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A note on collocation methods for Volterra integro-differential equations with weakly singular kernels

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Spline collocation methods can be used to solve Volterra integro-differential equations with weakly singular kernels. In order to obtain optimal convergence behavior, collocation on suitably graded meshes was considered by H. Brunner [1]. This work extends his results to more practical values of the grading exponent.

1. Introduction

This short note is concerned with collocation approximations for Volterra integro-differential equations (VIDEs)

$$y'(t)=f(t, y(t))+\int_0^t (t-s)^{-\alpha}k(t, s, y(s))\,\mathrm{d}s, \qquad t\in I:=[0, T], \tag{1.1}$$

with $0<\alpha<1$, and with given initial condition $y(0)=y_0$. For ease of exposition the linear counterpart of (1.1),

$$y'(t)=a(t)y(t)+b(t)+\int_0^t (t-s)^{-\alpha}K(t, s)y(s)\,\mathrm{d}s, \qquad t\in I, \tag{1.2}$$

will be employed in the analysis of the principle properties of the collocation approximations; the extension to nonlinear equations is straightforward (cf. [1, p. 225]).

High-order numerical methods for VIDEs with weakly singular kernels may be found in [1, 2, 6, 7, 8]. In this note we shall consider collocation methods for VIDE (1.1), based on Brunner's approach [1]. The following method and notation were introduced in [1]. Collocation methods generate, as approximations to the solution of (1.1), elements of the polynomial spline space

$$S_m^{(0)}(Z_N):=\{u\in C(I): u|_{\sigma_n}:=u_n\in \pi_m,\ 0\leqslant n\leqslant N-1\},$$

associated with a given mesh sequence $\Pi_N: 0=t_0<t_1<\cdots<t_N=T,\ N\geqslant 1$, of the interval I. Here, π_m is the set of (real) polynomials of degree not exceeding m (with $m\geqslant 1$), $\sigma_n:=[t_n, t_{n+1}]$ $(0\leqslant n\leqslant N-1)$, and $Z_N:=\{t_n: 1\leqslant n\leqslant N-1\}$. In other words, $S_m^{(0)}(Z_N)$ is the space of piecewise continuous polynomials of degree m with (possibly) jump discontinuities in the first derivative at the interior points Z_N. The quantity h, $h=\max\{h_n:=t_{n+1}-t_n,\ 0\leqslant n\leqslant N-1\}$, is often called the

diameter of the mesh sequence Π_N. If $h_n \equiv T/N$ for all $0 \leqslant n \leqslant N-1$, then the grid Π_N is called a uniform mesh.

The desired approximation to y is the element $u \in S_m^{(0)}(Z_N)$ satisfying

$$u'(t) = f(t, u(t)) + \int_0^t (t-s)^{-\alpha} k(t, s, u(s))\, ds, \qquad t \in X(N), \tag{1.3}$$

with

$$X(N) := \{t_n + c_j h_n : 0 \leqslant c_1 < \cdots < c_m \leqslant 1, 0 \leqslant n \leqslant N-1\},$$

where $\{c_j\}_{j=1}^m$ are collocation parameters. For the linearized VIDE (1.2), the collocation equations (1.3) can be written as

$$u'(t) = a(t)u(t) + b(t) + \int_0^t (t-s)^{-\alpha} K(t, s) u(s)\, ds, \qquad t \in X(N). \tag{1.4}$$

If the mesh points $\{t_n\}_{n=0}^N$ are given by

$$t_n := \left(\frac{n}{N}\right)^r T, \qquad (0 \leqslant n \leqslant N), \tag{1.5}$$

then Π_N is called a graded mesh; and the grading exponent r will always satisfy $r \geqslant 1$.

Brunner [1] considered the above collocation methods for VIDE (1.1). He found that the use of a uniform mesh leads, due to the nonsmooth nature of the exact solutions, to convergence of order less than one, regardless of the degree of the approximating spline functions. However, if a graded mesh of the form (1.5) with $r = m/(1-\alpha)$ is used, then optimal convergence behavior can be obtained. In his work he also points out that the use of graded meshes, with the grading exponent $m/(1-\alpha)$, has a practical limitation since the initial stepsize becomes very small as N is increased. As an example, if we assume that $m = 4$ and $\alpha = \frac{1}{2}$, then we have to start the collocation method on a subinterval whose length is of order N^{-8}. It is obvious, even for moderate values of N, that this may create serious round-off errors in subsequent calculations. Moreover, if $\alpha \to 1-$, then the value of r used in [1] tends to infinity which prevents one from obtaining meaningful approximations for graded meshes, even when working in double or extended precision.

