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Gradient bounds for a thin film epitaxy equation

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Abstract

We consider a gradient flow modeling the epitaxial growth of thin films with slope selection. The surface height profile satisfies a nonlinear diffusion equation with biharmonic dissipation. We establish optimal local and global wellposedness for initial data with critical regularity. To understand the mechanism of slope selection and the dependence on the dissipation coefficient, we exhibit several lower and upper bounds for the gradient of the solution in physical dimensions $d \le 3$.

Keywords: Epitaxy; Thin film; Maximum principle; Gradient bound

1. Introduction

Let $\nu > 0$. Consider

$$\partial_t h = \nabla \cdot ((|\nabla h|^2 - 1)\nabla h) - \nu \Delta^2 h \tag{1.1}$$

and the 1D version

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$$h_t = (h_x^3 - h_x)_x - \nu h_{xxxx}. \tag{1.2}$$

Eq. (1.1) is a nonlinear diffusion equation which models the epitaxial growth of thin films. It is posed on the spatial domain Ω which can either be the whole space $\mathbb{R}^d$, the L-periodic torus ($L > 0$ is a parameter corresponding to the size of the system) $\mathbb{R}^d/L\mathbb{Z}^d$, or a finite domain in $\mathbb{R}^d$ with suitable boundary conditions. In this work for simplicity we shall be mainly concerned with the 2π-periodic case $\Omega = \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ but our results can be easily generalized to other settings. The function $h = h(t,x) : \mathbb{R} \times \Omega \to \mathbb{R}$ represents the scaled height of a thin film and $\nu > 0$ is positive parameter which is sometimes called the diffusion coefficient. Typically in numerical simulations one is interested in the regime where ν is small so that the nonlinear effects become dominant. The 1D version (1.2) is connected with the Cahn–Hilliard equation:

$$\partial_t u = \Delta(u^3 - u) - \nu\Delta^2 u$$

through the identification $u = \partial_x h$. This connection breaks down for dimension $d \geq 2$.

Define the energy

$$E(h) = \int_\Omega \Big(\frac{1}{4}(|\nabla h|^2 - 1)^2 + \frac{\nu}{2}|\Delta h|^2\Big) dx. \tag{1.3}$$

The equation (1.1) can be regarded as a gradient flow of the energy functional $E(h)$ in $L^2(\Omega)$. In fact, it is easy to check that

$$\frac{d}{dt}E(h) = -\|\partial_t h\|_2^2, \tag{1.4}$$

i.e. the energy is always decreasing in time as far as smooth solutions are concerned. Alternatively one can derive the energy law from (1.1) by multiplying both sides by $\partial_t h$ and integrating by parts. The first term in (1.3) models the Ehrlich–Schowoebel effect [3,12,13]. Formally speaking it forces the slope of the thin film $|\nabla h| \approx 1$. For this reason Eq. (1.1) is often called the growth equation with slope selection. On the other hand, in the literature there are also models "without slope selection", such as

$$\partial_t h = -\nabla \cdot \big(\frac{1}{1 + |\nabla h|^2}\nabla h\big) - \nu\Delta^2 h. \tag{1.5}$$

Heuristically speaking, if in (1.5) the slope $|\nabla h|$ is small, then

$$\frac{1}{1 + |\nabla h|^2} \approx 1 - |\nabla h|^2$$

and one recovers the nonlinearity in (1.1). However this line of argument seems only reasonable when $|\nabla h| \ll 1$ which is a typical transient regime and not so appealing physically. Indeed the long time interfacial dynamics governed by (1.1) and (1.5) can be quite different, see for example the discussion in [5]. The second term in (1.3) corresponds to the fourth-order diffusion in (1.1). It has a stabilizing effect both theoretically and numerically.

Eq. (1.1) can also be viewed as regularized version of the equation

$$\partial_t h = \nabla \cdot ((|\nabla h|^2 - 1)\nabla h). \tag{1.6}$$

The wellposedness of (1.6) is a rather subtle issue. In light of recent developments [1,2], one should expect generic illposedness although the underlying mechanism will be different. However as it turns out, if there is a smooth solution to (1.6) on some finite time interval, then it must admit some form of a maximum principle. We record it here as

Proposition 1.1 *(Maximum principle for smooth solutions to (1.6)). Let the dimension $d \geq 1$ and $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ be the usual 2π-periodic torus. Let $T > 0$ and assume $h \in C_t^1 C_x^2([0, T] \times \mathbb{T}^d)$ is a classical solution to (1.6). Then*

$$\|\nabla h(t, \cdot)\|_\infty \leq \max\{\|\nabla h(0, \cdot)\|_\infty, 1\}, \qquad \forall\, 0 \leq t \leq T. \tag{1.7}$$

If the dimension $d = 1$, then a better bound is available:

$$\|\partial_x h(t, \cdot)\|_\infty \leq \max\{\|\partial_x h(0, \cdot)\|_\infty, \frac{1}{\sqrt{3}}\}, \qquad \forall\, 0 \leq t \leq T. \tag{1.8}$$

We stress that Proposition 1.1 is a *conditional* result, namely it assumes the existence of a smooth solution. On the other hand the wellposedness of classical solutions to the regularized equation (1.1) is much easier to obtain thanks to the fourth order dissipation on the right hand side. In the Fourier space, the biharmonic operator $-\Delta^2$ seems to offer much stronger dissipation and damping effect than the usual Laplacian operator, as can be seen from studying the linear equations

$$\partial_t h = Ah, \quad A = \Delta \text{ or } -\Delta^2.$$

Since equation (1.1) can be viewed as a regularized version of (1.6), it is very natural to stipulate that solutions to (1.1) should behave much better than those to (1.6) from a general perspective. From this heuristics, it is very tempting to expect that Proposition 1.1 also holds for (1.1). Preliminary numerical experiments seem to support this, thus

Conjecture 1. *Let $\nu > 0$. For general smooth initial data h_0, the corresponding solution $h = h(t, x)$ to (1.1) satisfies the bound*

$$\|\nabla h(t)\|_\infty \leq \max\{\|\nabla h_0\|_\infty, 1\}, \qquad \forall\, t > 0.$$

A weaker form of Conjecture 1 is the following:

Conjecture 2. *Let $\nu > 0$. For general smooth initial data h_0, the corresponding solution $h = h(t, x)$ to (1.1) satisfies the bound*

$$\|\nabla h(t)\|_\infty \leq \max\{\|\nabla h_0\|_\infty, \alpha_d\}, \qquad \forall\, t > 0,$$

where $\alpha_d > 0$ is a constant depending only on the dimension d.

Perhaps a better formulation of Conjecture 2 is that $\|\nabla h(t)\|_\infty \le F(\|\nabla h_0\|_\infty, d)$ for some function F independent of (ν, d). The main point in both Conjecture 1 and Conjecture 2 is that the constants in the upper bounds of $\|\nabla h\|_\infty$ are *independent of* ν. If true these gradient bounds can lead to better stability estimates of numerical algorithms (see [15,10,16,14,7–9]).

On the other hand, it is not so difficult to extract a ν-dependent upper bound on $\|\nabla h\|_\infty$, see Corollary 1.2 below.

Perhaps a bit surprisingly, the goal of this paper is to disprove Conjecture 1. Conjecture 2 is still open at the time of this writing. However we shall give a lower bound for the constant in Conjecture 2. Namely, we shall show that $\alpha_d \ge C_d > 1$ for some explicit constant C_d depending on the dimension d.

To make the paper self-contained, we first establish local and global wellposedness for (1.1). For H^2 initial data in dimensions $d = 1, 2, 3$, a fairly satisfactory wellposedness theory has been worked out in [5] using energy estimates and Galerkin approximation. By using the method of mild solutions, our Theorem 1.1 below slightly refines this wellposedness result and allows initial data to be in the "critical" space $H^{\frac{d}{2}}$ which in particular contains H^2 for $d \le 3$. Note that although (1.1) is *not* scale-invariant, in high frequency approximation, one can regard (1.1) as

$$\partial_t h = \nabla \cdot (|\nabla h|^2 \nabla h) - \nu \Delta^2 h. \tag{1.9}$$

To invoke scaling analysis, one can consider (1.9) posed on the whole space $\mathbb{R}^d$. If $h(t, x)$ is a solution to (1.9), then for any $\lambda > 0$,

$$h_\lambda(t, x) = h(\lambda^4 t, \lambda x)$$

is also a solution. From this one can deduce that the critical space for (1.9) is $L_x^\infty(\mathbb{R}^d)$ or $\dot{H}_x^{\frac{d}{2}}(\mathbb{R}^d)$. Thus we have

Theorem 1.1 *(Improved local wellposedness). Let the dimension $d \ge 1$. Consider (1.1) on the 2π-periodic torus $\mathbb{T}^d$ with $\nu > 0$. Let $s_d = d/2$. For any initial data $h_0 \in H^{s_d}(\mathbb{T}^d)$, there exist $T_0 = T(h_0) > 0$ and a unique local solution $h \in C_t^0 H_x^{s_d}$ with $t^{\frac14} \nabla h \in C_t^0 C_x^0$, $t^{\frac14} h \in C_t^0 H_x^{s_d+1}$. Moreover $h(t) \in H_x^m$ for all $m \ge 1$, $0 < t < T_*$, where $0 < T_* \le \infty$ is the maximal lifespan of the local solution. In particular $h(t) \in C_x^\infty$ for all $0 < t < T_*$. If h_0 has mean zero, then $h(t)$ also has mean zero for all $0 < t < T_*$.*

As is well-known, the long time dynamics is dictated by conserved quantities (or conservation laws). For (1.1), the energy dissipation law (1.4) gives a priori H^2 control of the solution with mean zero. Note that if h has mean zero, then $\|h\|_2$ is controlled by $\|\Delta h\|_2$ thanks to the Poincaré inequality. Or one can just prove it directly using the Fourier series. The space H^2 is subcritical in dimensions $d \le 3$ since the corresponding critical space is $H^{\frac{d}{2}}$. Thus

Corollary 1.1 *(Global wellposedness for $d \le 3$). Let the dimension $d = 1, 2, 3$. Consider (1.1) on the 2π-periodic torus $\mathbb{T}^d$ with $\nu > 0$. For any initial data $h_0 \in H^{\frac{d}{2}}(\mathbb{T}^d)$ with mean zero, the corresponding solution $h = h(t, x)$ to (1.1) obtained in Theorem 1.1 exists globally in time.*

Remark 1.1. An interesting open problem is to show the global wellposedness of (1.1) in dimension $d = 4$. In that case H^2 is the critical space.

