Front. Math. China 2012, 7(1): 69–84
DOI 10.1007/s11464-012-0170-0

RESEARCH ARTICLE

Convergence analysis of Jacobi spectral collocation methods for Abel-Volterra integral equations of second kind

Xianjuan LI[1], **Tao TANG**[2]

1 College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350108, China
2 Department of Mathematics, Hong Kong Baptist University, Hong Kong, China

Abstract This work is to analyze a spectral Jacobi-collocation approximation for Volterra integral equations with singular kernel $\varphi(t,s)=(t-s)^{-\mu}$. In an earlier work of Y. Chen and T. Tang [J. Comput. Appl. Math., 2009, 233: 938–950], the error analysis for this approach is carried out for $0<\mu<1/2$ under the assumption that the underlying solution is smooth. It is noted that there is a technical problem to extend the result to the case of Abel-type, i.e., $\mu=1/2$. In this work, we will not only extend the convergence analysis by Chen and Tang to the Abel-type but also establish the error estimates under a more general regularity assumption on the exact solution.

Keywords Jacobi spectral collocation method, Abel-Volterra integral equation, convergence analysis
MSC 35Q99, 35R35, 65M12, 65M70

1 Introduction

We consider the linear Abel-Volterra integral equations (AVIEs) of the second kind:

$$y(t)=g(t)+\int_0^t (t-s)^{-1/2}K(t,s)y(s)\mathrm{d}s,\quad t\in I, \tag{1}$$

where $I=[0,T]$, the function $g\in C(I)$, $y(t)$ is the unknown function, and $K\in C(I\times I)$ with $K(t,t)\neq 0$ for $t\in I$. Several numerical methods have been proposed for (1) (see, e.g., [3,4,8,22,23]).

The numerical treatment of AVIEs (1) is not simple, mainly due to the fact that the solutions of (1) usually have a weak singularity at $t=0$, even when

Received October 29, 2010; accepted July 29, 2011
Corresponding author: Tao TANG, E-mail: ttang@math.hkbu.edu.hk

the inhomogeneous term $g(t)$ is regular. For any positive integer m, if both g and K have continuous derivatives of order m, as discussed in [4], there exists a function $Z = Z(t, v)$ possessing continuous derivatives of order m, such that the solution of (1) can be written as $y(t) = Z(t, \sqrt{t})$. As this will be the standing point of this paper, the detailed regularity result of (1) is given below.

Lemma 1 [4] *Assume that $g \in C^m(I)$ and $K \in C^m(I \times I)$ with $K(t,t) \neq 0$ on $I = [0, T]$. Then, the regularity of the unique solution of the weakly singular AVIE* (1) *can be described by*

$$y \in C^m(0,T] \cap C(I), \quad \text{with } |y'(t)| \leqslant Ct^{-1/2} \text{ for } t \in (0,T]; \tag{2}$$

$$y(t) = \sum_{(j,k)} \gamma_{j,k} t^{j+\frac{k}{2}} + Y_m(t), \quad t \in I, \tag{3}$$

where $(j,k) := \{(j,k)\colon j,k$ are non-negative integers, $j + \frac{k}{2} < m\}$, $\gamma_{j,k}$ are some constants, and $Y_m(\cdot) \in C^m(I)$.

The above result implies that near $t = 0$ the m-th derivative of the solution $y(t)$ behaves like $y^{(m)}(t) \sim t^{\frac{1}{2}-m}$, which indicates that $y \notin C^m(I)$. Several methods have been proposed to recover high order convergence properties for (1) using collocation type methods, see, e.g., [3,8,22,23] and using multi-step method, see, e.g., [13]. We point out that for (1) without the singular kernel, spectral methods and the corresponding error analysis have been provided recently [24]; see also [1,2,25] for spectral methods to pantograph-type delay differential equations. In both cases, the underlying solutions are smooth.

We will first mention the difference between this work and the one in [7] where a Jacobi spectral collocation method is proposed for the weakly singular Volterra integral equations. In [7], to handle the nonsmoothness of the underlying solutions, *both* function transformation and variable transformation are used. However, it is found that the function transformation (see also [8]) generally makes the resulting equations and the corresponding approximations more complicated. The present approach only makes one transformation, i.e., coordinate transformation, but not the transformation for the unknown function. Moreover, in [6], we also studied convergence analysis of the Jacobi spectral-collocation methods for the Volterra integral equations with the singular kernel $\varphi(t,s) = (t-s)^{-\mu}$ for $0 < \mu < 1/2$ under the assumption that the underlying solution is smooth. Note that $0 < \mu < 1/2$ means that the Abel type kernel is not included.

In this work, we will consider the case that the exact solutions of (1) are nonsmooth. This case may occur when the source function g in (1) is smooth; see, e.g., [4, Theorem 6.1.11]. In this case, although the Jacobi-collocation spectral method can be implemented in a straightforward manner, the relevant polynomial approximation theory cannot be employed directly to obtain the desired convergence results, see, e.g., [21]. However, for the case of Abel kernel in (1) we can overcome this difficulty by taking a simple variable transformation so that the resulting equation possesses a smooth solution. With a more elegant proof technique, we can not only extend the convergence analysis in [6] to the Abel kernel type but also establish the error estimates under a more general regularity assumption on the exact solution of (1).

This paper is organized as follows. In Section 2, we outline the spectral approaches for (1). Some lemmas useful for establishing the convergence results will be provided in Section 3. The convergence analysis will be carried out in Section 4. Section 5 gives the numerical experiments.

2 Jacobi-collocation methods

We first introduce some notations. Let $\Lambda = [-1, 1]$. In order to discretize problem (1), we define $\mathscr{P}_N$ as the polynomials spaces of degree less than or equal to N. As defined in [5], let us denote $J_N^{\alpha,\beta}(x)$ the Jacobi polynomial of degree N with respect to weight

$$w^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad -1 < \alpha, \beta < 1.$$

Let $x_i^{\alpha,\beta}$ be the points of the Gauss-Jacobi (GJ) quadrature formula, defined by

$$J_{N+1}^{\alpha,\beta}(x_i^{\alpha,\beta}) = 0, \quad i = 0, \dots, N,$$

arranged by increasing order: $x_0^{\alpha,\beta} < x_1^{\alpha,\beta} < \cdots < x_N^{\alpha,\beta}$. The associated weights of the GJ quadrature formula are denoted by $\omega_i^{\alpha,\beta}$, $0 \leqslant i \leqslant N$.