In this note, we shall show that if r is slightly greater than $m/(2-\alpha)$ and if $u \in S_m^{(0)}(Z_N)$ is the collocation solution corresponding to the graded mesh (1.5), then $\|y-u\|_\infty = O(N^{-m})$ and $y'(t) - u'(t) = O(N^{-m})$ if t is away from the origin. The grading exponent suggested in this work is smaller than the one given in [1]. For the example mentioned above ($\alpha = \frac{1}{2}$ and $m = 4$), the initial stepsize is of order about $N^{-2 \cdot 7}$ (compare N^{-8} given by [1]). Moreover, for any $\alpha \in (0, 1)$, we have $m/(2-\alpha) < m$. Hence we can use a graded mesh even when α is very close to 1.

2. Main results

The main results of this section can be established using arguments similar to those used in [1]. The only modification needed will be an application of the following lemma in place of Lemma 3.3 of [1].

LEMMA 1 Let $m \geq 1$ and $0 \leq \mu \leq m+1$. If the sequence of points $\{t_n\}$ defines a graded mesh, then for any $s \geq 1$,

$$h_i^{m+1} t_i^{s(2-\alpha)-\mu} = O(N^{\gamma-(m+1)} i^{-\gamma}) \qquad (1 \leq i \leq N), \tag{2.1}$$

$$h_i^{m+1} = O(N^{\gamma-(m+1)} i^{-\gamma}) \qquad (1 \leq i \leq N), \tag{2.2}$$

where $\gamma := -r(2-\alpha) + m + 1$.

Proof. Since $t_i = i^r T N^{-r}$, it follows that

$$h_i = t_{i+1} - t_i = i^r T N^{-r}[(1+i^{-1})^r - 1] < r 2^{r-1} T i^{r-1} N^{-r}$$

for $1 \leq i \leq N-1$. Thus we obtain, for $0 \leq \mu \leq m+1$ and $s \geq 1$, that

$$h_i^{m+1} t_i^{s(2-\alpha)-\mu} \leq C i^{-\theta} N^{\theta-(m+1)} \tag{2.3}$$

where C is a constant independent of N and

$$\theta := -r[m+1+s(2-\alpha)-\mu] + m + 1. \text{ For } 0 \leq \mu \leq m+1 \text{ and } s \geq 1,$$

$$\theta \leq -r[m+1+(2-\alpha)-(m+1)] + m + 1 \leq -r(2-\alpha) + m + 1 = \gamma.$$

Noting that

$$\left(\frac{N}{i}\right)^{\theta} \leq \left(\frac{N}{i}\right)^{\gamma}, \quad \text{for} \quad 1 \leq i \leq N, \tag{2.4}$$

we can obtain (2.1) by combining (2.3) and (2.4). A similar proof technique yields (2.2). □

We shall estimate the error function $e(t) := y(t) - u(t)$ and its derivative $e'(t)$. The restriction of e to the subinterval σ_n, $0 \leq n \leq N-1$, will be denoted by e_n.

THEOREM 1 Let the functions a, b and K in (1.2) be m-times continuously differentiable functions, and assume that b and K do not vanish identically. If $u \in S_m^{(0)}(Z_N)$ is the collocation approximation defined by (1.4), and if the underlying mesh sequence Π_N consists of graded meshes of the form (1.5), with grading exponent r, $r > m/(2-\alpha)$, then for any collocation parameters $\{c_j\}$ with $0 \leq c_1 < \cdots < c_m \leq 1$, the resulting error $e := y - u$ satisfies

(i) $$\|e\|_\infty = O(N^{-m}), \tag{2.5}$$

(ii) $$e_n'(t) = O\left(N^{-m}\left(\frac{N}{n}\right)^{m-r(1-\alpha)}\right), \qquad (1 \leq n \leq N-1), \tag{2.6}$$

(iii) $$e_0'(t) = O(N^{-r(1-\alpha)}). \tag{2.7}$$

Proof. Let C denote a positive constant, independent of N and h, possibly with different values at different places. By using the same procedure and the same notation as [1] (pp. 226–233) we can obtain

$$|\beta_{n0}| \leqslant \sum_{i=0}^{n-1} \|\beta_i\|_1 + Ch_0^{2-\alpha} + \sum_{i=0}^{n-1} h_i^{m+1} |R_i(1)|, \qquad 1 \leqslant n \leqslant N-1, \tag{2.8}$$

$$\|\beta_n\|_1 \leqslant Ch \sum_{i=0}^{n-1} \|\beta_i\| + h_n z_n, \qquad 1 \leqslant n \leqslant N-1, \tag{2.9}$$