The following corollary gives gradient bounds on h. For simplicity we assume the initial data $h_0 \in H^2(\mathbb{T}^d)$ so that the energy is well-defined. By using the smoothing effect one can also treat the case $h_0 \in H^{\frac{d}{2}}(\mathbb{T}^d)$ with the help of Theorem 1.1. However the bounds in that case have slightly worse dependence on ν (for initial transient time when the smoothing effect takes place). We shall not dwell on this subtle issue here and focus instead on the long time bounds. In Corollary 1.2 below, we shall only consider the case when the diffusion coefficient ν is not so large (the physically relevant case is $\nu \to 0$), which we denote by the notation $0 < \nu \lesssim 1$. It means $0 < \nu \le \nu_0$ where $\nu_0 > 0$ is some constant of order 1. The numerical value of ν_0 is not so important. For example one can just take $\nu_0 = 1$.

Corollary 1.2 *(Gradient bounds for $d \le 3$). Let the dimension $d = 1, 2, 3$. Consider (1.1) on the 2π-periodic torus $\mathbb{T}^d$ with $0 < \nu \lesssim 1$. Assume $h_0 \in H^2(\mathbb{T}^d)$ with mean zero. Let $h = h(t, x)$ be the corresponding global solution to (1.1). Denote*

$$E_0 = \int_{\mathbb{T}^d} \Big(\frac{1}{2}\nu|\Delta h_0|^2 + \frac{1}{4}(|\nabla h_0|^2 - 1)^2\Big)dx.$$

Then ∇h admits the following bounds: for some absolute constants C_1, C_2, $C_3 > 0$,

$$\sup_{0\le t<\infty} \|\nabla h(t)\|_\infty \le C_1 \nu^{-\frac{1}{6}} E_0^{\frac{1}{6}} (E_0^{\frac{1}{6}} + 1), \qquad \textit{if } d = 1;$$

$$\sup_{1\lesssim t<\infty} \|\nabla h(t)\|_\infty \le C_2 (\frac{E_0}{\nu})^{\frac{1}{2}} |\log(\frac{E_0 + 1}{\nu})|, \qquad \textit{if } d = 2;$$

$$\sup_{1\lesssim t<\infty} \|\nabla h(t)\|_\infty \le C_3 \nu^{-\frac{3}{2}} (E_0 + 1)^{\frac{3}{2}}, \qquad \textit{if } d = 3.$$

Similarly for some absolute constants $C_2' > 0$, $C_3' > 0$,

$$\sup_{0\le t\lesssim 1} \|\nabla h(t) - \nabla e^{-\nu t \Delta^2} h_0\|_\infty \le C_2' \cdot (\frac{E_0}{\nu})^{\frac{1}{2}} |\log(\frac{E_0 + 1}{\nu})|, \qquad \textit{if } d = 2;$$

$$\sup_{0\le t\lesssim 1} \|\nabla h(t) - \nabla e^{-\nu t \Delta^2} h_0\|_\infty \le C_3' \nu^{-\frac{3}{2}} (E_0 + 1)^{\frac{3}{2}}, \qquad \textit{if } d = 3.$$

Remark 1.2. The above gradient bound for $d = 1$ follows trivially from energy law and interpolation inequalities. It does not use the dynamics at all. On the other hand the proof of the bounds for $d = 2, 3$ uses the mild formulation of the equation together with energy law. In terms of the dependence on ν the bounds here seem not optimal. See for example Proposition 5.1–5.2 in §5 for more refined results.

To disprove Conjecture 1, we shall use two different methods. The first method (see Theorem 1.2 and Corollary 1.3 below) gives a weak lower bound approximately of the form $1 + O(\nu)$ (with $O(\nu) > 0$). Even though this already settles Conjecture 1 in the negative, the obtained lower bound approaches to 1 as ν tend to zero which is the drawback of the construction. On the other hand, the second method (see Theorem 1.3) gives a ν-independent lower bound which also yields a lower bound for the constant α_d in Conjecture 2. It is quite possible that these bounds can be improved further.

We now introduce the first construction. To elucidate the main idea, we first state the 1D version.

Theorem 1.2. *Consider (1.2) with $\nu > 0$ and 2π-periodic boundary condition. There exists a family $\mathcal{A}$ of smooth initial data such that the following holds:*

(1) *For any $h_0 \in \mathcal{A}$, we have $\int_{\mathbb{T}} h_0(x)dx = 0$ and $\|\partial_x h_0\|_\infty < 1$.*
(2) *For any $h_0 \in \mathcal{A}$, there exists $t_0 > 0$ (depending on h_0) such that the corresponding solution to (1.2) satisfies*

$$\|\partial_x h(t_0, \cdot)\|_\infty > 1.$$

It is relatively straightforward to generalize the construction in Theorem 1.2 to the equation (1.1) in all dimensions.

Corollary 1.3. *Let the dimension $d \geq 1$ and $\mathbb{T}^d$ be the usual 2π-periodic torus. Consider (1.1) with $\nu > 0$ and on $(t, x) \in [0, \infty) \times \mathbb{T}^d$. There exists a family $\mathcal{A}$ of smooth initial data such that the following holds:*

(1) *For any $h_0 \in \mathcal{A}$, we have $\int_{\mathbb{T}^d} h_0(x)dx = 0$ and $\|\partial_x h_0\|_\infty < 1$.*
(2) *For any $h_0 \in \mathcal{A}$, there exists $t_0 > 0$ (depending on h_0) such that the corresponding solution to (1.1) satisfies*

$$\|\nabla h(t_0, \cdot)\|_\infty > 1.$$

We now introduce the second construction. The key idea builds on examining the linear evolution $e^{-\nu t\Delta^2}$, and treating the nonlinear part as a correction.

Theorem 1.3. *Let the dimension $d \geq 1$ and $\mathbb{T}^d$ be the usual 2π-periodic torus. Consider (1.1) with $\nu > 0$ and on $(t, x) \in [0, \infty) \times \mathbb{T}^d$. There exists a constant $C_d > 1$ depending only on the dimension d, such that for any $\epsilon > 0$, there exists $h_0 \in C^\infty(\mathbb{T}^d)$ for which the following hold:*

(1) *$\int_{\mathbb{T}^d} h_0(x)dx = 0$ and $\|\nabla h_0\|_\infty < 1$.*
(2) *There exists $t_0 > 0$ such that the corresponding solution to (1.2) satisfies*

$$\|\nabla h(t_0, \cdot)\|_\infty > C_d - \epsilon.$$

Remark 1.3. Let $f(x) = \frac{1}{(2\pi)^d} \int\limits_{\mathbb{R}^d} e^{-|\xi|^4} e^{i\xi\cdot x} d\xi$. The constant C_d in Theorem 1.3 is given by $C_d = \|f\|_{L^1_x(\mathbb{R}^d)} > 1$.

Remark 1.4. One can also consider the following version of (1.1) with fractional dissipation:

$$\partial_t h = \nabla \cdot ((|\nabla h|^2 - 1)\nabla h) - \nu|\nabla|^\gamma h, \tag{1.10}$$

where $\gamma > 2$ controls the "order" of dissipation. For $h : \mathbb{T}^d \to \mathbb{R}$, $|\nabla|^\gamma$ can be defined on the Fourier side as

$$\widehat{|\nabla|^\gamma h}(k) = |k|^\gamma \hat{h}(k), \qquad k \in \mathbb{Z}^d.$$

The L^∞-maximum principle holds for the fractional heat propagator $e^{-t|\nabla|^\gamma}$ for $0 \le \gamma \le 2$. The behavior of $e^{-t|\nabla|^\gamma}$ for $\gamma < 2$ and the heat operator $e^{t\Delta}$ can be quite different, see for example [6] for a discussion in the (Littlewood–Paley) frequency-localized context. In the wider setting one can even consider operators of the form $\mathcal{A} = |\nabla|^\gamma / \log^\beta(\lambda + |\nabla|)$ (for $0 \le \gamma \le 2$, $\beta \ge 0$ and $\lambda > 1$) and establish a new generalized maximum principle (see [4]) for the drift equation

$$\partial_t \theta + v \cdot \nabla\theta = -\mathcal{A}\theta,$$

where v is a given arbitrary external velocity field transporting the scalar quantity θ. On the other hand, in the regime $\gamma > 2$, the L^∞-maximum principle is no longer expected since the corresponding fundamental solution may change signs. Based on this, an analogue of Theorem 1.3 is expected to hold for (1.10) when $\gamma > 2$. In that case the constant C_d is replaced by

$$C_{d,\gamma} = \|\mathcal{F}^{-1}(e^{-|\xi|^\gamma})\|_{L_x^1(\mathbb{R}^d)} > 1.$$

2. Notation and preliminaries

In this section we collect some notation and preliminaries used in this paper.

For any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we use the Japanese bracket notation $\langle x \rangle = \sqrt{1 + x_1^2 + \cdots + x_d^2}$.

We denote by $\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$ the 2π-periodic torus.

Let $\Omega = \mathbb{R}^d$ or $\mathbb{T}^d$, $d \ge 1$. For any function $f : \Omega \to \mathbb{R}$, we use $\|f\|_{L^p} = \|f\|_{L^p(\Omega)}$ or sometimes $\|f\|_p$ to denote the usual Lebesgue L^p norm for $1 \le p \le \infty$. If $f = f(x, y) : \Omega_1 \times \Omega_2 \to \mathbb{R}$, we shall denote by $\|f\|_{L_x^{p_1} L_y^{p_2}}$ to denote the mixed-norm:

$$\|f\|_{L_x^{p_1} L_y^{p_2}} = \Big\| \|f(x, y)\|_{L_y^{p_2}(\Omega_2)} \Big\|_{L_x^{p_1}(\Omega_1)}.$$

In a similar way one can define other mixed-norms such as $\|f\|_{C_t^0 H_x^m}$ etc.