It is known that the spectral methods have been an efficient tool for solving the differential equations with smooth solutions. Thus, in order to make it practical, by variable transformation

$$t = z^2, \quad z = \sqrt{t}, \quad s = w^2, \quad w = \sqrt{s},$$

we change the weakly singular problem (1) as follows:

$$\overline{u}(z) = \overline{f}(z) + \int_0^z (z-w)^{-1/2}\overline{K}(z,w)\overline{u}(w)\mathrm{d}w, \quad z \in [0, \sqrt{T}\,], \tag{4}$$

where

$$\overline{f}(z) = g(z^2), \quad \overline{K}(z,w) = 2(z+w)^{-1/2}wK(z^2,w^2),$$

and

$$\overline{u}(z) = y(z^2) \tag{5}$$

is the smooth solution of equation (4).

For the sake of applying the theory of orthogonal polynomials conveniently, by the linear transformation

$$z = \frac{\sqrt{T}\,(1+x)}{2}, \quad w = \frac{\sqrt{T}\,(1+\tau)}{2},$$

letting

$$u(x)=\overline{u}\Big(\frac{\sqrt{T}\,(1+x)}{2}\Big),\quad f(x)=\overline{f}\Big(\frac{\sqrt{T}\,(1+x)}{2}\Big),$$

the weakly singular problem (4) can be rewritten as follows:

$$u(x)=f(x)+\int_{-1}^{x}(x-\tau)^{-1/2}\widetilde{K}(x,\tau)u(\tau)\mathrm{d}\tau,\quad x\in\Lambda, \tag{6a}$$

where

$$\widetilde{K}(x,\tau)=\Big(\frac{\sqrt{T}}{2}\Big)^{1/2}\overline{K}\Big(\frac{\sqrt{T}}{2}\,(1+x),\frac{\sqrt{T}}{2}\,(1+\tau)\Big). \tag{6b}$$

The Jacobi-collocation method for Eq. (6a) is to find $u_N\in\mathscr{P}_N$ such that for all $0\leqslant j\leqslant N$,

$$u_N(x_i^{\alpha,\beta})=f(x_i^{\alpha,\beta})+\int_{-1}^{x_i^{\alpha,\beta}}(x_i^{\alpha,\beta}-\tau)^{-1/2}\widetilde{K}(x_i^{\alpha,\beta},\tau)u_N(\tau)\mathrm{d}\tau,\quad 0\leqslant i\leqslant N. \tag{7}$$

In order to obtain the higher-order accuracy for the VIEs problem, the main difficulty is to compute the integral term in (7). We rewrite the integral term in (7) into the form:

$$\begin{aligned}&\int_{-1}^{x_i^{\alpha,\beta}}(x_i^{\alpha,\beta}-\tau)^{-1/2}\widetilde{K}(x_i^{\alpha,\beta},\tau)u_N(\tau)\mathrm{d}\tau\\&\quad=\int_{-1}^{1}(1-\theta)^{-1/2}K_1(x_i^{\alpha,\beta},\tau_i(\theta))u_N(\tau_i(\theta))\mathrm{d}\theta,\end{aligned} \tag{8}$$

by the linear variable transformation

$$\tau=\tau_i(\theta)=\frac{1+x_i^{\alpha,\beta}}{2}\,\theta+\frac{x_i^{\alpha,\beta}-1}{2},\quad \theta\in\Lambda, \tag{9}$$

where

$$K_1(x_i^{\alpha,\beta},\tau_i(\theta))=\Big(\frac{1+x_i^{\alpha,\beta}}{2}\Big)^{1/2}\widetilde{K}(x_i^{\alpha,\beta},\tau_i(\theta)). \tag{10}$$

We discretize the integral term by the Gauss quadrature formula relative to the Jacobi weight $\rho(x)=\omega^{-\frac{1}{2},0}(x)$, and therefore, the full collocation scheme (with numerical integration) becomes

$$u_N(x_i^{\alpha,\beta})=f(x_i^{\alpha,\beta})+\sum_{k=0}^{N}K_1(x_i^{\alpha,\beta},\tau_i(\theta_k))u_N(\tau_i(\theta_k))\rho_k,\quad 0\leqslant i\leqslant N, \tag{11}$$

where

$$\theta_k=x_k^{-\frac{1}{2},0},\quad \rho_k=\omega_k^{-\frac{1}{2},0},\quad k=0,1,\ldots,N.$$

Since the exact solution of problem (1) can be written as $y(t) = u(x)$, we can define $y_N(t) = u_N(x),\ t \in I,\ x \in \Lambda$, as the approximated solution of problem (1).

Throughout the paper, C will denote a generic positive constant that is independent of N but which will depend on the length T and on bounds for the given functions f, $\widetilde{K}$.