(see (3.13) and (3.18) of [1]), where

$$|R_i(1)| \leqslant C \max \{t_i^{s(2-\alpha)-\mu};\ 1 \leqslant s \leqslant m,\ 0 \leqslant \mu \leqslant m+1\}. \tag{2.10}$$

Using Lemma 1 we can show that the last term of (2.9) is to be bounded by $O(N^{-(m+1)+\gamma} n^{-\gamma})$, with $\gamma = -r(2-\alpha) + m + 1 < 1$. Then a result of [1, p. 232] gives

$$\|\beta_n\|_1 = O(h_n z_n) = O(N^{-(m+1)+\gamma} n^{-\gamma}). \tag{2.11}$$

Since $\gamma < 1$, (2.8), (2.10), (2.11) and Lemma 1 lead to

$$|\beta_{n0}| \leqslant C \sum_{i=1}^{n-1} N^{-(m+1)+\gamma} i^{-\gamma} + Ch_0^{2-\alpha} = O(N^{-m}). \tag{2.12}$$

The above estimates (2.11)–(2.12) and (3.7)–(3.8) of [1, p. 228] lead to (2.5)–(2.7). □

Theorem 1 suggests that if the grading exponent r is greater than $m/(2-\alpha)$, then we can obtain the optimal convergence rate for the error function e itself. It can also be seen from (2.6) that $e_n'(t) = O(N^{-m})$ if t is away from the origin. Moreover, by setting $m = 1$ in Theorem 1 we can obtain the following results.

COROLLARY 1 Let the functions a, b and K in (1.2) be continuously differentiable in their corresponding domains, and assume that b and K do not vanish identically. If $u \in S_1^{(0)}(Z_N)$ is the collocation approximation defined by (1.4), and if the underlying mesh sequence Π_N is a uniform mesh, then for any choice of a collocation parameter $c_1 \in [0, 1]$, the resulting error $e := y - u$ satisfies

(i) $$\|e\|_\infty = O(N^{-1}), \tag{2.13}$$

(ii) $$e_n'(t) = O\left(N^{-1}\left(\frac{N}{n}\right)^{\alpha}\right), \qquad (1 \leqslant n \leqslant N-1), \tag{2.14}$$

(iii) $$e_0'(t) = O(N^{-(1-\alpha)}). \tag{2.15}$$

The following theorem is concerned with collocation in the space $S_m^{(0)}(Z_n)$, with $m \geqslant 2$, using a uniform mesh. It can be established in a similar way to the proof of Theorem 1. Dixon [4] shows that, away from the origin, the error in product integration and collocation schemes for second-kind Volterra integral equations is of order $2-\alpha$. The following theorem can also be obtained by a similar proof technique to that given in [4].

THEOREM 2 Let the functions a, b and K in (1.2) be subject to the conditions stated in Theorem 1. If $u \in S_m^{(0)}(Z_N)$, $m \geqslant 2$, is the collocation approximation defined by (1.4), and if the underlying mesh sequence Π_N is a uniform mesh, then for any choice of collocation parameters $0 \leq c_1 < \cdots < c_m \leq 1$, the resulting error $e := y - u$ satisfies

(i) $$\|e\|_\infty = O(N^{-(2-\alpha)}), \tag{2.16}$$

(ii) $$e_n'(t) = O\left(N^{-(2-\alpha)} + N^{-m}\left(\frac{N}{n}\right)^{\alpha+m-1}\right), \quad (1 \leq n \leq N-1), \tag{2.17}$$

(iii) $$e_0'(t) = O(N^{-(1-\alpha)}). \tag{2.18}$$

Theorem 2 suggests that if the mesh sequence Π_N is uniform and $m \geq 2$, then for any choice of the collocation parameters $\{c_j\}$ the global convergence rates of collocation approximations are

$$\|e\|_\infty = O(N^{-(2-\alpha)}), \qquad \|e'\|_\infty = O(N^{-(1-\alpha)}). \tag{2.19}$$