For any two quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant C on other parameters or constants are usually clear from the context and we will often suppress this dependence. We denote $X \lesssim_{Z_1, \cdots, Z_m} Y$ if $X \le CY$ where the constant C depends on the parameters $Z_1, \cdots, Z_m$.

We adopt the following convention for Fourier transform pair on $\mathbb{R}^d$:

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx,$$

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\cdot\xi} d\xi.$$

Sometimes the inverse Fourier transform is denoted as $\mathcal{F}^{-1}$.

Also for $f : \mathbb{T}^d \to \mathbb{R}$, and $k \in \mathbb{Z}^d$, we denote the Fourier coefficient

$$\hat{f}(k) = \int_{\mathbb{T}^d} f(x) e^{-ik\cdot x} dx.$$

Of course (under suitable conditions) f can be recovered from the Fourier series:

$$f(x) = \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{Z}^d} \hat{f}(k) e^{ik\cdot x}.$$

Note that if we regard f as a periodic function on $\mathbb{R}^d$, then

$$(\mathcal{F}f)(\xi) = \sum_{k\in\mathbb{Z}^d} \hat{f}(k)\delta(\xi - k), \tag{2.11}$$

where δ is the usual Dirac delta distribution on $\mathbb{R}^d$.

For $f : \mathbb{T}^d \to \mathbb{R}$ and $s \geq 0$, we define the H^s-norm and $\dot{H}^s$-norm of f as

$$\|f\|_{H^s} = \Big(\sum_{k\in\mathbb{Z}^d} (1 + |k|^{2s})|\hat{f}(k)|^2\Big)^{\frac{1}{2}}, \quad \|f\|_{\dot{H}^s} = \Big(\sum_{k\in\mathbb{Z}^d} |k|^{2s}|\hat{f}(k)|^2\Big)^{\frac{1}{2}},$$

provided of course the above sums are finite. If f has mean zero, then $\hat{f}(0) = 0$ and in this case

$$\|f\|_{H^s} \sim \Big(\sum_{k\in\mathbb{Z}^d} |k|^{2s}|\hat{f}(k)|^2\Big)^{\frac{1}{2}}.$$

Occasionally we will need to use the Littlewood–Paley (LP) frequency projection operators. To fix the notation, let $\phi_0 \in C_c^\infty(\mathbb{R}^d)$ and satisfy

$$0 \leq \phi_0 \leq 1, \quad \phi_0(\xi) = 1 \text{ for } |\xi| \leq 1, \quad \phi_0(\xi) = 0 \text{ for } |\xi| \geq 2.$$

Let $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$ which is supported in $1/2 \leq |\xi| \leq 2$. For any $f \in \mathcal{S}'(\mathbb{R}^d)$, $j \in \mathbb{Z}$, define

$$\widehat{\Delta_j f}(\xi) = \phi(2^{-j}\xi)\hat{f}(\xi),$$
$$\widehat{S_j f}(\xi) = \phi_0(2^{-j}\xi)\hat{f}(\xi), \qquad \xi \in \mathbb{R}^d.$$

We recall the Bernstein estimates/inequalities: for $1 \leq p \leq q \leq \infty$,

$$\||\nabla|^s \Delta_j f\|_{L^p(\mathbb{R}^d)} \sim 2^{js}\|\Delta_j f\|_{L^p(\mathbb{R}^d)}, \qquad s \in \mathbb{R};$$
$$\|S_j f\|_{L^q(\mathbb{R}^d)} + \|\Delta_j f\|_{L^q(\mathbb{R}^d)} \lesssim 2^{jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

We also need the Bernstein inequalities for periodic functions. Let $f : \mathbb{T}^d \to \mathbb{R}$ be a smooth function and "lift" f to be a periodic function on $\mathbb{R}^d$. Then in this way $f \in \mathcal{S}'(\mathbb{R}^d)$ and one can define $\Delta_j f$ for any $j \in \mathbb{Z}$. By expressing $\Delta_j f$ in terms of a convolution integral, it is easy to check that $\Delta_j f$ is also a periodic function on $\mathbb{R}^d$ and thus can be identified as a function on $\mathbb{T}^d$. A more "direct" way is just to use (2.11) and recognize $\Delta_j f$ as (on the Fourier side) the partial sum of δ-distributions in a dyadic block. It is then natural to expect that the following "Bernstein"-type inequalities hold (note that the norms are evaluated on $\mathbb{T}^d$): for any $1 \le p \le q \le \infty$,

$$\||\nabla|^s \Delta_j f\|_{L^p(\mathbb{T}^d)} \sim 2^{js} \|\Delta_j f\|_{L^p(\mathbb{T}^d)}, \qquad s \in \mathbb{R}; \tag{2.12}$$

$$\|\Delta_j f\|_{L^q(\mathbb{T}^d)} \lesssim 2^{jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{T}^d)}, \qquad j \in \mathbb{Z}; \tag{2.13}$$

$$\|S_j f\|_{L^q(\mathbb{T}^d)} \lesssim 2^{jd(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{T}^d)}, \qquad j \ge -2. \tag{2.14}$$

If f has mean zero (so that $\hat{f}(0) = 0$), then one does not need the condition $j \ge -2$ (since $S_j f = 0$ for $j < -2$). Although these inequalities are standard, we include the proof here for the sake of completeness.

Proof of (2.12)–(2.14). We shall only prove (2.12)–(2.13). The proof of (2.14) is similar to (2.13).

First we deal with (2.12). For some Schwartz function ψ ($\psi = \mathcal{F}^{-1}(|\xi|^s \phi(\xi))$), we have

$$\begin{aligned}(|\nabla|^s \Delta_j f)(x) &= 2^{js} \int_{\mathbb{R}^d} 2^{jd} \psi(2^j(x-y)) f(y)dy \\ &= 2^{js} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{T}^d} 2^{jd} \psi(2^j(x-y+2\pi k)) f(y)dy \\ &= 2^{js} \int_{\mathbb{T}^d} \tilde{\psi}_j(x-y) f(y)dy,\end{aligned}$$

where $\tilde{\psi}_j(z) = \sum_{k \in \mathbb{Z}^d} 2^{jd} \psi(2^j(z+2\pi k))$ is a periodic function on $\mathbb{R}^d$ (and thus can be identified as a function on $\mathbb{T}^d$). By using Young's inequality on $\mathbb{T}^d$, we get

$$\||\nabla|^s \Delta_j f\|_{L^p(\mathbb{T}^d)} \lesssim 2^{js} \|\tilde{\psi}_j\|_{L^1(\mathbb{T}^d)} \|f\|_{L^p(\mathbb{T}^d)}.$$

Easy to check that

$$\|\tilde{\psi}_j\|_{L^1(\mathbb{T}^d)} \le 2^{jd} \|\psi(2^j z)\|_{L^1_z(\mathbb{R}^d)} = \|\psi\|_{L^1(\mathbb{R}^d)} \lesssim 1.$$

Therefore

$$\||\nabla|^s \Delta_j f\|_{L^p(\mathbb{T}^d)} \lesssim 2^{js} \|f\|_{L^p(\mathbb{T}^d)}.$$

By using a fattened projection $\tilde{\Delta}_j = \sum_{l=-2}^{2} \Delta_{j-l}$ (and noting that $\Delta_j f = \tilde{\Delta}_j \Delta_j f$), one can then derive (2.12).

Next we derive (2.13). By Young's inequality, we have

$$\|\Delta_j f\|_{L^q(\mathbb{T}^d)} \lesssim \|\tilde{\psi}_j\|_{L^r(\mathbb{T}^d)} \|f\|_{L^p(\mathbb{T}^d)},$$

where $\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}$. By (2.11) and $\hat{f}(0) = 0$, easy to check that $\Delta_j f = 0$ if $j < -2$. Therefore we may assume without loss of generality that $j \geq -2$. Then by using the fact that ψ is Schwartz, we get

$$\begin{aligned}
&\|\sum_{k\in\mathbb{Z}^d} 2^{jd}\psi(2^j(z+2\pi k))\|_{L_z^r(\mathbb{T}^d)} \\
&\lesssim \sum_{|k|\leq 100} 2^{jd}\|\psi(2^j(z+2\pi k))\|_{L_z^r(\mathbb{T}^d)} + \sum_{|k|>100} 2^{jd}\langle 2^j k\rangle^{-100d} \\
&\lesssim 2^{jd}\|\psi(2^j z)\|_{L_z^r(\mathbb{R}^d)} + 1 \lesssim 2^{jd}2^{-j\frac{d}{r}}.
\end{aligned}$$

Thus (2.13) is proved. □

3. Proof of Proposition 1.1

For $0 \leq t \leq T$, consider $f(t,x) = |\nabla h(t,x)|^2$. Note that

$$\partial_t h = (f-1)\Delta h + \nabla f \cdot \nabla h.$$

Clearly $\partial_t \nabla h = (\Delta h)\nabla f + (f-1)\Delta\nabla h + \sum_{j=1}^d \partial_j \nabla h \partial_j f + \sum_{j=1}^d \partial_j h \partial_j \nabla f$.

Therefore

$$\begin{aligned}
\frac{1}{2}\partial_t f &= \nabla h \cdot \partial_t \nabla h \\
&= \Delta h(\nabla h\cdot\nabla f) + (f-1)(\nabla h)\cdot(\Delta\nabla h) + \sum_{j=1}^d (\nabla h\cdot\partial_j\nabla h)\partial_j f \\
&\quad + \sum_{j=1}^d \partial_j h(\nabla h\cdot\partial_j\nabla f) \\
&= \Delta h(\nabla h\cdot\nabla f) + (f-1)(\nabla h)\cdot(\Delta\nabla h) + \frac{1}{2}|\nabla f|^2 + \sum_{j,k=1}^d \partial_j h\partial_k h\partial_{jk} f. \qquad (3.15)
\end{aligned}$$

By definition, it is easy to check that

$$\Delta f = 2\nabla\Delta h\cdot\nabla h + 2\sum_{k,j=1}^d (\partial_k\partial_j h)^2.$$

Therefore $\nabla h \cdot \Delta\nabla h = \frac{1}{2}\Delta f - \sum_{k,j=1}^d (\partial_k\partial_j h)^2$.