3 Some useful lemmas

We first introduce some notations that will be used throughout the paper. Let $L^2_{\omega^{\alpha,\beta}}(\Lambda)$ be the space of measurable functions whose square is Lebesgue integrable in Λ relative to the weight function $\omega^{\alpha,\beta}(x)$. The inner product and norm of $L^2_{\omega^{\alpha,\beta}}(\Lambda)$ are defined by

$$\left.\begin{aligned}(u,v)_{\omega^{\alpha,\beta},\Lambda} &= \int_\Lambda u(x)v(x)\omega^{\alpha,\beta}(x)\mathrm{d}x,\\ \|u\|_{\omega^{\alpha,\beta},\Lambda} &= \sqrt{(u,u)_{\omega^{\alpha,\beta},\Lambda}},\end{aligned}\right. \qquad \forall\ u,v \in L^2_{\omega^{\alpha,\beta}}(\Lambda).$$

For a non-negative integer m, define

$$H^m_{\omega^{\alpha,\beta}}(\Lambda) := \{v;\ \|v\|_{m,\omega^{\alpha,\beta}} < \infty\},$$

with

$$|v|_{m,\omega^{\alpha,\beta}} = \|\partial_x^m v\|_{\omega^{\alpha,\beta}}, \quad \|v\|_{m,\omega^{\alpha,\beta}} = \left(\sum_{k=0}^m |v|^2_{k,\omega^{\alpha,\beta}}\right)^{1/2}.$$

Particularly, let

$$\omega^c(x) = \omega^{-\frac{1}{2},-\frac{1}{2}}(x)$$

be the Chebyshev weight function.

In bounding from the above approximation error, only some of the $L^2_{\omega^{\alpha,\beta}}$-norms appearing on the right-hand side of above norm enter into play. Thus, it is convenient to introduce the seminorms

$$|v|_{H^{m;N}_{\omega^{\alpha,\beta}}(\Lambda)} = \left(\sum_{k=\min(m,N+1)}^m \|\partial_x^k v\|^2_{L^2_{\omega^{\alpha,\beta}}(\Lambda)}\right)^{1/2}.$$

Hereafter, in cases where no confusion would arise, the domain symbol Λ may be dropped from the notations.

In the purpose of carrying out an error analysis to the spectral collocation method, we introduce two approximation operator as follows. First, we define the Lagrange interpolation operator $I_N^{\alpha,\beta}\colon \mathscr{C}(\Lambda) \to \mathscr{P}_N(\Lambda)$, by $\forall\ v \in \mathscr{C}(\Lambda),\ I_N^{\alpha,\beta} v \in \mathscr{P}_N(\Lambda)$, such that

$$I_N^{\alpha,\beta} v(x_i^{\alpha,\beta}) = v(x_i^{\alpha,\beta}), \quad 0 \leqslant i \leqslant N,$$

see, e.g., [5,20]. The Lagrange interpolating polynomial can be written in the form

$$I_N^{\alpha,\beta}v(x) = \sum_{i=0}^{N} v(x_i^{\alpha,\beta})h_i(x),$$

where $h_i(x)$ is the Lagrange interpolation basis function associated with $x_i^{\alpha,\beta}$.

Then, for all $v \in H^m_{\omega^{\alpha,\beta}}(\Lambda),\ m \geqslant 1$, the following optimal error estimates hold (see [5]):

$$\|v - I_N^{\alpha,\beta}v\|_{\omega^{\alpha,\beta}} \leqslant CN^{-m}|v|_{H^{m;N}_{\omega^{\alpha,\beta}}}. \tag{12}$$

Next, we introduce a discrete inner product. For any $u, v \in \mathscr{C}(\Lambda)$, define

$$(u,v)_N = \sum_{i=0}^{N} u(x_i^{\alpha,\beta})v(x_i^{\alpha,\beta})\omega_i^{\alpha,\beta}. \tag{13}$$

For Gauss-Jacobi quadrature formula, the error estimate is well known [5]: $\forall\ \phi \in \mathscr{P}_N$,

$$|(v,\phi)_{\omega^{\alpha,\beta}} - (v,\phi)_N| \leqslant CN^{-m}|v|_{H^{m;N}_{\omega^{\alpha,\beta}}}\|\phi\|_{\omega^{\alpha,\beta}}, \quad v \in H^m_{\omega^{\alpha,\beta}}(\Lambda),\ m \geqslant 1. \tag{14}$$

From [14], we have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the zeros of the Jacobi polynomials.

Lemma 2 *Let $\{h_j(x)\}_{j=0}^N$ be the N-th Lagrange interpolation polynomials associated with the Gauss points of the Jacobi polynomials. Then*

$$\|I_N^{\alpha,\beta}\|_\infty := \max_{x\in\Lambda}\sum_{j=0}^{N}|h_j(x)| = \begin{cases} O(\log N), & -1 < \alpha,\beta \leqslant -\dfrac{1}{2}, \\ O(N^{\gamma+\frac{1}{2}}), & \gamma = \max(\alpha,\beta),\ \textit{otherwise}. \end{cases} \tag{15}$$

As demonstrated in [7], we have the following result.

Lemma 3 *Suppose that $L \geqslant 0,\ 0 < \mu < 1$, and $v(x)$ is a non-negative, locally integrable function defined on Λ satisfying*

$$u(x) \leqslant v(x) + L\int_{-1}^{x}(x-\tau)^{-\mu}u(\tau)\mathrm{d}\tau.$$

Then there exists a constant $C = C(\mu)$ such that

$$u(x) \leqslant v(x) + CL\int_{-1}^{x}(x-\tau)^{-\mu}v(\tau)\mathrm{d}\tau, \quad -1 \leqslant x < 1.$$

From now on, for $r \geqslant 0$ and $\kappa \in [0,1]$, $\mathscr{C}^{r,\kappa}(\Lambda)$ will denote the space of functions whose r-th derivatives are Hölder continuous with exponent κ, endowed with the usual norm:

$$\|v\|_{r,\kappa} = \max_{0\leqslant k\leqslant r}\max_{x\in\Lambda}|\partial_x^k v(x)| + \max_{0\leqslant k\leqslant r}\sup_{\substack{x,y\in\Lambda\\ x\neq y}}\frac{|\partial_x^k v(x) - \partial_x^k v(y)|}{|x-y|^\kappa}.$$

When $\kappa = 0$, $\mathscr{C}^{r,0}(\Lambda)$ denotes the space of functions with r continuous derivatives on Λ, which is also commonly denoted by $\mathscr{C}^r(\Lambda)$, and with norm $\|\cdot\|_r$.

