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Letter Section

The proof of an inequality arising in a reaction-diffusion study in a small cell

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Abstract

The monotonicity of a sequence arising in the convergence proof of a product integration scheme for a Volterra integro-differential equation is demonstrated.

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In a paper in this journal Dixon [1] studied a nonlinear weakly singular Volterra integro-differential equation arising from a reaction-diffusion process in a small cell and attempted to demonstrate the convergence of a product integration method. The argument, however, depends crucially upon [1, second part of Lemma 5.2]. However, this is incomplete, based on the incorrect assumption that $f(i) > f(i+1)$, where

$$f(i) = (i^{1/2} - (i-1)^{1/2})\mathrm{e}^{-n^2/(i\Delta t)},$$

n is any positive integer, and Δt is sufficiently small.

We shall require the following notation. Let $t_i = i\,\Delta t$, $i = 0, 1, 2, \ldots, N$, with $N\,\Delta t = T > 0$. Define

$$\gamma^l(i) = \frac{1}{\Delta t}\left(1 + 2\sum_{n=1}^{l} \mathrm{e}^{-n^2/t_i}\right) \int_0^{\Delta t} \frac{\mathrm{d}s}{(t_i - s)^{1/2}}, \quad i = 1, 2, \ldots, N, \tag{1}$$

where l can be any positive integer or infinity.

The result of this note is: $\forall N > 0$, $\exists$ a finite L such that when $l > L$,

$$\gamma^l(i) > \gamma^l(i+1), \quad i = 1, \ldots, N-1. \tag{2}$$

The demonstration of this result is necessary for Dixon's convergence argument.

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First we shall prove that

$$\gamma^{\infty}(i) > \gamma^{\infty}(i+1), \quad i = 1, \dots, N-1. \tag{3}$$

This can be achieved by showing that

$$k(t) = \frac{1}{\sqrt{\pi t}}\left(1 + 2\sum_{n=1}^{\infty} e^{-n^2/t}\right), \quad t > 0,$$

is monotonic decreasing.

One form of the functional equation of the so-called theta function reads (see, e.g., [2, p.15])

$$1 + 2\psi(x) = \left(1 + 2\psi\left(\frac{1}{x}\right)\right)x^{-1/2}, \tag{4}$$

where $\psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$; taking $x = \pi t$ in (4) yields

$$\frac{1}{\sqrt{\pi t}}\left(1 + 2\sum_{n=1}^{\infty} e^{-n^2/t}\right) = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}. \tag{5}$$

Since $k(t)$ is the left-hand side of (5), it is clear that $k(t)$ is monotonic decreasing.

To see how this implies (3), consider the following. Note that for $t > 0$ and $a > 0$, $(1 + a/t)^{1/2}$ is also monotonic decreasing, so for $s \in [(i-1)\,\Delta t, i\,\Delta t)$,

$$\frac{(s + \Delta t)^{1/2}}{s^{1/2}} > \frac{((i+1)\,\Delta t)^{1/2}}{(i\,\Delta t)^{1/2}}.$$

Hence,

$$k(i\,\Delta t)\frac{(i\,\Delta t)^{1/2}}{s^{1/2}} > k((i+1)\,\Delta t)\frac{((i+1)\,\Delta t)^{1/2}}{(s + \Delta t)^{1/2}}, \tag{6}$$

that is,

$$\left(1 + 2\sum_{n=1}^{\infty} e^{-n^2/(i\,\Delta t)}\right)\frac{1}{\Delta t\, s^{1/2}} > \left(1 + 2\sum_{n=1}^{\infty} e^{-n^2/((i+1)\,\Delta t)}\right)\frac{1}{\Delta t\,(s + \Delta t)^{1/2}}. \tag{7}$$

Integrating both sides of (7) from $(i-1)\,\Delta t$ to $i\,\Delta t$ with respect to s gives

$$\gamma^{\infty}(i) > \gamma^{\infty}(i+1), \quad i = 1, \dots, N-1,$$

so,

$$\min_i\{\gamma^{\infty}(i) - \gamma^{\infty}(i+1)\} = 2\delta > 0.$$

But $\gamma^l(i)$ is a monotone increasing convergent series. Hence there exists an L such that when $l > L$,

$$\gamma^{\infty}(i) - \gamma^l(i) < \delta, \quad i = 1, \dots, N-1.$$

Thus for $i = 1, \dots, N-1$ and $l > L$,

$$\gamma^l(i) - \gamma^l(i+1) > (\gamma^{\infty}(i) - \delta) - \gamma^{\infty}(i+1) > \delta.$$

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