Plugging this expression into (3.15), we then obtain

$$\begin{aligned}\frac{1}{2}\partial_t f &= \frac{1}{2}(f-1)\Delta f - (f-1)\sum_{k,j=1}^{d}(\partial_k\partial_j h)^2 + \Delta h(\nabla h\cdot\nabla f)\\&\quad + \frac{1}{2}|\nabla f|^2 + \sum_{k,j=1}^{d}\partial_j h\partial_k h\partial_j\partial_k f.\end{aligned}$$

Now let $\epsilon > 0$ be a small parameter which will tend to zero later. Consider the auxiliary function

$$f^{\epsilon}(t,x) = f(t,x) - \epsilon t, \quad \forall 0 \le t \le T,\ x \in \mathbb{T}^d.$$

Note the equation for f^{ϵ} reads as

$$\begin{aligned}\frac{1}{2}\partial_t f^{\epsilon} &= -\frac{1}{2}\epsilon + \frac{1}{2}(f^{\epsilon}+\epsilon t-1)\Delta f^{\epsilon} - (f^{\epsilon}+\epsilon t-1)\sum_{k,j=1}^{d}(\partial_k\partial_j h)^2\\&\quad + \Delta h(\nabla h\cdot\nabla f^{\epsilon}) + \frac{1}{2}|\nabla f^{\epsilon}|^2 + \sum_{k,j=1}^{d}\partial_j h\partial_k h\partial_j\partial_k f^{\epsilon}.\end{aligned} \tag{3.16}$$

Since f^{ϵ} is a continuous function on the compact domain $[0,T]\times\mathbb{T}^d$, it must achieve its maximum at some point (t_*, x_*), i.e.

$$\max_{0\le t\le T,\ x\in\mathbb{T}^d} f^{\epsilon}(t,x) = f^{\epsilon}(t_*,x_*) =: M_{\epsilon}.$$

We discuss several cases.

Case 1. $0 < t_* \le T$ and $M_{\epsilon} > 1$. In this case observe that

$$\begin{aligned}&\nabla f^{\epsilon}(t_*,x_*) = 0, \quad \Delta f^{\epsilon}(t_*,x_*) \le 0,\\&\sum_{k,j=1}^{d} c_j c_k(\partial_j\partial_k f^{\epsilon})(t_*,x_*) \le 0, \quad \text{for any } (c_1,\cdots,c_d)\in\mathbb{R}^d.\end{aligned}$$

Therefore by (3.16) and the fact that $M_{\epsilon} > 1$, we have

$$\begin{aligned}\frac{1}{2}(\partial_t f^{\epsilon})(t,x)\Big|_{(t_*,x_*)} &\le -\frac{1}{2}\epsilon + \frac{1}{2}(M_{\epsilon}+\epsilon t_*-1)(\Delta f^{\epsilon})(t_*,x_*)\\&\quad - (M_{\epsilon}+\epsilon t-1)\sum_{k,j=1}^{d}(\partial_k\partial_j h)^2\\&\le -\frac{1}{2}\epsilon < 0.\end{aligned}$$

This obviously contradicts to the fact that $0 < t_* \le T$ and (t_*, x_*) is a maximum. Hence Case 1 is impossible.

Case 2. $0 < t_* \le T$ and $M_\epsilon \le 1$. In this case we obtain the bound

$$\max_{0\le t\le T,\, x\in\mathbb{T}^d} f(t,x) \le \epsilon T + 1.$$

Case 3. $t_* = 0$. Clearly then

$$\max_{0\le t\le T,\, x\in\mathbb{T}^d} f(t,x) \le \max_{x\in\mathbb{T}^d} f(0,x) + \epsilon T.$$

Concluding from all cases and sending ϵ to zero, we obtain (1.7).

In the case dimension $d = 1$, the proof of (1.8) is similar. Set $g = h_x$. Note that

$$\partial_t g = (g^3 - g)_{xx} = (3g^2 - 1)g_{xx} + 6g(g_x)^2.$$

Clearly $(3g^2 - 1)g_{xx}$ is elliptic when $3g^2 > 1$, whence

$$\|g(t)\|_\infty \le \max\{\|g(0)\|_\infty, \frac{1}{\sqrt{3}}\}, \qquad \forall t \ge 0.$$

4. Proof of Theorem 1.1

Lemma 4.1. *Let $\nu > 0$ and $L = -\nu\Delta^2$. Then for any integer $m \ge 1$ and any $t > 0$, we have*

$$\|D^m e^{tL} f\|_{L_x^\infty(\mathbb{T}^d)} \lesssim_{\nu,d,m} (1 + t^{-\frac{m}{4}})\|f\|_{H_x^{d/2}(\mathbb{T}^d)}; \tag{4.17}$$

similarly for any integer $m \ge 0$ and any $t > 0$,

$$\|D^m e^{tL} f\|_{L_x^\infty(\mathbb{T}^d)} \lesssim_{\nu,d,m} t^{-\frac{m}{4}}\|f\|_{L_x^\infty(\mathbb{T}^d)}, \tag{4.18}$$

$$\|D^m e^{tL} f\|_{L_x^2(\mathbb{T}^d)} \lesssim_{\nu,d,m} (1 + t^{-\frac{m}{4}})\|f\|_{L_x^2(\mathbb{T}^d)}. \tag{4.19}$$

In the above D^m denotes any differential operator of order m. For example D^2 can be any one of the operators $\partial_{x_i x_j}$, $1 \le i, j \le d$.

If f has mean zero, then (4.17) and (4.19) can be improved as:

$$\|D^m e^{tL} f\|_\infty \lesssim_{\nu,d,m} t^{-\frac{m}{4}}\|f\|_{H^{\frac{d}{2}}}, \quad \forall m \ge 1,\ t > 0, \tag{4.20}$$

$$\|D^m e^{tL} f\|_2 \lesssim_{\nu,d,m} t^{-\frac{m}{4}}\|f\|_2, \qquad \forall m \ge 0,\ t > 0. \tag{4.21}$$

Proof. We first show (4.17). Define $\langle\nabla\rangle = \sqrt{1-\Delta}$. Clearly

$$D^m e^{tL} f = D^m e^{tL}\langle\nabla\rangle^{-\frac{d}{2}}\langle\nabla\rangle^{\frac{d}{2}} f = K_1 * (\langle\nabla\rangle^{\frac{d}{2}} f)$$

where $*$ denotes the usual convolution and K_1 is the kernel corresponding to $D^m e^{tL}\langle\nabla\rangle^{-\frac{d}{2}}$. Then

$$\|D^m e^{tL} f\|_{L_x^\infty(\mathbb{T}^d)} \lesssim \|K_1\|_{L_x^2(\mathbb{T}^d)} \|f\|_{H_x^{\frac{d}{2}}(\mathbb{T}^d)}.$$

Now since $m \ge 1$,

$$\|K_1\|_{L_x^2}^2 \lesssim \sum_{k\in\mathbb{Z}^d} e^{-2\nu t|k|^4} |k|^{2m} \cdot \langle k\rangle^{-d} \lesssim 1 + \sum_{k\neq 0} e^{-2\nu t|k|^4} |k|^{2m-d} \lesssim 1 + t^{-\frac{m}{2}}.$$

Thus (4.17) follows easily.

For (4.18), we can regard f as a periodic function on $\mathbb{R}^d$. Then using the fact that for any multi-index α with $|\alpha| = m$, $\|\mathcal{F}^{-1}(\xi^\alpha e^{-t|\xi|^4})\|_{L_x^1(\mathbb{R}^d)} \lesssim t^{-\frac{m}{4}}$, we get

$$\|D^m e^{tL} f\|_{L_x^\infty(\mathbb{T}^d)} = \|D^m e^{tL} f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim t^{-\frac{m}{4}} \|f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim t^{-\frac{m}{4}} \|f\|_{L_x^\infty(\mathbb{T}^d)}.$$

Similarly one can prove (4.19) by computing everything on the Fourier side.

In the case f has mean zero, we note that $\hat{f}(0) = 0$, and (4.20)–(4.21) follows easily. □

Proof of Theorem 1.1. This is more or less a standard application of the theory of mild solutions. Therefore we shall only sketch the details.

We recast (1.1) into the mild form (alternatively one can also construct the mild solution by considering $L = -\nu\Delta^2 - \Delta$ as the linear part and taking e^{tL} as the linear propagator):

$$\begin{aligned} h(t) &= e^{-t\nu\Delta^2} h_0 + \sum_{j=1}^d \int_0^t \partial_j e^{-(t-s)\nu\Delta^2} ((|\nabla h|^2 - 1)\partial_j h)(s) ds \\ &=: e^{-t\nu\Delta^2} h_0 + \Phi(h)(t). \end{aligned}$$

Fix $h_0 \in H^{d/2}(\mathbb{T}^d)$. Define $h^{(0)} = e^{-t\nu\Delta^2} h_0$, and for $j \ge 1$,

$$h^{(j)}(t) = e^{-t\nu\Delta^2} h_0 + \Phi(h^{(j-1)})(t).$$

For $T > 0$, introduce the Banach space

$$X_T = \left\{ h \in C_t^0 H_x^{\frac{d}{2}}([0,T] \times \mathbb{T}^d) : t^{\frac{1}{4}} \nabla h \in C_t^0 C_x^0,\ t^{\frac{1}{4}} h \in C_t^0 H_x^{\frac{d}{2}+1} \right\}$$

with the norm

$$\|h\|_{X_T} = \|h\|_{C_t^0 H_x^{\frac{d}{2}}} + \|t^{\frac{1}{4}} \nabla h\|_{L_{t,x}^\infty} + \|t^{\frac{1}{4}} h\|_{C_t^0 H_x^{\frac{d}{2}+1}}.$$

For convenience denote the seminorm

$$\|h\|_{Y_T} = \|t^{\frac{1}{4}} \nabla h\|_{L_{t,x}^\infty} + \|t^{\frac{1}{4}} h\|_{C_t^0 H_x^{\frac{d}{2}+1}}.$$

We shall show that for sufficiently small $T > 0$ (depending on the profile of h_0), the iterates $h^{(j)}$, $j \ge 0$ form a Cauchy sequence in the set

$$B_T = \{h \in X_T : \|h\|_{X_T} \le 2\|h_0\|_{H^{\frac{d}{2}}(\mathbb{T}^d)}, \ \|h\|_{Y_T} \le 2\epsilon_1 \|h_0\|_{H^{\frac{d}{2}}(\mathbb{T}^d)}\},$$

where $\epsilon_1 > 0$ is a sufficiently small constant depending only on (ν, d) and $\|h_0\|_{H^{\frac{d}{2}}}$.