We shall make use of a result of Ragozin [17,18] (see also [10]), which states that, for non-negative integer r and $\kappa \in (0,1)$, there exists a constant $C_{r,\kappa} > 0$ such that for any function $v \in \mathscr{C}^{r,\kappa}(\Lambda)$, there exists a polynomial function $\mathscr{T}_N v \in \mathscr{P}_N$ such that

$$\|v - \mathscr{T}_N v\|_\infty \leqslant C_{r,\kappa} N^{-(r+\kappa)} \|v\|_{r,\kappa}. \tag{16}$$

Actually, as stated in [17,18], $\mathscr{T}_N$ is a linear operator from $\mathscr{C}^{r,\kappa}(\Lambda)$ into $\mathscr{P}_N$.

We further define a linear, weakly singular integral operator $\mathscr{M}$:

$$\mathscr{M} v = \int_{-1}^{x} (x-\tau)^{-1/2} \widetilde{K}(x,\tau) v(\tau) \mathrm{d}\tau. \tag{17}$$

Below we will show that $\mathscr{M}$ is compact as an operator from $\mathscr{C}(\Lambda)$ to $\mathscr{C}^{0,\kappa}(\Lambda)$ provided that the index κ satisfies $0 < \kappa < 1/2$. The proof of the following lemma can be found in [7].

Lemma 4 *Let $\mathscr{M}$ be defined by* (17). *Then, for any function $v \in \mathscr{C}(\Lambda)$, there exists a positive constant C, which is dependent on $\|\widetilde{K}\|_{0,\kappa}$, such that*

$$\|\mathscr{M} v\|_{0,\kappa} \leqslant C \|v\|_\infty, \quad 0 < \kappa < \frac{1}{2}, \tag{18}$$

where $\|\cdot\|_\infty$ is the standard norm in $\mathscr{C}(\Lambda)$.

4 Convergence analysis

The objective of this section is to analyze the approximation scheme (11). First, we derive the error estimate in L^∞ norm of the Jacobi collocation method.

4.1 Error estimate in L^∞

Theorem 1 *Let u be the exact solution to the Volterra integral equation* (6), *which is assumed to be sufficiently smooth. Let the approximated solution u_N be obtained by using the spectral collocation scheme* (11). *If $u \in H^m_{\omega^{\alpha,\beta}}(\Lambda) \cap H^m_{\omega^c}(\Lambda)$ $(m \geqslant 1)$, then*

$$\|u - u_N\|_\infty \leqslant \begin{cases} C N^{\frac{1}{2}-m} \log N (|u|_{H^{m;N}_{\omega^c}} + N^{-1/2} K^* \|u\|_\infty), & -1 < \alpha, \beta \leqslant -\frac{1}{2}, \\ C N^{1+\gamma-m} (|u|_{H^{m;N}_{\omega^c}} + N^{-1/2} K^* \|u\|_\infty), & \gamma = \max\{\alpha,\beta\} < 0, \end{cases} \tag{19}$$

for N sufficiently large, where

$$K^* = \max_{0 \leqslant i \leqslant N} |K_1(x_i^{\alpha,\beta}, \tau_i(\cdot))|_{H^{m;N}_\rho}. \tag{20}$$

Proof First, we use the weighted inner product to rewrite (6) as

$$u(x_i^{\alpha,\beta}) = f(x_i^{\alpha,\beta}) + (K_1(x_i^{\alpha,\beta}, \tau_i(\cdot)), u(\tau_i(\cdot)))_\rho, \quad 0 \leqslant i \leqslant N. \tag{21}$$

By using the discrete inner product (13), the numerical scheme (11) can be written as

$$u_N(x_i^{\alpha,\beta}) = f(x_i^{\alpha,\beta}) + (K_1(x_i^{\alpha,\beta}, \tau_i(\cdot)), u_N(\tau_i(\cdot)))_N, \quad 0 \leqslant i \leqslant N. \tag{22}$$

Subtracting (22) from (21) gives

$$\begin{aligned} u(x_i^{\alpha,\beta}) - u_N(x_i^{\alpha,\beta}) &= (K_1(x_i^{\alpha,\beta}, \tau_i(\cdot)), e(\tau_i(\cdot)))_\rho + I_{i,2} \\ &= \int_{-1}^{x_i^{\alpha,\beta}} (x_i^{\alpha,\beta} - \tau)^{-1/2} \widetilde{K}(x_i^{\alpha,\beta}, \tau) e(\tau) \mathrm{d}\tau + I_{i,2} \end{aligned} \tag{23}$$

for $0 \leqslant i \leqslant N$, where $e(x) = u(x) - u_N(x)$ is the error function, and

$$I_{i,2} = (K_1(x_i^{\alpha,\beta}, \tau_i(\cdot)), u_N(\tau_i(\cdot)))_\rho - (K_1(x_i^{\alpha,\beta}, \tau_i(\cdot)), u_N(\tau_i(\cdot)))_N.$$

Multiplying $h_i(x)$ on both sides of the error equation (23) and summing up from $i = 0$ to $i = N$ yield

$$I_N^{\alpha,\beta} u - u_N = I_N^{\alpha,\beta}\Big(\int_{-1}^{x} (x-\tau)^{-1/2} \widetilde{K}(x,\tau) e(\tau) \mathrm{d}\tau\Big) + \sum_{i=0}^{N} I_{i,2} h_i(x). \tag{24}$$

Consequently,

$$e(x) = \int_{-1}^{x} (x-\tau)^{-1/2} \widetilde{K}(x,\tau) e(\tau) \mathrm{d}\tau + I_1 + I_2 + I_3, \tag{25}$$

where

$$I_1 = u - I_N^{\alpha,\beta} u, \quad I_2 = \sum_{i=0}^{N} I_{i,2} h_i(x),$$

$$I_3 = I_N^{\alpha,\beta}\Big(\int_{-1}^{x} (x-\tau)^{-1/2} \widetilde{K}(x,\tau) e(\tau) \mathrm{d}\tau\Big) - \int_{-1}^{x} (x-\tau)^{-1/2} \widetilde{K}(x,\tau) e(\tau) \mathrm{d}\tau.$$

