $u(x, t)$.

The following theorem follows from Lemmas 2.1-3.1.

Theorem 3.1 *The solution $u(x, t)$ is C^k smooth in some neighborhood of (x_0, t_0) if and only if there is a unique non-degenerate minimizer for $F(x_0, t_0, \bullet)$.*

It is known that

$$\widetilde{\mathbb{R}}^n = \{y \in \mathbb{R}^n |\ g(y) > \inf_{x\in\mathbb{R}^n} g(x)\}$$

is an open subset and is the union of path connected components R_i, i.e.,

$$\widetilde{\mathbb{R}}^n = \{y \in \mathbb{R}^n |\ g(y) > \inf_{x\in\mathbb{R}^n} g(x)\} = \bigcup R_i. \tag{3.6}$$

Theorem 3.2 *Assume $g \in C^k$ and $|Dg(y)| \to 0$ as $|y| \to \infty$. Let R_i be the path connected component of $\widetilde{\mathbb{R}}^n$ on which initial function does not attain its minimum. Then $S_i = \mathscr{S}(R_i)$ is a path connected component of the set singularity points S and never vanishes for $t > t_i$, where t_i is the formation time of S_i. Moreover, there exists one-to-one correspondence $S_i \leftrightarrow R_i$ and $S = \bigcup S_i$.*

Proof: We have $g(y) = \inf_{x\in\mathbb{R}^n} g(x)$ and the characteristic emanating from y has slope $DH(0)$ for $y \in \partial R_i$, where ∂R_i is the boundary of R_i. It follows from Lemma 2.1 that for $y \in \partial R_i$ $Dg(y)$ is the unique non-degenerate minimizer for $F(x, t, \bullet)$, where $x = y+tDH(0)$. Note that R_i and R_j are both path connected components of $\widetilde{\mathbb{R}}^n$, and so R_i and R_j disjoint. Therefore, a valid segment of characteristic from R_i and a valid one from $R_j (i \neq j)$ can not intersect with each other.

For each $y \in \widetilde{\mathbb{R}}^n$, it is known that $\mathscr{S}(y) = (x_s(y), t_s(y))$, where

$$\begin{aligned} &x_s(y) = y + t_s(y)DH(Dg(y)),\\ &t_s(y) = \max\{t|\ u(x, t) = F(x, t, Dg(y)), (x, t) \in C\},\end{aligned}$$

where $C = \{(x, t)|x = y + tDH(Dg(y))\}$. Furthermore

$$\begin{aligned} \mathscr{S}(\widetilde{\mathbb{R}}^n) &= \{(x_s(y), t_s(y))|\ x_s(y) = y + t_s(y)DH(Dg(y)),\ y \in \widetilde{\mathbb{R}}^n\}\\ &= \bigcup \mathscr{S}(R_i) = \bigcup S_i = S, \end{aligned} \tag{3.7}$$

where $S_i = \mathscr{S}(R_i)$. We have $S_i \bigcap S_j = \emptyset\ (i \neq j)$ since valid segments of characteristic from R_i and R_j can not intersect with each other. Thus $S_i = \mathscr{S}(R_i)$ is a path connected component of the set of the singularity points since the singularity mapping $\mathscr{S}$ is continuous and R_i is path connected.

Next we will show each path connected component S_i never vanishes as t increases. Suppose S_i vanishes before $t = T < \infty$. Then

$$\begin{aligned} &\text{for each point } (x, t) \in \Pi_i \textstyle\bigcap \{t \geq T\},\ \text{there exists a}\\ &\text{valid segment of characteristic passing through } (x, t), \end{aligned} \tag{3.8}$$

where

$$\Pi_i = \{(x, t)|\ x = y + tDH(Dg(y)),\ y \in R_i,\ t > 0\}.$$

We see $\Pi_i \bigcap \Pi_j = \emptyset$ if $i \neq j$ since valid segments of characteristic from R_i and R_j can not intersect with each other. Using this fact, we claim that

$$\begin{aligned} &\text{the valid segment of characteristic must emanate from}\\ &\text{a point } y \text{ satisfying } g(y) = \inf\nolimits_{y\in\mathbb{R}^n} g(y). \end{aligned} \tag{3.9}$$

Otherwise, it follows from Lemmas 2.2 and 2.3 that the characteristic emanating from y will meet at the singularity point $(x_s(y), t_s(y))$. This contradicts to the fact that S_i is a path connected component.

The assertion (3.9) suggests that all valid segments of characteristics passing through $\Pi_i \bigcap \{(x,t), t = T\}$ have slope $DH(0)$. Therefore, the valid segments of characteristics mentioned above will cover $\Pi_i \bigcap \{(x,t), 0 < t < T\}$ and therefore meet S_i. This contradicts to the fact that S_i is a set of singularity points. Hence the theorem is proved. ■

The above result is dependent on the initial data, while it is independent of the particular forms of the equations as long as the Hamiltonian is convex. Using Theorem 3.2 we have the following corollary.