We shall only verify that $h^{(j)} \in B_T$ and omit the contraction argument since it is quite similar.

Consider first $j = 0$. For $h_0 \in H^{\frac{d}{2}}(\mathbb{T}^d)$, obviously

$$\|e^{-\nu\Delta^2 t} h_0\|_{C_t^0 H_x^{\frac{d}{2}}} \le \|h_0\|_{H^{\frac{d}{2}}}.$$

By Lemma 4.1 and a density argument, we have for $h_0 \in H^{\frac{d}{2}}$,

$$\lim_{t\to 0+} \|t^{\frac14} \nabla e^{-\nu t\Delta^2} h_0\|_{L_x^\infty} = 0, \qquad \lim_{t\to 0+} \|t^{\frac14} e^{-\nu t\Delta^2} h_0\|_{H_x^{\frac{d}{2}+1}} = 0.$$

Thus for $T > 0$ sufficiently small,

$$\|h^{(0)}\|_{X_T} \le \frac{3}{2}\|h_0\|_{H^{\frac{d}{2}}}, \qquad \|h^{(0)}\|_{Y_T} \le \epsilon_1 \|h_0\|_{H^{\frac{d}{2}}},$$

where ϵ_1 will be taken sufficiently small (depending on (ν, d) and $\|h_0\|_{H^{\frac{d}{2}}}$) later when we verify the estimates for $h^{(j)}$, $j \ge 1$.

Now inductively assume $h^{(j-1)} \in B_T$. To show $h^{(j)} \in B_T$, it suffices for us to check

$$\|\Phi(h^{(j-1)})\|_{X_T} \le \epsilon_1 \|h_0\|_{H^{\frac{d}{2}}}.$$

To simplify notation, in the computation below we shall drop the superscript $(j-1)$ and write $\Phi(h^{(j-1)})$ simply as $\Phi(h)$. We also write $\lesssim_{\nu,d}$ simply as $\lesssim$.

Note that without loss of generality we can assume $t \lesssim 1$, so that when applying Lemma 4.1, we have $1 + t^{-\frac{m}{4}} \lesssim t^{-\frac{m}{4}}$ (i.e. the constant 1 is not needed). Now by Lemma 4.1, we have

$$\begin{aligned}
\|\Phi(h)(t)\|_{H_x^{\frac{d}{2}}} &\lesssim \Big\| \int_0^t \langle\nabla\rangle^{\frac{d}{2}} \nabla \cdot e^{-(t-s)\nu\Delta^2} \big(|\nabla h|^2 - 1)\nabla h\big)(s) ds \Big\|_2 \\
&\lesssim \int_0^t (t-s)^{-\frac14} \|\langle\nabla\rangle^{\frac{d}{2}} \nabla h(s)\|_2 ds \\
&\quad + \int_0^t (t-s)^{-\frac14} \|\langle\nabla\rangle^{\frac{d}{2}} \big(|\nabla h(s)|^2 \nabla h(s)\big)\|_2 ds \\
&\lesssim \int_0^t (t-s)^{-\frac14} s^{-\frac14} ds \cdot \|s^{\frac14} h(s)\|_{C_s^0 H_x^{\frac{d}{2}+1}} \\
&\quad + \int_0^t (t-s)^{-\frac14} s^{-\frac34} ds \cdot \|s^{\frac14} h(s)\|_{C_s^0 H_x^{\frac{d}{2}+1}} \cdot \|s^{\frac14} \nabla h(s)\|^2_{L_s^\infty L_x^\infty}
\end{aligned}$$

$$\lesssim t^{\frac{1}{2}} \|s^{\frac{1}{4}} h(s)\|_{C_s^0 H_x^{\frac{d}{2}+1}} + \|s^{\frac{1}{4}} h(s)\|_{C_s^0 H_x^{\frac{d}{2}+1}} \cdot \|s^{\frac{1}{4}} \nabla h(s)\|_{L_s^\infty L_x^\infty}^2$$

$$\lesssim t^{\frac{1}{2}} \|h_0\|_{H^{\frac{d}{2}}} + \|h\|_{Y_t}^3.$$

Thus for $T > 0$ sufficiently small and ϵ_1 sufficiently small,

$$\|\Phi(h)\|_{C_t^0 H_x^{\frac{d}{2}}([0,T]\times\mathbb{T}^d)} \le \frac{\epsilon_1}{10} \|h_0\|_{H^{\frac{d}{2}}}.$$

Similarly easy to check that

$$\|t^{\frac{1}{4}} \Phi(h)(t)\|_{C_t^0 H_x^{\frac{d}{2}+1}([0,T]\times\mathbb{T}^d)} + \|t^{\frac{1}{4}} \nabla \Phi(h)(t)\|_{L_{t,x}^\infty([0,T]\times\mathbb{T}^d)} \le \frac{\epsilon_1}{5} \|h_0\|_{H^{\frac{d}{2}}}.$$

Thus

$$\|\Phi(h)\|_{X_T} \le \epsilon_1 \|h_0\|_{H^{\frac{d}{2}}}.$$

We have finished the proof of existence and uniqueness of a solution in the Banach space X_T. $\square$

The smoothing estimate of $h(t)$ for $t > 0$ is utterly standard. For example if we know $h \in L_t^\infty H_x^m([t_0, t_1] \times \mathbb{T}^d)$ on some time interval $[t_0, t_1]$, then for $t \in (t_0, t_1]$,

$$\left\| D^{m+1} \int_{t_0}^{t} \nabla \cdot e^{-(t-s)\nu\Delta^2} ((|\nabla h|^2 - 1)\nabla h)(s) ds \right\|_2$$

$$\lesssim \int_{t_0}^{t} (t-s)^{-\frac{3}{4}} \|(|\nabla h(s)|^2 - 1)\nabla h(s)\|_{H^{m-1}} ds$$

$$\lesssim \int_{t_0}^{t_1} (t-s)^{-\frac{3}{4}} ds \cdot \|h\|_{L_s^\infty H_x^m}$$

$$+ \int_{t_0}^{t_1} (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} ds \cdot \|h\|_{L_s^\infty H_x^m} \cdot \|s^{\frac{1}{4}} \nabla h\|_{L_s^\infty L_x^\infty}^2.$$

This shows that h has higher regularity H_x^{m+1} on $(t_0, t_1]$ (the linear part

$$e^{-(t-t_0)\nu\Delta^2} h(t_0) \in H_x^{m+1}$$

only for $t \in (t_0, t_1]$). We omit further details.

5. Proof of Corollary 1.1 and Corollary 1.2

Proof of Corollary 1.1. Let the dimension $d \le 3$.

We first assume that the initial data $h_0 \in H^4(\mathbb{T}^d)$ with mean zero. Denote the corresponding solution obtained by Theorem 1.1 as h. To bound $\|\partial_t h\|_2$, we need to control $\|\partial^2 h \cdot \partial h \cdot \partial h\|_2 \lesssim \|\partial^2 h\|_2 \|\partial h\|_\infty^2 \lesssim \|h\|_{H^4}^2$. The H^4 regularity is used to control $\|\nabla h\|_\infty$. It is then easy to check that $h \in C_t^0 H_x^4 \cap C_t^1 L_x^2$ and

$$\frac{d}{dt} E = -\|\partial_t h\|_2^2, \tag{5.22}$$

where

$$E(t) = \frac{1}{2}\nu \|\Delta h(t)\|_2^2 + \frac{1}{4}\int_{\mathbb{T}^d} (|\nabla h(t)|^2 - 1)^2 dx.$$

Alternatively to avoid the issue of differentiability, one can interpret (5.22) as the integral formulation: $E(t_2) = E(t_1) - \int_{t_1}^{t_2} \|\partial_t h\|_2^2 dt$ for any $0 \le t_1 < t_2$.

From energy conservation we get $\|h(t)\|_{H^2} \lesssim \|h_0\|_{H^2}$ for any $t > 0$. Now for H^2 initial data (recall the critical space in Theorem 1.1 is $H^{d/2}$ and $d/2 < 2$ for $d \le 3$), the lifespan of the local solution depends on the H^2-norm of the initial data. Thanks to this fact and the estimate $\|h(t)\|_{H^2} \lesssim \|h_0\|_{H^2}$, the corresponding local solution can be continued for all time by a standard argument. This concludes the proof of global wellposedness under the assumption that $h_0 \in H^4$. □

Now let $h_0 \in H^{\frac{d}{2}}(\mathbb{T}^d)$ with mean zero. By Theorem 1.1, there exists a local solution h on $[0, T_0]$ for some $T_0 > 0$ depending on h_0. Let $h_1 = h(T_0/2)$. By Theorem 1.1, $h_1 \in H^m$ for all $m \ge 1$. In particular $h_1 \in H^4$. Now with h_1 as initial data, the corresponding solution can be denoted as $\tilde{h}(t) = h(t + T_0/2)$. One can then repeat the argument described in the previous paragraph to obtain global wellposedness.