It follows from the Gronwall inequality in Lemma 3 that

$$\|e\|_\infty \leqslant C(\|I_1\|_\infty + \|I_2\|_\infty + \|I_3\|_\infty). \tag{26}$$

Let $I_N^c u \in \mathscr{P}_N$ denote the interpolant of u at the Chebyshev Gauss points. From [5, (5.5.28)], the interpolation error estimate in the maximum norm is given by

$$\|u - I_N^c u\|_\infty \leqslant C N^{\frac{1}{2}-m} |u|_{H_{\omega^c}^{m;N}}. \tag{27}$$

By using (27), Lemma 2, and noting that

$$I_N^{\alpha,\beta} p(x) = p(x), \quad \forall\, p(x) \in \mathscr{P}_N,$$

we obtain

$$\begin{aligned}
\|I_1\|_\infty &= \|u - I_N^{\alpha,\beta} u\|_\infty \\
&= \|u - I_N^c u + I_N^{\alpha,\beta}(I_N^c u) - I_N^{\alpha,\beta} u\|_\infty \\
&\leqslant \|u - I_N^c u\|_\infty + \|I_N^{\alpha,\beta}(I_N^c u - u)\|_\infty \\
&\leqslant (1 + \|I_N^{\alpha,\beta}\|_\infty)\|u - I_N^c u\|_\infty \\
&\leqslant \begin{cases} CN^{\frac{1}{2}-m} \log N |u|_{H^{m;N}_{\omega^c}}, & -1 < \alpha, \beta \leqslant -\dfrac{1}{2}, \\ CN^{1+\gamma-m} |u|_{H^{m;N}_{\omega^c}}, & \gamma = \max\{\alpha,\beta\}, \text{ otherwise.} \end{cases}
\end{aligned} \tag{28}$$

Next, using the integration error estimates (14) for Jacobi-Gauss polynomials quadrature, we have

$$\begin{aligned}
\max_{0\leqslant i\leqslant N} |I_{i,2}| &\leqslant CN^{-m} \max_{0\leqslant i\leqslant N} |K_1(x_i^{\alpha,\beta}, \tau_i(\cdot))|_{H^{m;N}_\rho} \max_{0\leqslant i\leqslant N} \|u_N(\tau_i(\cdot))\|_\rho \\
&\leqslant CN^{-m} K^*(\|e\|_\infty + \|u\|_\infty),
\end{aligned} \tag{29}$$

where K^* is defined as in (20). Hence, by combining with Lemma 2, yields

$$\begin{aligned}
\|I_2\|_\infty &= \Big\| \sum_{i=0}^{N} I_{i,2} h_i(x) \Big\|_\infty \\
&\leqslant C \max_{0\leqslant i\leqslant N} |I_{i,2}| \max_{x\in\Lambda} \sum_{j=0}^{N} |h_j(x)| \\
&\leqslant \begin{cases} CN^{-m} \log N K^*(\|e\|_\infty + \|u\|_\infty), & -1 < \alpha, \beta \leqslant -\dfrac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-m} K^*(\|e\|_\infty + \|u\|_\infty), & \gamma = \max\{\alpha,\beta\}, \text{ otherwise,} \end{cases}
\end{aligned} \tag{30}$$

for sufficiently large N.

We now estimate the third term I_3. It follows from (16), Lemmas 2 and 4 that

$$\begin{aligned}
\|I_3\|_\infty &= \|(I_N^{\alpha,\beta} - I)\mathscr{M} e\|_\infty \\
&= \|(I_N^{\alpha,\beta} - I)(\mathscr{M} e - \mathscr{T}_N \mathscr{M} e)\|_\infty \\
&\leqslant (1 + \|I_N^{\alpha,\beta}\|_\infty)\|\mathscr{M} e - \mathscr{T}_N \mathscr{M} e\|_\infty \\
&\leqslant C(1 + \|I_N^{\alpha,\beta}\|_\infty) N^{-\kappa} \|\mathscr{M} e\|_{0,\kappa} \\
&\leqslant \begin{cases} CN^{-\kappa} \log N \|e\|_\infty, & -1 < \alpha, \beta \leqslant -\dfrac{1}{2}, \\ CN^{\frac{1}{2}+\gamma-\kappa} \|e\|_\infty, & \gamma = \max\{\alpha,\beta\}, \text{ otherwise,} \end{cases}
\end{aligned} \tag{31}$$

where in the last step, we have used Lemma 4 under the condition $0 < \kappa < 1/2$.

It is clear that

$$\|I_3\|_\infty \leqslant \frac{1}{3}\,\|e\|_\infty \tag{32}$$

under the following assumption:

$$\begin{cases} 0 < \kappa < \dfrac{1}{2}, & -1 < \alpha,\beta \leqslant -\dfrac{1}{2}, \\ \dfrac{1}{2} + \gamma < \kappa < \dfrac{1}{2}, & \gamma = \max\{\alpha,\beta\} < 0, \end{cases} \tag{33}$$

provided that N is sufficiently large. Combining (26), (28), (30), and (32) gives the desired estimate (19). □

4.2 Error estimate in weighted L^2 norm

To prove the error estimate in weighted L^2 norm, we need the generalized Hardy's inequality with weights (see, e.g., [9,12,19]).

Lemma 5 *For all measurable function $f \geqslant 0$, the generalized Hardy's inequality*

$$\Big(\int_a^b |(Tf)(x)|^q u(x)\mathrm{d}x\Big)^{1/q} \leqslant C\Big(\int_a^b |f(x)|^p v(x)\mathrm{d}x\Big)^{1/p}$$

holds if and only if

$$\sup_{a<x<b}\Big(\int_x^b u(t)\mathrm{d}t\Big)^{1/q}\Big(\int_a^x v^{1-p'}(t)\mathrm{d}t\Big)^{1/p'} < \infty, \quad p' = \frac{p}{p-1},$$

for the case $1 < p \leqslant q < \infty$. Here, T is an operator of the form

$$(Tf)(x) = \int_a^x k(x,t)f(t)\mathrm{d}t$$

with $k(x,t)$ a given kernel, u and v weight functions, and $-\infty \leqslant a < b \leqslant \infty$.

