J Sci Comput
DOI 10.1007/s10915-015-0069-5

Numerical Solutions for Weakly Singular Volterra Integral Equations Using Chebyshev and Legendre Pseudo-Spectral Galerkin Methods

Xianjuan Li[1] · Tao Tang[2,3] · Chuanju Xu[4]

Received: 3 October 2014 / Revised: 20 June 2015 / Accepted: 1 July 2015

Abstract In this paper we present and analyze Chebyshev and Legendre pseudo-spectral methods for the second kind Volterra integral equations with weakly singular kernel $(x-s)^{-\mu}, 0<\mu<1$. The proposed methods are based on the Gauss-type quadrature formula for approximating the integral operators involved in the equations. The present work is an extension of the earlier proposed spectral Jacobi–Galerkin method for the second kind Volterra integral equations with regular kernels (Xie et al. in J Sci Comput 53(2):414–434, [21]). A detailed convergence analysis is carried out, and several error estimates in L^∞ and L^2_ω norms are obtained. Numerical examples are considered to verify the theoretical predictions.

Xianjuan Li: supported by National NSF of China (11201077), NSF of Fujian Province (2012J01007), and Start-up fund of Fuzhou University (0460022456).
Tao Tang: partially supported by Hong Kong Research Grant Council GIF Grants and Hong Kong Baptist University FRG Grants.
Chuanju Xu: partially supported by National NSF of China (Grant Numbers 11471274, 11421110001, and 91130002).

✉ Chuanju Xu
cjxu@xmu.edu.cn

Xianjuan Li
xjli@fzu.edu.cn

Tao Tang
ttang@math.hkbu.edu.hk

1 College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350108, China

2 Department of Mathematics & Institute for Computational and Theoretical Studies, Hong Kong Baptist University, Kowloon Tong, Hong Kong

3 Present Address: Department of Mathematics, South University of Science and Technology, Shenzhen 518055, Guangdong, China

4 School of Mathematical Sciences and Fujian Provincial Key Laboratory of Mathematical Modeling and High Performance Scientific Computing, Xiamen University, Xiamen 361005, China

Published online: 15 July 2015

1 Introduction

Volterra integral equations (VIEs) find application in many disciplines, such as electromagnetic scattering, demography, viscoelastic materials, insurance mathematics, etc. They have been subject of many theoretical and numerical investigations. Among the numerical methods for VIEs, the spectral approximations have been attracting more attention recently. The latest progress in this direction includes but not exclusive to: the collocation methods for Volterra integral and related functional equations, see, e.g., [2], the Jacobi spectral-collocation method for VIEs with a weakly singular kernel [6], spectral Petrov–Galerkin methods for the second kind Volterra type integro-differential equations [20], a spectral Jacobi-collocation approximation for VIEs with Abel type singular kernel [11], a spectral collocation method for weakly singular VIEs with pantograph delays [22], a spectral method for Volterra functional integro-differential equations with delays and smooth kernel [18].

In particular, Chen and Tang [6] proposed and analyzed a Jacobi-collocation spectral method for the second kind VIEs with a weakly singular kernel. Some function transformations and variable transformations are employed to change the equation into a new VIE defined on the standard interval $[-1, 1]$, so that the solution of the new equation possesses better regularity and the orthogonal polynomial theory can be applied accordingly. Li and Tang [11] considered the special case of VIEs with weakly singular kernel $(t-s)^{-1/2}$. For this special case, a variable transformation was applied to get a new VIE with regular kernel. Zhang et al. [22] investigated the VIEs of second kind with weakly singular kernel and pantograph delays. They also proposed to use function and variable transformations to convert the equation into VIEs with pantograph delays so that the Jacobi orthogonal polynomials could be applied.

The work more relevant to the present one is given by Xie et al. in [21] which investigated a spectral Jacobi–Galerkin approach for second kind VIEs. The Gauss–Legendre quadrature formula was used to approximate the integral operator. A rigorous error estimate was given under the assumption that both the kernel function and the source function are smooth. In [7], the authors investigated a spectral Jacobi-collocation method for VIEs with a kernel of the form $(t-s)^{-\alpha}k(t,s)$, $0<\alpha<1$. The error estimate was obtained under the assumption that the solution is smooth.