Proof of Corollary 1.2. The 1D case. Note that by energy law we have $E(t) \le E_0$. Thus

$$\|\partial_{xx} h(t)\|_2 \lesssim \frac{1}{\sqrt{\nu}}\sqrt{E_0}, \quad \|\partial_x h(t)\|_4 \lesssim E_0^{\frac{1}{4}} + 1.$$

By using the Gagliardo–Nirenberg interpolation inequality, we have

$$\|\partial_x h\|_\infty \lesssim \|\partial_x h\|_4^{\frac{2}{3}} \|\partial_{xx} h\|_2^{\frac{1}{3}}.$$

Therefore

$$\|\partial_x h(t)\|_\infty \lesssim \nu^{-\frac{1}{6}} E_0^{\frac{1}{6}} (E_0^{\frac{1}{6}} + 1).$$

The 2D case. We first perform a short time estimate. Let $0 < \epsilon < 1$ which will be taken sufficiently small. Consider

$$h(t) = e^{-\nu t \Delta^2} h_0 + \int_0^t \nabla \cdot e^{-\nu(t-s)\Delta^2} (|\nabla h|^2 - 1) \nabla h(s) ds.$$

Easy to check that in 2D, $\||\nabla|^{1+\frac{\epsilon}{100}} h\|_\infty \lesssim \||\nabla|^{2+2\epsilon} h\|_{2-\epsilon}$ (recall h has mean zero). Then

$$\begin{aligned}
\||\nabla|^{2+2\epsilon} h(t)\|_{2-\epsilon} &\lesssim \||\nabla|^{2\epsilon} e^{-\nu t \Delta^2} |\nabla|^2 h_0\|_{2-\epsilon} \\
&\quad + \int_0^t \||\nabla|^{2+2\epsilon} \nabla \cdot e^{-\nu(t-s)\Delta^2} ((|\nabla h|^2 - 1)\nabla h)(s) ds\|_{2-\epsilon} ds \\
&\lesssim (\nu t)^{-2\epsilon} \|h_0\|_{H^2} + \int_0^t (\nu(t-s))^{-\frac{3+2\epsilon}{4}} (\|h(s)\|_{H^2}^3 + \|h(s)\|_{H^2}) ds \\
&\lesssim (\nu t)^{-2\epsilon} (\frac{E_0}{\nu})^{\frac{1}{2}} + \nu^{-\frac{3+2\epsilon}{4}} t^{\frac{1-2\epsilon}{4}} ((\frac{E_0}{\nu})^{\frac{1}{2}} + (\frac{E_0}{\nu})^{\frac{3}{2}}).
\end{aligned}$$

In the above when bounding the nonlinearity, we used the estimate

$$\||\nabla h|^2 \nabla h\|_{2-\epsilon} \lesssim \|\nabla h\|_2 \|\nabla h\|_{\frac{2-\epsilon}{\epsilon}}^2 \lesssim \|h\|_{H^2}^3.$$

Thus for $t \sim 1$ and $0 < \nu \lesssim 1$, we get

$$\||\nabla|^{1+\frac{\epsilon}{100}} h(t)\|_\infty \lesssim (\frac{E_0 + 1}{\nu})^{10}.$$

By repeating the same analysis with $t \gg 1$ and h_0 replaced by $h(t-1)$ (note that only $\|h\|_{H^2}$ enters the analysis), we get for all $t \gtrsim 1$

$$\||\nabla|^{1+\frac{\epsilon}{100}} h(t)\|_\infty \lesssim (\frac{E_0 + 1}{\nu})^{10}.$$

Now note that $\|h(t)\|_{H^2} \lesssim (\frac{E_0}{\nu})^{\frac{1}{2}}$. Using Littlewood–Paley decomposition (note that $S_{-2}\nabla h = 0$), we get

$$\begin{aligned}
\|\nabla h(t)\|_{L^\infty(\mathbb{T}^2)} &\lesssim \sum_{-2 \le j \le j_0} \|\Delta_j \nabla h\|_{L^\infty(\mathbb{T}^2)} + \sum_{j > j_0} \|\Delta_j \nabla h\|_{L^\infty(\mathbb{T}^2)} \\
&\lesssim (j_0 + 3)\|h\|_{H^2} + 2^{-j_0 \frac{\epsilon}{100}} \||\nabla|^{1+\frac{\epsilon}{100}} h\|_\infty \\
&\lesssim (j_0 + 3)(\frac{E_0}{\nu})^{\frac{1}{2}} + 2^{-j_0 \frac{\epsilon}{100}} (\frac{E_0 + 1}{\nu})^{10}.
\end{aligned}$$

Optimizing in j_0, we get

$$\sup_{1 \lesssim t < \infty} \|\nabla h(t)\|_\infty \lesssim (\frac{E_0}{\nu})^{\frac{1}{2}} |\log(\frac{E_0 + 1}{\nu})|.$$

Now to obtain the estimate for $t \lesssim 1$, we simply note that for $t \lesssim 1$, by repeating the analysis before,

$$\||\nabla|^{1+\frac{\epsilon}{100}}(h(t) - e^{-\nu t\Delta^2}h_0)\|_\infty \lesssim \left(\frac{E_0+1}{\nu}\right)^{10}.$$

On the other hand,

$$\|h(t) - e^{-\nu t\Delta^2}h_0\|_{H^2} \lesssim \|h\|_{H^2} + \|h_0\|_{H^2} \lesssim (\frac{E_0}{\nu})^{\frac{1}{2}}.$$

Thus we obtain the same bound for $h(t) - e^{-\nu t\Delta^2}h_0$.

This finishes the estimate for the 2D case.

The 3D case. We shall again perform a short time estimate. Write

$$\nabla h(t) = e^{-\nu t\Delta^2}\nabla h_0 + \int_0^t \nabla\nabla\cdot e^{-\nu(t-s)\Delta^2}((|\nabla h|^2 - 1)\nabla h)(s)ds.$$

It is easy to check that

$$\|e^{-\nu\Delta^2 t}\nabla h_0\|_{L_x^\infty(\mathbb{T}^3)} \lesssim (\nu t)^{-\frac{1}{8}}\|h_0\|_{H_x^2(\mathbb{T}^3)}.$$

We then get for $t \lesssim 1$,

$$\begin{aligned}\|\nabla h(t)\|_\infty &\lesssim (\nu t)^{-\frac{1}{8}}\|h_0\|_{H^2} + \int_0^t (\nu(t-s))^{-\frac{7}{8}}(\|\nabla h(s)\|_6^3 + \|\nabla h(s)\|_2)ds \\ &\lesssim t^{-\frac{1}{8}}\nu^{-\frac{5}{8}}E_0^{\frac{1}{2}} + \nu^{-\frac{7}{8}}t^{\frac{1}{8}}(\nu^{-\frac{3}{2}}E_0^{\frac{3}{2}} + 1).\end{aligned}$$

Choosing $t \sim \nu^7$ then yields $\|\nabla h(t)\|_\infty \lesssim \nu^{-\frac{3}{2}}(E_0^{\frac{3}{2}} + 1)$. For general $t \gg \nu^7$, we can replace h_0 by $h(t - \nu^7)$ and repeat the above analysis. This ends the estimate for the 3D case. □

The following proposition shows that in 1D, there exists initial data such that the corresponding solution obeys uniform in time gradient bounds which are independent of ν.

Proposition 5.1. *Let the dimension $d = 1$. Consider (1.2) on the 2π-periodic torus $\mathbb{T}$ with $0 < \nu \lesssim 1$. Assume $h_0 \in H^2(\mathbb{T})$ with mean zero and let $h = h(t,x)$ be the corresponding global solution to (1.2). Denote*

$$E_0 = \int_{\mathbb{T}^d} \Big(\frac{1}{2}\nu|\partial_{xx}h_0|^2 + \frac{1}{4}(|\partial_x h_0|^2 - 1)^2\Big)dx.$$

Then for all $t > 0$ and some absolute constant $C_1 > 0$,

$$\|\partial_x h(t)\|_\infty \le C_1 \max\{1, \nu^{-\frac{1}{6}}E_0^{\frac{1}{3}}\}. \tag{5.23}$$

For each $0 < \nu \lesssim 1$, there exists a family $\mathcal{A}_\nu$ of initial data, such that if $h_0 \in \mathcal{A}_\nu$, then $E_0 \lesssim \sqrt{\nu}$, and the corresponding solution satisfies

$$\|\partial_x h(t)\|_\infty \le B_1, \qquad \forall t \ge 0,$$

where $B_1 > 0$ is an absolute constant. (In particular, it is independent of ν.)

Proof of Proposition 5.1. We first show (5.23). Denote $\|h_x\|_\infty = A$ and $g = h_x^2 - 1$. If $A \le 2$ we are done. Now assume $A > 2$, then obviously $A^2 \lesssim \|g\|_\infty$. Now by Gagliardo–Nirenberg interpolation, we get

$$A^2 \lesssim \|g\|_\infty \lesssim \|g\|_2^{\frac{1}{2}} \|\partial_x g\|_2^{\frac{1}{2}} \lesssim \|g\|_2^{\frac{1}{2}} \|\partial_{xx} h\|_2^{\frac{1}{2}} \|\partial_x h\|_\infty^{\frac{1}{2}} \lesssim \|g\|_2^{\frac{1}{2}} \|\partial_{xx} h\|_2^{\frac{1}{2}} A^{\frac{1}{2}}.$$

Thus

$$A \lesssim \|g\|_2^{\frac{1}{3}} \|\partial_{xx} h\|_2^{\frac{1}{3}} \lesssim E_0^{\frac{1}{6}} (\frac{E_0}{\nu})^{\frac{1}{6}} \lesssim \nu^{-\frac{1}{6}} E_0^{\frac{1}{3}}.$$

We now show that there exists initial data h_0 such that $E_0 \lesssim \sqrt{\nu}$. The idea is to mollify the "sawtooth"-type profile and add a δ-cap ($\delta \approx \sqrt{\nu}$) around each tips of the sawtooth. To this end, let $L_0 \ge 3$ be an integer and define

$$g_0(x) = \int_0^x \operatorname{sgn}(\sin(L_0 \tau)) d\tau, \qquad x \in [-\pi, \pi],$$

where sgn is the usual sign function:

$$\operatorname{sgn}(z) = \begin{cases} 1, & z > 0, \\ 0, & z = 0, \\ -1, & z < 0. \end{cases}$$

The value of L_0 is not important as long as it is independent of ν.