From [15, Theorem 1], we have the following weighted mean convergence result of Lagrange interpolation based at the zeros of Jacobi polynomials.

Lemma 6 *For every bounded function $v(x)$, there exists a constant C independent of v such that*

$$\sup_N \Big\| \sum_{j=0}^N v(x_j)h_j(x)\Big\|_{L^2_{\omega^{\alpha,\beta}}(\Lambda)} \leqslant C\|v\|_\infty,$$

where $h_i(x)$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $x_i^{\alpha,\beta}$, $i = 0,1,\ldots,N$.

Theorem 2 *Let u be the exact solution to the Volterra integral equation (6), which is assumed to be sufficiently smooth. Let the approximated solution u_N*

be obtained by using the spectral collocation scheme (11). *Assume that* $u \in H^m_{\omega^{\alpha,\beta}}(\Lambda) \cap H^m_{\omega^c}(\Lambda)$, $m \geqslant 1$. *Then, for* N *sufficiently large,*

$$\|u - u_N\|_{\omega^{\alpha,\beta}} \leqslant \begin{cases} CN^{-m}(U_2 + N^{\frac{1}{2}-\kappa}\log N U_1), & -1 < \alpha, \beta \leqslant -\dfrac{1}{2}, \\ CN^{-m}(U_2 + N^{1+\gamma-\kappa}U_1), & \gamma = \max\{\alpha, \beta\} < 0, \end{cases} \tag{34}$$

for any κ *satisfying* (33), *where*

$$U_1 = K^*|u|_{H^{m;N}_{\omega^c}}, \quad U_2 = |u|_{H^{m;N}_{\omega^{\alpha,\beta}}} + K^*\|u\|_\infty, \tag{35}$$

and K^* *is defined by* (20).

Proof By using the generalization of Gronwall's inequality in Lemma 3, it follows from (25) that

$$e(x) \leqslant C\int_{-1}^{x}(x-\tau)^{-1/2}\widetilde{K}(x,\tau)(I_1 + I_2 + I_3)(\tau)\mathrm{d}\tau + I_1 + I_2 + I_3. \tag{36}$$

By the generalized Hardy's inequality in Lemma 5, we obtain that

$$\begin{aligned}\|e\|_{\omega^{\alpha,\beta}} &\leqslant C\Big\|\int_{-1}^{x}(x-\tau)^{-1/2}\widetilde{K}(x,\tau)(I_1 + I_2 + I_3)(\tau)\mathrm{d}\tau\Big\|_{\omega^{\alpha,\beta}} \\ &\quad + C(\|I_1\|_{\omega^{\alpha,\beta}} + \|I_2\|_{\omega^{\alpha,\beta}} + \|I_3\|_{\omega^{\alpha,\beta}}) \\ &\leqslant C(\|I_1\|_{\omega^{\alpha,\beta}} + \|I_2\|_{\omega^{\alpha,\beta}} + \|I_3\|_{\omega^{\alpha,\beta}}).\end{aligned}$$

Now, by applying (12), we obtain that

$$\|I_1\|_{\omega^{\alpha,\beta}} = \|u - I_N^{\alpha,\beta}u\|_{\omega^{\alpha,\beta}} \leqslant CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}. \tag{37}$$

By using Lemma 6 and (29), we have

$$\|I_2\|_{\omega^{\alpha,\beta}} = \Big\|\sum_{i=0}^{N} I_{i,2}h_i(x)\Big\|_{\omega^{\alpha,\beta}} \leqslant C\max_{0\leqslant i\leqslant N}|I_{i,2}| \leqslant CN^{-m}K^*(\|e\|_\infty + \|u\|_\infty). \tag{38}$$

Finally, it follows from Lemmas 6, 4, and (16) that

$$\begin{aligned}\|I_3\|_{\omega^{\alpha,\beta}} &= \|(I_N^{\alpha,\beta} - I)\mathscr{M}e\|_{\omega^{\alpha,\beta}} \\ &= \|(I_N^{\alpha,\beta} - I)(\mathscr{M}e - \mathscr{T}_N\mathscr{M}e)\|_{\omega^{\alpha,\beta}} \\ &\leqslant C\|\mathscr{M}e - \mathscr{T}_N\mathscr{M}e\|_\infty \\ &\leqslant CN^{-\kappa}\|\mathscr{M}e\|_{0,\kappa} \\ &\leqslant CN^{-\kappa}\|e\|_\infty,\end{aligned} \tag{39}$$

where, in the last step, we used Lemma 4 for any $\kappa \in (0, 1/2)$. By the convergence result in Theorem 1, we obtain that

$$\|I_3\|_{\omega^{\alpha,\beta}} \leqslant \begin{cases} CN^{\frac{1}{2}-m-\kappa} \log N(|u|_{H^{m;N}_{\omega^c}} + N^{-1/2} K^* \|u\|_\infty), & -1 < \alpha, \beta \leqslant -\frac{1}{2}, \\ CN^{1+\gamma-m-\kappa}(|u|_{H^{m;N}_{\omega^c}} + N^{-1/2} K^* \|u\|_\infty), & \gamma = \max\{\alpha, \beta\} < 0, \end{cases} \tag{40}$$

for N sufficiently large and any κ satisfying (33). The desired estimate (34) is obtained by combining (37), (38), and (40). □

Finally, we can derive the main result of this paper, i.e., the error estimates for the numerical solutions to the AVIE (1).