Corollary 3.1 *The domain of dependence of a point* $(x,t) \in \Pi_i$ *is* $R_i \bigcap B(x, rt)$*, where* $r = \sup_{y \in \mathbb{R}^n} |DH(Dg(y))|$*. The domain of influence of a point* $y \in R_i$ *is*

$$\Pi_i \bigcap \{(y + \xi t, t) : \ |\xi| \le r\}.$$

4 Concluding Remarks

This paper is concerned with the Hamilton-Jacobi equations of multidimensional space variables with convex Hamiltonian. Using Hopf's formula (I), we studied the differentiability of the solutions. For any given point, we give a sufficient and necessary condition under which the solutions are C^k smooth in some neighborhood of the point. We also study the characteristics of the HJ equations. It is shown that there are only two kinds of characteristics, one never touches the singularity, and the other touches the singularity in a finite time. The sufficient and necessary condition under which the characteristic never touches the point of singularity is given. It is also shown that there exists an one-to-one correspondence between the path connected components of the set of singularity points and the path connected components of the subset of $\mathbb{R}^n \times \{t = 0\}$ on which the initial function does not attain its minimum. A path connected component of the set of singularity points never terminates at a finite time.

In the second paper [16], we will consider the regularity and global structure of the HJ solutions with convex initial data under the assumptions that the initial function $g \in C^k$ is strictly convex with general Hamiltonian H. The solutions to the Hamilton-Jacobi equations are given by Hopf's formula (II) as analogue to Hopf's formula (I). In this case, we also obtain the regularity results on the differentiability of the solutions similar to the ones obtained in this work. We will show that there exists a one-to-one correspondence between the path connected components of set of singularity points and the path connected components of the set

$$\{(Dg(y), convH(Dg(y)))|\ y \in \mathbb{R}^n\} \backslash \{(Dg(y), H(Dg(y)))|\ y \in \mathbb{R}^n\},$$

where $convH$ is the convex hull of H,

$$convH(x) = \inf\Big\{ \sum_{i=1}^m \lambda_i H(x_i) \mid \sum_{i=1}^m \lambda_i x_i = x, \ \sum_{i=1}^m \lambda_i = 1, \ \lambda_i \ge 0\}.$$

That each path connected component of the set of singularity points never terminates at a finite time is also proved.

Acknowledgments.

The first author would like to thank Professor Pingwen Zhang for his support and encouragement. His research was supported by China Postdoctoral Science Foundation under contract 148028. The second author was partially supported by CERG Grants of Hong Kong Research Grant Council, FRG grants of Hong Kong Baptist University, and International Research Team of Complex Systems of Chinese Academy of Sciences. The third author was supported by National Natural Science Foundation of China under contract 10671116.

References

[1] E.N. Barron, P. Cannasa, R. Jensen and C. Sinestrari, Regularity of Hamilton-Jacobi equations when forward is backward, *Indiana. Univ. Math. J.*, **48** (1999), 395-409.

[2] M. Bardi and L.C. Evans, On Hopf's formulas for solutions of Hamilton-Jacobi equations, *Nonlinear Anal., Theory, Methods & Applications*, **8** (1984), 1373-1381.

[3] E. Conway and E. Hopf, Hamilton's theory and generalized solutions of the Hamilton-Jacobi equation, *J. Math. Mech.*, **13** (1964), 939-986.

[4] L.C. Evans, Partial Differential Equations, American Mathematical Society, 1998.

[5] E. Hopf, Generalized solutions of nonlinear equations of first order, *J. Math. Mech.* **14** (1965) 951-973.

[6] N. Hoang, The regularity of generalized solutions of Hamilton-Jacobi equations, *Nonlinear Anal.* **59** (2004) 745-757

[7] S.N. Kruzkov, Generalized solutions of nonlinear first order equations with several variables. II. *Math. Sbornik*, **1** (1967), 93-116.

[8] B. Li and J. Wang, The global qualitative study of solutions to a conservation law, *Sci. Special Math. Issue* (1979), 12-24 (in Chinese).

[9] C.-T. Lin, E. Tadmor L^1 Stability and error estimates for approximate Hamilton-Jacobi solutions, *Numer.Math.*, **87** 701-735 (2001)

[10] P. L. Lions, Generalized Solutions of Hamilton-Jacobi Equations, Pitman Advance Publishing Program, 1982.

[11] D.G. Schaeffer, A regularity theorem for conservation laws, *Adv. Math.* **11** (1973), 368-386

[12] T. Tang, J. Wang, and Y. Zhao, On the piecewise smoothness of entropy solutions to scalar conservation laws for a large class of initial data, *J. Hyperbol. Differ. Eq.* **4** (2007), pp. 369-389.

[13] T.D. Van, N. Hoang, N.D. Thai Son, Explicit global Lipschitz solutions to first-order nonlinear partial differential equations, *Viet.J.Math*, **27** (2) (1999) 93-114

[14] T.D. Van, N. Hoang, M.Tsuji, On Hopf's formula for Lipschitz solutions of the Cauchy problem for Hamilton-Jacobi equations, *Nonlinear Anal.* **29** (10) (1997) 1145-1159.

[15] T.D. Van, M.Tsuji, N.D. Thai Son, The characteristic method and its generalizations for first order nonlinear PDEs, *Chapman* & *Hall/CRC, London/Boca Raton, Fl,* 2000

[16] Y. Zhao, T. Tang and J.-H. Wang, Regularity and global structure of solutions to Hamilton-Jacobi equations II. Cconvex initial data. In preparation.