In this paper we consider the second kind VIEs with weakly singular kernels, and propose a spectral method based on Chebyshev and Legendre polynomial approximation. Since the solutions of the underlying equation are not smooth, it is natural to consider the numerical method based on a weighted weak formulation and carry out the error analysis in suitable weighted Sobolev spaces. The present work is an extension of the method proposed in [21] for regular kernels, but the extension requires much technicality due to the involvement of the singular factor $(t-s)^{\alpha}$. In particular, the error analysis in the weighted Sobolev space uses quite different space and analysis tools. As compared to Chen and Tang [6] and Chen et al. [7] who considered collocation type spectral methods, the present work is of Galerkin type, which is set up in a more general framework. More importantly, some useful tools for Galerkin-type approximations can be employed to derive optimal error estimates. It is also worthwhile to emphasize that the convergence rates obtained in the last theorem of this paper are better than those in [7].

The outline of the paper is as follows. In Sect. 2, the spectral and pseudo-spectral Galerkin approaches are presented for VIEs. Some lemmas useful for the convergence analysis are provided in Sect. 3. The convergence analysis is given in Sect. 4 for spectral and pseudo-spectral Jacobi–Galerkin methods. Numerical experiments are carried out in Sect. 5, which are used to validate the theoretical results obtained in Sect. 4.

2 Chebyshev/Legendre Spectral Galerkin Methods

Given the source function $g(x)$, we consider the following Volterra equation:

$$u(x) + Su(x) = g(x), \quad x \in \Lambda := (-1, 1). \tag{2.1}$$

In (2.1), the integral operator S is defined by

$$Su(x) = \int_{-1}^{x} (x-s)^{-\mu} K(x,s)u(s)\mathrm{d}s, \quad 0 < \mu < 1, \tag{2.2}$$

where the function $K(x,s)$ is assumed to be smooth.

Now we describe the numerical method for solving (2.1). We first define $\mathcal{P}_N$ as the polynomial spaces of degree less than or equal to N. We denote by $J_N^{\alpha,\beta}(x)$ the Jacobi polynomial of degree N with respect to the weight function $\omega^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$, $-1 < \alpha, \beta < 1$. Let $x_i^{\alpha,\beta}$ be the points of the Gauss–Jacobi (GJ) quadrature formula, defined by

$$J_{N+1}^{\alpha,\beta}\left(x_i^{\alpha,\beta}\right) = 0, \quad i = 0, \ldots, N,$$

arranged by increasing order: $x_0^{\alpha,\beta} < x_1^{\alpha,\beta} < \cdots < x_N^{\alpha,\beta}$. The associated weights of the GJ quadrature formula are denoted by $\omega_i^{\alpha,\beta}$, $0 \le i \le N$. Particularly, we use $\omega(x)$ to denote the Chebyshev or Legendre weight function and use ω_i to denote the associated weights of the Gauss quadrature formula. It is well known that the numerical quadrature

$$\int_{-1}^{1} u(x)(1-x)^\alpha(1+x)^\beta \mathrm{d}x \simeq \sum_{i=0}^{N} u\left(x_i^{\alpha,\beta}\right)\omega_i^{\alpha,\beta} \tag{2.3}$$

is exact for all functions $u \in \mathcal{P}_{2N+1}(\Lambda)$.