Now around each local maxima or minima of g_0, easy to check that g_0' change its sign from -1 to 1, or 1 to -1. At the maxima (minima), g_0' is undefined. One can then mollify g_0 therein within a δ-neighborhood. Denote the mollified function as g_δ. Then

$$E(g_\delta) = \int_{\mathbb{T}} \Big(\frac{1}{2} \nu |\partial_{xx} g_\delta|^2 + \frac{1}{4} (|\partial_x g_\delta|^2 - 1)^2 \Big) dx \lesssim_{L_0} \nu \cdot \frac{1}{\delta^2} \cdot \delta + \delta.$$

Choosing $\delta \sim \sqrt{\nu}$ then yields $E(g_\delta) \lesssim_{L_0} \sqrt{\nu}$. □

Proposition 5.2. *Let the dimension $d = 1$. Consider (1.2) on the 2π-periodic torus $\mathbb{T}$ with $0 < \nu \lesssim 1$. Assume $h_0 \in H^{\frac{1}{2}}(\mathbb{T})$ with mean zero and let $h = h(t,x)$ be the corresponding global solution to (1.2). Then*

$$\limsup_{t \to \infty} \|\partial_x h(t)\|_\infty \le K_0, \tag{5.24}$$

where K_0 is a constant depending only on the initial data h_0. If in additional h_0 is even in x, then (5.24) can be improved to

$$\limsup_{t\to\infty} \|\partial_x h(t)\|_\infty \le 1. \tag{5.25}$$

Remark 5.1. Recall that in the 1D case, the equation (1.2) can be transformed into the usual Cahn–Hilliard equation via the change of variable $u = \partial_x h$. The convergence to steady states (and consequently gradient bounds) can be obtained using the Łojasiewicz–Simon inequality (cf. [11]). Our proof below however does not appeal to this theory and gives an alternative approach.

Proof of Proposition 5.2. First observe that by using Theorem 1.1 and a shift in time we may assume $h_0 \in H^{10}(\mathbb{T})$. By using the Duhamel formula

$$h(t) = e^{-\nu t\partial_x^4} h_0 + \int_0^t e^{-\nu(t-s)\partial_x^4} \partial_x((h_x^2 - 1)h_x)(s)ds,$$

the energy law, and the exponential (in time) decay of the propagator $e^{-\nu(t-s)\partial_x^4}$ (acting on mean-zero functions), it is not difficult to derive that

$$\sup_{t\ge 0} \|h(t)\|_{H^{10}(\mathbb{T})} \lesssim_{\nu,E_0} 1. \tag{5.26}$$

This estimate will be used below.

Step 1: we show that $\lim_{t\to\infty} \|\partial_t h\|_\infty = 0$. Denote $g = \partial_t h$, then g satisfies the equation $\partial_t g = \partial_x((3h_x^2 - 1)g_x) - \nu\partial_x^4 g$. Consider $t > t_0$, where t_0 will be picked later. We have

$$\begin{aligned} g(t) &= e^{-\nu(t-t_0)\partial_x^4} g(t_0) + \int_{t_0}^t \partial_x e^{-\nu(t-s)\partial_x^4}((3h_x^2 - 1)g_x)(s)ds \\ &= e^{-\nu(t-t_0)\partial_x^4} g(t_0) + \int_{t_0}^t \partial_{xx} e^{-\nu(t-s)\partial_x^4}((3h_x^2 - 1)g)(s)ds \\ &\quad - \int_{t_0}^t \partial_x e^{-\nu(t-s)\partial_x^4}(6h_{xx}h_x g)(s)ds. \end{aligned} \tag{5.27}$$

Now note that for any function $\tilde{g} : \mathbb{T} \to \mathbb{R}$ (not necessarily having mean zero), one has for $m \ge 1$,

$$\|\partial_x^m e^{-\nu t\partial_x^4} \tilde{g}\|_2 \lesssim_{m,\nu} e^{-\nu t/100} t^{-\frac{m}{4}} \|\tilde{g}\|_2.$$

Here the point is that since $m \ge 1$, $\tilde{g}$ can be replaced by $\tilde{g} - \bar{\tilde{g}}$ ($\bar{\tilde{g}}$ denotes the mean of $\tilde{g}$) and $|\bar{\tilde{g}}| \lesssim \|\tilde{g}\|_2$.

Now continuing from (5.27), we get (by using (5.26))

$$\|g(t)\|_2 \lesssim_{\nu,E_0} \|g(t_0)\|_2 + \int_{t_0}^{t} (t-s)^{-\frac{1}{2}} e^{-\nu(t-s)/100} \|g(s)\|_2 ds$$
$$+ \int_{t_0}^{t} (t-s)^{-\frac{1}{4}} e^{-\nu(t-s)/100} \|g(s)\|_2 ds. \tag{5.28}$$

By using the energy law, we have $\int_0^\infty \|g(s)\|_2^2 ds < \infty$. Thus one can find t_0 sufficiently large such that $\|g(t_0)\|_2 \ll 1$ and also $\int_{t_0}^\infty \|g(s)\|_2^2 ds \ll 1$. By (5.26), we also have $\sup_{s\ge 0} \|g(s)\|_2 \lesssim 1$. These estimates with (5.28) and an ϵ-δ argument (One needs to split the time interval in (5.28). For s close to t, we use the smallness of the time interval and the estimate $\|g(s)\|_2 \lesssim 1$. For s away from t, use $\int_{t_0}^\infty \|g(s)\|_2^2 ds \ll 1$.) then easily yield

$$\lim_{t\to\infty} \|g(t)\|_2 = 0.$$

Interpolating the above estimate with (5.26) (recall $g(t) = \partial_t h = (h_x^3 - h_x)_x - \nu \partial_x^4 h$), we get

$$\lim_{t\to\infty} \|\partial_t h\|_\infty = 0. \tag{5.29}$$

Step 2: we show (5.25). Easy to check that the even symmetry is propagated in time. Denote $f = \partial_x h$. Then

$$\partial_x \Big(f^3 - f - \nu f_{xx} \Big) = \partial_t h.$$

In view of the even symmetry of h, we have $f(t, x=0) \equiv 0$, $\partial_{xx} f(t, x=0) \equiv 0$. Thus

$$(f^2 - 1) f - \nu \partial_{xx} f = \int_0^x (\partial_t h)(t, y) dy.$$

A simple maximum principle argument together with (5.29) then yield (5.25).

Finally the proof of (5.24) is similar. In the general case, observe that (since $f = \partial_x h$)

$$\frac{1}{2\pi} \int_{\mathbb{T}} (f^3 - f - \nu f_{xx}(t,x)) dx = \underbrace{\frac{1}{2\pi} \int_{\mathbb{T}} f^3(t,x) dx}_{:=m(t)}.$$

By the Mean Value Theorem, there exists $x_0 \in [-\pi, \pi]$ such that

$$f^3(t,x_0) - f(t,x_0) - \nu f_{xx}(t,x_0) = m(t).$$

We then have

$$f^3 - f - \nu f_{xx} = \int_{x_0}^{x} (\partial_t h)(t, y)dy + m(t).$$

Now observe that

$$|m(t)| \lesssim \|\partial_x h(t)\|_3^3 \lesssim 1 + \int_{\mathbb{T}} (h_x^2 - 1)^2 dx \lesssim 1 + E_0,$$

where E_0 is the initial energy. The bound (5.24) then again follows from a maximum principle argument using this estimate. □

6. Proof of Theorem 1.2 and Corollary 1.3

The following perturbation lemma is more or less standard. It follows from the local theory and we omit the proof.

Proposition 6.1 *(Finite time stability of solutions). Let $\nu > 0$ in (1.1). Let $u_0 \in H^k$, $k > d/2$ and u be the corresponding solution. Let $T > 0$ be given and assume u has lifespan bigger than $[0, T]$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that the following holds:*

For any $v_0 \in H^k$, $k > d/2$ with $\|v_0 - u_0\|_{H^k} < \delta$, there exists a solution v to (1.1) corresponding to the initial data v_0 and has lifespan containing $[0, T]$. Furthermore we have

$$\max_{0 \le t \le T} \|v(t) - u(t)\|_{H^k} < \epsilon.$$

In particular by shrinking δ further if necessary, we have

$$\max_{0 \le t \le T} \|\nabla v(t) - \nabla u(t)\|_\infty < \epsilon.$$

We now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Step 1. We first show that there exists a smooth solution w to (1.2) with initial data w_0 such that $\|w_0'\|_\infty = 1$ and for some $t_* > 0$, $C_1 > 1$

$$\|\partial_x w(t_*)\|_\infty > C_1 > 1. \tag{6.30}$$

Let $\eta > 0$ be sufficiently small and w_0 be a smooth 2π-periodic function with mean zero (Here one can choose w_0 such that it is odd in x when regarded as a function on $\mathbb{R}$. This in turn easily implies that w_0 has mean zero on $[-\pi, \pi]$.) such that

$$\begin{aligned} &w_0(x) = x - \eta x^5, \quad |x| < \eta, \\ &|w_0'(x)| < 1, \quad \eta \le |\xi| \le \pi. \end{aligned} \tag{6.31}$$

Denote by $w = w(t, x)$ the corresponding solution to (1.2). Observe that

$$w_0'(x) = 1 - 5\eta x^4, \quad \text{for } |x| < \eta.$$

Obviously it follows that $|w_0'(x)| \le 1$ with equality holding only at $x = 0$ (and its 2π-periodic images). By a direct calculation, we have for $|x| < \eta$,

$$(\partial_x w_0)^3 - \partial_x w_0 = (1 - 5\eta x^4)^3 - (1 - 5\eta x^4) = O(x^4).$$

Clearly it holds that

$$\partial_{xx}\Big((\partial_x w_0)^3 - \partial_x w_0\Big)\Big|_{x=0} = 0.$$

Now since

$$\partial_t(w_x) = (w_x^3 - w_x)_{xx} - \nu\partial_x^5 w,$$

we have

$$(\partial_t\partial_x w)(0,0) = ((\partial_x w_0)^3 - \partial_x w_0)_{xx}\Big|_{x=0} - \nu\partial_x^5 w_0\Big|_{x=0} = 120\nu\eta > 0.$$

Since $A(t) = (\partial_x w)(t,0)$ is a continuously differentiable function of t with $A(0) = 1$, $A'(0) > 0$, obviously (6.30) holds.

Step 2. The perturbation argument.

Let $\phi \in C_c^\infty(\{x : |x| < \eta\})$ be a fixed smooth cut-off function with $\phi(x) = 1$ for $|x| < \frac{\eta}{2}$. Let ϕ be even in x and let

$$v_0^\delta(x) = w_0(x) - \delta x\phi(x).$$

Note that v_0^δ is odd in x and still has mean zero.