Brunner [1] presented some numerical results for (1.2) for $m = 2$ and $\alpha = \frac{1}{3}, \frac{1}{2}$ and $\frac{2}{3}$. His numerical results, obtained by using a uniform mesh, give convergence rates of $\|e\|_\infty$ as $1{\cdot}65(\alpha = \frac{1}{3})$, $1{\cdot}47(\alpha = \frac{1}{2})$ and $1{\cdot}29(\alpha = \frac{2}{3})$; they are in good agreement with the theoretical estimate (2.19) (convergence rate $2 - \alpha$). Also, the numerical calculations give the convergence rates for $\|e'\|_\infty$ as $0{\cdot}65(\alpha = \frac{1}{3})$, $0{\cdot}48(\alpha = \frac{1}{2})$ and $0{\cdot}31(\alpha = \frac{2}{3})$; they are also in good agreement with the theoretical estimate (2.19) (convergence rate $1 - \alpha$).

3. Numerical example

For numerical verification of the results stated in Section 2, we consider

$$y'(t) = a(t)y(t) + b(t) + \int_0^t \lambda(t-s)^{-\alpha} y(s)\,ds, \qquad y(0) = 0, \quad 0 \leq t \leq T,$$

with $a(t) \equiv -1$, $\lambda = -1$, and with $b(t)$ chosen so that $y(t) = t^{2-\alpha}$. This equation was tested in [1] in which a detailed description of the discretization of the collocation equations (1.3) and (1.4) was provided.

The grading exponents used in our calculations are $r_1 = 1$ (uniform mesh), $r_2 = m/(2-\alpha) + \frac{1}{3}$ (suggested by this work) and $r_3 = m/(1-\alpha)$ (suggested by [1]). The numerical results are obtained with $m = 3$ and with the collocation parameters $c_1 = 0{\cdot}1$, $c_2 = 0{\cdot}3$ and $c_3 = 0{\cdot}5$.

Firstly we consider the case when α is not too close to 1. In Table 1 we list the errors, $\|e\|_\infty$, and the computed rates of convergence for $\alpha = 0{\cdot}5$ and $T = 1$. The

TABLE 1
Errors and convergence rates with $\alpha = 0{\cdot}5$ and $T = 1 : \|e\|_\infty$

	$r = r_1$		$r = r_2$		$r = r_3$	
N	$\|e\|_\infty$	rate	$\|e\|_\infty$	rate	$\|e\|_\infty$	rate
10	$1{\cdot}69D-3$		$1{\cdot}11D-4$		$8{\cdot}54D-4$	
20	$5{\cdot}88D-4$	1·52	$1{\cdot}53D-5$	2·86	$9{\cdot}20D-5$	3·21
40	$2{\cdot}09D-4$	1·49	$2{\cdot}07D-6$	2·89	$1{\cdot}06D-5$	3·12
80	$7{\cdot}46D-5$	1·49	$2{\cdot}72D-7$	2·93	$1{\cdot}27D-6$	3·06

TABLE 2
Errors and convergence rates with $\alpha = 0{\cdot}5$ and $T = 1$: $|e(T)|$

N	$r = r_1$ $\|e(T)\|$	rate	$r = r_2$ $\|e(T)\|$	rate	$r = r_3$ $\|e(T)\|$	rate
10	6·66$D-5$		6·35$D-5$		8·54$D-4$	
20	1·98$D-6$	—	7·07$D-6$	3·17	9·20$D-5$	3·21
40	4·83$D-6$	—	8·29$D-7$	3·09	1·06$D-5$	3·12
80	2·41$D-6$	—	9·99$D-8$	3·05	1·27$D-6$	3·06

TABLE 3
Errors and convergence rates with $\alpha = 0{\cdot}5$ and $T = 1$: $|e'(T)|$

N	$r = r_1$ $\|e'(T)\|$	rate	$r = r_2$ $\|e'(T)\|$	rate	$r = r_3$ $\|e'(T)\|$	rate
10	9·81$D-4$		3·45$D-4$		7·20$D-3$	
20	3·04$D-4$	1·69	3·66$D-5$	3·24	8·52$D-4$	3·08
40	9·94$D-5$	1·61	4·13$D-6$	3·15	1·01$D-4$	3·08
80	3·37$D-5$	1·56	4·85$D-7$	3·09	1·22$D-5$	3·05

TABLE 4
Errors with $\alpha = 0{\cdot}9$ and $T = 5$: $\|e\|_\infty$ and $|e(T)|$