Theorem 3 *Let y and y_N be the exact solution and approximated solution of the Volterra integral equation* (1), *respectively. If the given data $g(t)$ and $K(t,s)$ in* (1) *belong to $C^m(I)$, then*

$$\|y - y_N\|_{L^\infty(I)} \leqslant \begin{cases} CN^{\frac{1}{2}-m} \log N(|y(t(\cdot))|_{H^{m;N}_{\omega^c}} + N^{-1/2} K^* \|y(t(\cdot))\|_\infty), & -1 < \alpha, \beta \leqslant -\frac{1}{2}, \\ CN^{1+\gamma-m}(|y(t(\cdot))|_{H^{m;N}_{\omega^c}} + N^{-1/2} K^* \|y(t(\cdot))\|_\infty), & \gamma = \max\{\alpha, \beta\} < 0, \end{cases}$$

and

$$\|y - y_N\|_{\omega^{\alpha,\beta}} \leqslant \begin{cases} CN^{-m}(U_2 + N^{\frac{1}{2}-\kappa} \log N U_1), & -1 < \alpha, \beta \leqslant -\frac{1}{2}, \\ CN^{-m}(U_2 + N^{1+\gamma-\kappa} U_1), & \gamma = \max\{\alpha, \beta\} < 0, \end{cases}$$

for any κ satisfying (33), *where*

$$U_1 = K^* |y(t(\cdot))|_{H^{m;N}_{\omega^c}}, \quad U_2 = |y(t(\cdot))|_{H^{m;N}_{\omega^{\alpha,\beta}}} + K^* \|y(t(\cdot))\|_\infty,$$

and K^* is defined by (20).

Remark 1 In this paper, we consider the spectral collocation methods based on the Jacobi-Gauss points corresponding to the weights $\omega^{\alpha,\beta}$, $-1 < \alpha, \beta < 0$. The reason for this consideration is that, we employ Lemmas 2 and 4 to estimate the bound for $\|I_3\|_\infty$ in Eq. (31). However, this is not the only case for the convergence results to hold. For example, in [11], for the special Chebyshev weight

$$\alpha = \beta = -\frac{1}{2},$$

Lemma 2 holds not only for the Gauss points but also for the Gauss-Lobatto points, i.e., the bound for Lebesgue constant $\|I_N^{CGL}\|_\infty = O(\log N)$, where I_N^{CGL} based on the Chevbyshev-Gauss-Lobatto points. This means that similar convergence results in Theorems 1 and 2 also hold for the Chebyshev-Gauss-Lobatto points.

5 Numerical experiments

5.1 Implementation

Here, we choose to use the Lagrangian polynomials as a basis of the approximation spaces. Let $\{h_i\colon i=0,\dots,N\}$ be the Lagrangian polynomials associated with GJ points $\{x_i^{\alpha,\beta}\colon i=0,\dots,N\}$. That is, $h_i(x)\in\mathscr{P}_N(\Lambda)$, such that $h_i(x_k^{\alpha,\beta})=\delta_{ik}$, where δ denotes the Kronecker function. It is seen that the set $\{h_i\colon i=0,\dots,N\}$ forms a basis of $\mathscr{P}_N(\Lambda)$:

$$\mathscr{P}_N(\Lambda)=\operatorname{span}\{h_i(x)\colon i=0,\dots,N\}.$$

Let $\{u_j=u_N(x_j^{\alpha,\beta})\}_{j=0}^N$. By expressing u_N in this basis

$$u_N(x)=\sum_{j=0}^N u_jh_j(x)\implies u_N(\tau(x_i^{\alpha,\beta},\theta_k))=\sum_{j=0}^N u_jh_j(\tau(x_i^{\alpha,\beta},\theta_k)),\tag{41}$$

the scheme (11) is equivalent to

$$u_i=f(x_i^{\alpha,\beta})+\sum_{j=0}^N u_j\bigg(\sum_{k=0}^N K_1(x_i^{\alpha,\beta},\tau(x_i^{\alpha,\beta},\theta_k))h_j(\tau(x_i^{\alpha,\beta},\theta_k))\rho_k\bigg),\qquad 0\leqslant i\leqslant N,\tag{42}$$

and we arrive at the matrix statement of (42):

$$(\boldsymbol{I}-\boldsymbol{A})U_N=\boldsymbol{F},\tag{43}$$

where $\boldsymbol{I}=\boldsymbol{I}_{N\times N}$ denotes the identity matrix,

$$\boldsymbol{F}=[f(x_0^{\alpha,\beta}),\dots,f(x_N^{\alpha,\beta})]^{\mathrm{T}},\quad U_N=[u_0,\dots,u_N]^{\mathrm{T}},$$

and $\boldsymbol{A}=(\boldsymbol{A}_{ij})_{N\times N}$ with

$$\boldsymbol{A}_{ij}=\sum_{k=0}^N K_1(x_i^{\alpha,\beta},\tau(x_i^{\alpha,\beta},\theta_k))h_j(\tau(x_i^{\alpha,\beta},\theta_k))\rho_k.$$

Noting that (43) is a nonsymmetric system, we use Bicgstab [16] iterative methods to solve it.

5.2 Numerical results

In this subsection, we present the numerical results obtained by the proposed collocation spectral method. The estimates in Theorem 3 indicates that the convergence of numerical solutions is exponential if the exact solution is smooth. To confirm the theoretical prediction, a numerical experiment is carried out by considering the following example.

Example 1 Consider the linear Abel-Volterra integral equations of the second kind:

$$y(t) = b(t) - \int_0^t (t-s)^{-1/2} y(s)\mathrm{d}s, \quad 0 \leqslant t \leqslant T, \tag{44}$$

with the exact solution

$$y(t) = \frac{\sin t}{\sqrt{t}}.$$

By calculation,

$$b(t) = \frac{\sin t}{\sqrt{t}} + \pi \sin \frac{t}{2}\, J_0\Big(\frac{t}{2}\Big),$$

where $J_0(z)$ is the Bessel function defined by

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-z^2)^k}{(k\,!)^2\, 4^k}.$$

This problem has the property stated at the beginning of this paper, i.e.,

$$y'(t) = \frac{\cos t}{\sqrt{t}} + \frac{\sin t}{\sqrt{t^3}} \sim \frac{1}{\sqrt{t}} \quad (t = 0^+),$$

which is singular at $t = 0^+$. In the theory presented in the previous section, our main concern is the regularity of the transformed solution. For the present problem, by employing nonlinear transformation $t = z^2$, the smooth solution

$$\overline{u}(z) = y(z^2) = \frac{\sin z^2}{z}$$

is obtained.