Clearly

$$\|v_0^\delta - w_0\|_{H^2} \le \delta\|x\phi(x)\|_{H^2} \le \text{const}\cdot\delta \tag{6.32}$$

and can be made arbitrarily small.

On the other hand for $|x| < \eta/2$,

$$\partial_x v_0^\delta(x) = \partial_x w_0(x) - \delta = 1 - 5\eta x^4 - \delta \le 1 - \delta.$$

For $\eta/2 \le |x| \le \pi$, since by construction we have

$$|\partial_x w_0(x)| \le 1 - \beta,$$

for some constant $\beta > 0$. Obviously by choosing $\delta > 0$ sufficiently small we can have

$$|\partial_x v_0^\delta(x)| \le 1 - \frac{\beta}{2}, \quad \forall\, \eta/2 \le |x| \le \pi.$$

Therefore we have shown

$$\|\partial_x v_0^\delta\|_\infty < 1.$$

Now let v^δ be the solution to (1.2) corresponding to initial data v_0^δ. By (6.32), (6.30) and Proposition 6.1, for $\delta > 0$ sufficiently small, we have

$$\|\partial_x v^\delta(t^*)\|_\infty > C_1' > 1,$$

where C_1' is another constant.

Define $\mathcal{A} = \{v_0^\delta : \delta \text{ is sufficiently small}\}$. This concludes our construction. $\square$

Proof of Corollary 1.3. The essential ideas are already in the proof of Theorem 1.2. Therefore we only sketch the necessary notational modifications.

Take $\eta > 0$ sufficiently small and $a = \frac{1}{\sqrt{d}}(1, \cdots, 1)^T$ (here d is the dimension). Note that by definition $|a| = 1$. We define a smooth function $w_0 \in C^\infty(\mathbb{T}^d)$ such that

$$w_0(x) = a \cdot x - \eta \sum_{j=1}^{d} x_j^5, \quad \text{for } |x| < \eta.$$

Let $D = [-\pi, \pi]^d$ be the fundamental domain of the torus $\mathbb{T}^d$. For $|x| \geq \eta$, $x \in D$, we simply require

$$|\nabla w_0(x)| < 1.$$

Take a radial $\phi \in C_c^\infty(\{x \in \mathbb{R}^d : |x| < \eta\})$ such that $\phi(x) \equiv 1$ for $|x| \leq \eta/2$.

For $\delta > 0$ sufficiently small, define

$$v_0^\delta x = w_0(x) - \delta \cdot (a \cdot x) \cdot \phi(x)$$

and

$$\mathcal{A} = \{v_0^\delta : \delta > 0 \text{ is sufficiently small}\}.$$

The set $\mathcal{A}$ is the desired family of initial data. $\square$

7. Proof of Theorem 1.3

In this section we give the proof of Theorem 1.3.

Proof of Theorem 1.3. Without loss of generality we assume the dimension $d = 1$. The case $d \geq 2$ can be proved with suitable modifications.

Fix $\epsilon > 0$. Let

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\xi^4} e^{i\xi \cdot x} d\xi.$$

Define

$$C_1 = \|f\|_{L_x^1(\mathbb{R})}, \quad A_1 = \|f''\|_{L_x^1(\mathbb{R})}.$$

Define $t_1 > 0$ such that

$$2C_1^3 \cdot A_1 \cdot \nu^{-\frac{1}{2}} \cdot 2t_1^{\frac{1}{2}} = \frac{\epsilon}{3}. \tag{7.33}$$

Step 1: We show that there exist $t_2 > 0$ with $t_2 \le t_1$ and $h_0 \in C^\infty(\mathbb{T})$ with mean zero such that $\|\partial_x h_0\|_\infty < 1$ and

$$\|e^{-\nu t_2 \partial_{xxxx}} \partial_x h_0\|_\infty > C_1 - \frac{\epsilon}{3}. \tag{7.34}$$

To show this, we first choose $\tilde{F}(t,x)$ to be an odd function of x which is 2π-periodic, and such that

$$\tilde{F}(t,x) = \begin{cases} \int_0^x \operatorname{sgn}(f(s/(\nu t)^{\frac{1}{4}}))ds, \quad 0 \le x \le t^{\frac{1}{5}}; \\ 0, \qquad t^{\frac{1}{5}} + |\int_0^{t^{\frac{1}{5}}} \operatorname{sgn}(f(s/(\nu t)^{\frac{1}{4}}))ds| \le x \le \pi; \\ \text{linear interpolation}, \quad t^{\frac{1}{5}} \le x \le t^{\frac{1}{5}} + |\int_0^{t^{\frac{1}{5}}} \operatorname{sgn}(f(s/(\nu t)^{\frac{1}{4}}))ds|. \end{cases}$$

Easy to check that for $t \le 1/2$ the function $\tilde{F}(t,x)$ is well-defined. Furthermore

$$\partial_x \tilde{F}(t,x) = \operatorname{sgn}(f(x/(\nu t)^{\frac{1}{4}})), \qquad a.e.\ |x| \le t^{\frac{1}{5}};$$

and $\|\partial_x \tilde{F}\|_\infty \le 1$. Define

$$\tilde{G}(t,x) = \Big(e^{-\nu t \partial_{xxxx}}(\partial_x \tilde{F}(t,\cdot))\Big)(t,x).$$

Then clearly if t is sufficiently small, then

$$\begin{aligned} |\tilde{G}(t,0)| &\ge \int\limits_{|x|\le t^{\frac{1}{5}}} |f(\frac{x}{(\nu t)^{\frac{1}{4}}})|(\nu t)^{-\frac{1}{4}}dx - \int\limits_{|x|> t^{\frac{1}{5}}} |f(\frac{x}{(\nu t)^{\frac{1}{4}}})|(\nu t)^{-\frac{1}{4}}dx \\ &= \|f\|_{L^1_x(\mathbb{R})} - 2\int\limits_{|x|> t^{\frac{1}{5}}} |f(\frac{x}{(\nu t)^{\frac{1}{4}}})|(\nu t)^{-\frac{1}{4}}dx \\ &= C_1 - 2\int\limits_{|x|>\nu^{-\frac{1}{4}} t^{-\frac{1}{20}}} |f(x)|dx \\ &> C_1 - \frac{\epsilon}{4}. \end{aligned}$$

In the last inequality above, we used the fact that f is a Schwartz function and the tail contribution to the integral can be made arbitrarily small (by taking t small).

Now take an even function $\psi \in C_c^\infty(\mathbb{R})$ such that $0 \le \psi \le 1$, $\psi(x) = 1$ for $|x| \le 1$ and $\int \psi = 1$. Define $\psi_\delta(x) = \delta^{-1}\psi(x/\delta)$ and

$$\tilde{F}_\delta(t,x) = (1-\delta)\cdot\Big(\psi_\delta * \tilde{F}(t,\cdot)\Big)(t,x),$$

where $*$ is the usual convolution on $\mathbb{R}$. Easy to check that $\|\partial_x \tilde{F}_\delta\|_\infty < 1$, $\tilde{F}_\delta$ is 2π-periodic, odd in x and has mean zero.

Define

$$\tilde{G}_\delta(t,x) = \Big(e^{-\nu t\partial_{xxxx}}(\partial_x \tilde{F}_\delta(t,\cdot))\Big)(t,x).$$

Obviously for δ sufficiently small, we have

$$|\tilde{G}_\delta(t,0)| > C_1 - \frac{\epsilon}{3}.$$

Thus (7.34) is achieved with $h_0(x) = \tilde{F}_\delta(t,x)$.

`Step 2`: Control of the nonlinear solution. We shall fix t_2 and h_0 from Step 1. With h_0 as initial data, let h be the corresponding solution to (1.2). We argue by contradiction and assume that

$$\sup_{0\le t\le t_2} \|\partial_x h(t,\cdot)\|_\infty \le C_1 - \epsilon. \tag{7.35}$$

Then

$$\|h_x^3 - h_x\|_\infty \le 2C_1^3, \qquad \forall\, 0 < t \le t_2.$$

Now since

$$\partial_x h(t) = e^{-\nu t\partial_{xxxx}}\partial_x h_0 + \int_0^t \partial_{xx} e^{-\nu s\partial_{xxxx}}\Big((h_x^3 - h_x)(t-s)\Big)ds,$$

we get

$$\|\partial_x h(t) - e^{-\nu t\partial_{xxxx}}\partial_x h_0\|_\infty \le \int_0^t \|\partial_{xx} e^{-\nu s\partial_{xxxx}}((h_x^3 - h_x)(t-s))\|_\infty ds.$$

Regard $(h_x^3 - h_x)$ as a 2π-periodic function on $\mathbb{R}$. Recall that $f''(x) = \mathcal{F}^{-1}(-\xi^2 e^{-\xi^4})$. Then

$$\begin{aligned}
&\|\partial_{xx} e^{-\nu s\partial_{xxxx}}((h_x^3 - h_x))\|_{L_x^\infty(\mathbb{T})} \\
&= \|\partial_{xx} e^{-\nu s\partial_{xxxx}}((h_x^3 - h_x))\|_{L_x^\infty(\mathbb{R})} \\
&\le \|\mathcal{F}^{-1}(-|\xi|^2 e^{-\nu s|\xi|^4})\|_{L_x^1(\mathbb{R})} \|h_x^3 - h_x\|_{L_x^\infty(\mathbb{R})} \\
&\le \|f''\|_{L_x^1(\mathbb{R})} \cdot (\nu s)^{-\frac{1}{2}} \cdot 2C_1^3 \\
&= A_1 \cdot (\nu s)^{-\frac{1}{2}} \cdot 2C_1^3.
\end{aligned}$$

Thus we obtain for $0 < t \le t_2$,

$$\|\partial_x h(t) - e^{-\nu t \partial_{xxxx}} \partial_x h_0\|_\infty \le A_1 \cdot 2\nu^{-\frac{1}{2}} t_2^{\frac{1}{2}} \cdot 2C_1^3.$$

Since $t_2 \le t_1$, by (7.33) and Step 1, we get

$$\|\partial_x h(t_2)\|_\infty > C_1 - \frac{\epsilon}{3} - \frac{\epsilon}{3} = C_1 - \frac{2\epsilon}{3}$$

which is an obvious contradiction to (7.35). □

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