N	$r = r_1$ $\|e\|_\infty$	$\|e(T)\|$	$r = r_2$ $\|e\|_\infty$	$\|e(T)\|$	$r = r_3$ $\|e\|_\infty$	$\|e(T)\|$
10	3·62$D-2$	1·40$D-4$	4·19$D-2$	4·19$D-2$	8·24$D-1$	8·24$D-1$
20	1·14$D-2$	1·50$D-5$	6·06$D-5$	2·70$D-5$	3·72$D-1$	3·72$D-1$
40	4·03$D-3$	5·48$D-6$	9·09$D-6$	6·29$D-6$	3·94$D+0$	3·94$D+0$
80	1·56$D-3$	2·18$D-6$	1·32$D-6$	5·85$D-7$	1·20$D+1$	1·20$D+1$

convergence rates predicted in Theorems 1 and 2 for the error function are confirmed by Table 1. It can also be seen that the numerical results obtained by using the grading exponent r_2 as suggested by the present work are more accurate than those obtained by using r_1 and r_3. For the derivative of the error function this work suggests that high orders of convergence may be observed at the right endpoint of the interval of integration. It was pointed out by Brunner [1] that in many practical applications one is more interested in generating a numerical approximation which is very accurate at the right endpoint $t = T$. In Tables 2 and 3 we list the errors $|e(T)|$ and $|e'(T)|$ and the corresponding computed rates of convergence, respectively. The convergence results of Theorems 1 and 2 for the derivative of the approximate solution are reflected in Table 3. Moreover, both Tables 2 and 3 indicate that numerical results at $t = T$ obtained by using the grading exponent r_2 are much more accurate than those obtained by using r_1 and r_3.

Finally we look at the convergence behavior of the collocation methods with graded grids when α is close to 1. For this purpose we consider the case $\alpha = 0{\cdot}9$ and $T = 5$. Table 4 shows the errors $\|e\|_\infty$ and $|e(T)|$. It can be seen from this table that graded meshes with the grading exponents given in Theorem 1 lead to very accurate numerical solutions. However, solutions with $r = r_3$ are divergent, because the initial stepsizes used in the calculations are too small.

4. Conclusion

This note is concerned with collocation methods for Volterra integro-differential equations with weakly singular kernels. By choosing a smaller grading exponent than that suggested in [1], we are able to obtain optimum rates of convergence while overcoming the practical problems encountered using previous grading exponents. The grading exponent suggested by this note is not greater than m (the polynomial degree) which is independent of α. This is in contrast to the theories of spline collocation methods for Volterra integral equations of the second kind. The main disadvantage of using a graded mesh to solve Volterra integro-differential (or integral) equations with weakly singular kernels is that $O(N^2)$ instead of $O(N)$ quadrature weights have to be computed for the kernel $(t-s)^{-\alpha}$. This increase of computational work is not present for the approaches in [3] and [5]. However, computational time is in general not too expensive for the one-dimensional calculations. Moreover, since the collocation method is easy to implement and can produce very accurate results by using reasonably small numbers of collocation points, it is still considered to be a useful tool in solving Volterra equations.

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References

[1] BRUNNER, H. 1986 Polynomial spline collocation methods for Volterra integro-differential equations with weakly singular kernels. *IMA J. Numer. Anal.* **6,** 221–239.
[2] BRUNNER, H., & TANG, T. 1989 Polynomial spline collocation methods for the nonlinear Basset equation. *Computers Math. Applic.* **18,** 449–457.
[3] HAIRER, E., LUBICH, C., & SCHLICHTE, M. 1985 Fast numerical solution of weakly singular Volterra integral equations. *J. Comput. Appl. Math.* **23,** 87–98.
[4] DIXON, J. 1985 On the order of the error in discretization methods for weakly singular second kind Volterra integral equations with nonsmooth solutions. *BIT* **25,** 624–634.
[5] EGGERMONT, P. P. B. 1989 Uniform error estimates of Galerkin methods for monotone Abel-Volterra integral equations on the half-line. *Math. Comp.* **53,** 157–189.
[6] HAIRER, E., & MAASS, P. 1987 Numerical methods for singular nonlinear integro-differential equations. *Appl. Numer. Math.* **3,** 243–256.
[7] MCKEE, S. 1979 The analysis of variable step, variable coefficient linear multistep methods for solving singular integro-differential equation arising from the diffusion of discrete particles in a turbulent fluid. *J. Inst. Math. Applic.* **23,** 373–388.
[8] MCKEE, S., & STOKES, A. 1983 Product integration methods for the nonlinear Basset equation. *SIAM J. Numer. Anal.* **20,** 143–160.