The main purpose is to check the convergence behavior of numerical solutions with respect to the polynomial degrees N for several α and β. In Fig. 1, we plot the $L^2_{\omega^{\alpha,\beta}}$-errors and L^∞-errors in semi-log scale. To confirm the

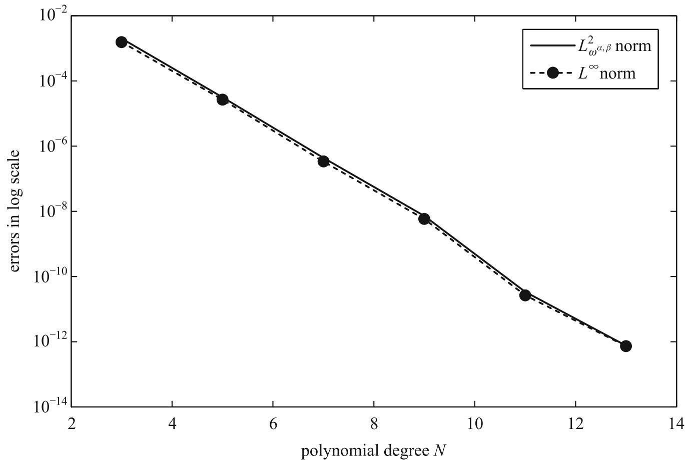

Fig. 1 L^∞ and $L^2_{\omega^{\alpha,\beta}}$ errors versus the polynomial degree N $(\alpha = -1/2,\ \beta = -1/2)$

theoretical prediction, we plot the errors as functions of the polynomial degrees N for $\alpha = -1/2$, $\beta = -1/2$. As expected, the errors show an exponential decay, since in this semi-log representation one observes that the error variations are essentially linear versus the degrees of polynomial. This indicates that the convergence of the spectral collocation method is exponential.

Acknowledgements This work was are supported by the FRG Grant of Hong Kong Baptist University and the RGC Grants provided by Research Grant Council of Hong Kong, China.

References

1. Ali I, Brunner H, Tang T. A spectral method for pantograph-type delay differential equations and its convergence analysis. J Comput Math, 2009, 27: 254–265
2. Ali I, Brunner H, Tang T. Spectral methods for pantograph-type differential and integral equations with multiple delays. Front Math China, 2009, 4(1): 49–61
3. Brunner H. Polynomial spline collocation methods for Volterra integro-differential equations with weakly singular kernels. IMA J Numer Anal, 1986, 6: 221–239
4. Brunner H. Collocation Methods for Volterra Integral and Related Functional Equations Methods. Cambridge: Cambridge University Press, 2004
5. Canuto C, Hussaini M Y, Quarteroni A, Zang T A. Spectral Methods Fundamentals in Single Domains. Berlin: Springer-Verlag, 2006
6. Chen Y, Tang T. Spectral methods for weakly singular Volterra integral equations with smooth solutions. J Comput Appl Math, 2009, 233: 938–950
7. Chen Y, Tang T. Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel. Math Comp, 2010, 79: 147–167
8. Diogo T, McKee S, Tang T. Collocation methods for second-kind Volterra integral equations with weakly singular kernels. Proc Roy Soc Edinburgh Sext A, 1994, 124: 199–210
9. Gogatishvill A, Lang J. The generalized hardy operator with kernel and variable integral limits in Banach function spaces. J Inequal Appl, 1999, 4(1): 1–16
10. Graham I G, Sloan I H. Fully discrete spectral boundary integral methods for Helmholtz problems on smooth closed surfaces in $\mathbb{R}^3$. Numer Math, 2002, 92: 289–323
11. Hesthaven J S. From electrostatics to almost optimal nodal sets for polynomial interpolation in a simplex. SIAM J Numer Anal, 1998, 35: 655–676
12. Kufner A, Persson L E. Weighted Inequalities of Hardy Type. New York: World Scientific, 2003
13. Lubich Ch. Fractional linear multi-step methods for Abel-Volterra integral equations of the second kind. Math Comp, 1985, 45: 463–469
14. Mastroianni G, Occorsio D. Optimal systems of nodes for Lagrange interpolation on bounded intervals. A survey. J Comput Appl Math, 2001, 134: 325–341
15. Nevai P. Mean convergence of Lagrange interpolation. III. Trans Amer Math Soc, 1984, 282: 669–698
16. Quarteroni A, Valli A. Numerical Approximation of Partial Differential Equations. Berlin: Springer-Verlag, 1997
17. Ragozin D L. Polynomial approximation on compact manifolds and homogeneous spaces. Trans Amer Math Soc, 1970, 150: 41–53
18. Ragozin D L. Constructive polynomial approximation on spheres and projective spaces. Trans Amer Math Soc, 1971, 162: 157–170

19. Samko S G, Cardoso R P. Sonine integral equations of the first kind in $L_p(0, b)$. Fract Calc & Appl Anal, 2003, 6: 235–258
20. Shen J, Tang T. Spectral and High-Order Methods with Applications. Beijing: Science Press, 2006
21. Shen J, Tang T, Wang L -L. Spectral Methods: Algorithms, Analysis and Applications. Springer Series in Computational Mathematics, Vol 41. Berlin: Springer, 2011
22. Tang T. Superconvergence of numerical solutions to weakly singular Volterra integro-differential equations. Numer Math, 1992, 61: 373–382
23. Tang T. A note on collocation methods for Volterra integro-differential equations with weakly singular kernels. IMA J Numer Anal, 1993, 13: 93–99
24. Tang T, Xu X, Cheng J. On spectral methods for Volterra type integral equations and the convergence analysis. J Comput Math, 2008, 26: 825–837
25. Wan Z, Chen Y, Huang Y. Legendre spectral Galerkin methods for second-kind Volterra integral equations. Front Math China, 2009, 4(1): 181–193