We recall that the Lebesgue space $L^2_{\omega^{\alpha,\beta}}(\Lambda)$ is defined as

$$L^2_{\omega^{\alpha,\beta}}(\Lambda) = \left\{ u \text{ measurable over } \Lambda \text{ and } \int_\Lambda u^2(x)\omega^{\alpha,\beta}(x)\mathrm{d}x < \infty \right\},$$

which is equipped with the scalar product

$$\forall \phi, \psi \in L^2_{\omega^{\alpha,\beta}}(\Lambda), \quad (\phi, \psi)_{\omega^{\alpha,\beta}} = \int_\Lambda \omega^{\alpha,\beta}(x)\phi(x)\psi(x)\mathrm{d}x.$$

The associated norm is defined by $\|\phi\|_{0,\omega^{\alpha,\beta}} := (\phi,\phi)^{1/2}_{\omega^{\alpha,\beta}}$. If $\omega^{\alpha,\beta} \equiv 1$, the symbol $\omega^{\alpha,\beta}$ will be omitted from the subscript.

The Chebyshev (resp. Legendre) spectral Galerkin method for (2.1) is to find $u_N \in \mathcal{P}_N(\Lambda)$ such that

$$(u_N, v_N)_\omega + (Su_N, v_N)_\omega = (g, v_N)_\omega, \quad \forall v_N \in \mathcal{P}_N(\Lambda), \tag{2.4}$$

where $\omega(x) = \omega^{-\frac{1}{2},-\frac{1}{2}}(x)$ (resp. $\omega \equiv 1$).

In order to obtain high order accurate numerical solution of the problem (2.4), the integral term in (2.4) has to be evaluated with high accuracy. Next we propose a numerical quadrature to efficiently compute the term involving the integral operator S. First we notice

$$
\begin{aligned}
Su_N(x) &= \int_{-1}^{x} (x-s)^{-\mu} K(x,s)\, u_N(s)\mathrm{d}s \\
&= \left(\frac{1+x}{2}\right)^{1-\mu} \int_{-1}^{1} (1-\theta)^{-\mu} K\left(x, s_x(\theta)\right) u_N\left(s_x(\theta)\right) \mathrm{d}\theta \\
&= (1+x)^{1-\mu} \int_{-1}^{1} (1-\theta)^{-\mu} \bar{K}(x,\theta) u_N\left(s_x(\theta)\right) \mathrm{d}\theta,
\end{aligned}
\tag{2.5}
$$

where

$$
s_x(\theta) = \frac{1+x}{2}\theta + \frac{x-1}{2}, \quad \theta \in \Lambda, \qquad \bar{K}(x,\theta) = \frac{1}{2^{1-\mu}} K\left(x, s_x(\theta)\right). \tag{2.6}
$$

Thus

$$
\begin{aligned}
(Su_N, v_N)_\omega &= \int_{-1}^{1} \left(1-x^2\right)^{\alpha} (1+x)^{1-\mu} v_N(x) \int_{-1}^{1} (1-\theta)^{-\mu} \bar{K}\left(x,\theta\right) u_N\left(s_x(\theta)\right) \mathrm{d}\theta \mathrm{d}x \\
&= (\tilde{S}u_N, v_N)_\rho,
\end{aligned}
$$

where $\alpha = -\frac{1}{2}$ in the case $\omega(x) = \omega^{-\frac{1}{2},-\frac{1}{2}}(x)$, and $\alpha = 0 \quad \text{if } \omega \equiv 1$,

$$
\tilde{S}u_N = \frac{Su_N}{(1+x)^{1-\mu}}, \qquad \rho(x) = w^{\alpha,\alpha+1-\mu}(x), \qquad \bar{K}(x,\theta) = \frac{1}{2^{1-\mu}} K\left(x, s_x(\theta)\right). \tag{2.7}
$$

Now we approximate the integral terms by using the Gauss quadrature related to different Jacobi weights, i.e.,

$$
\begin{aligned}
&(u_N, v_N)_\omega \approx (u_N, v_N)_{N,\omega} := \sum_{i=0}^{N} u_N(x_i) v_N(x_i) \omega_i, \\
&\tilde{S}u_N \approx \tilde{S}_N u_N := \sum_{k=0}^{N} \bar{K}(x,\theta_k) u_N(s_x(\theta_k)) \varrho_k, \\
&(\tilde{S}u_N, v_N)_\rho \approx (\tilde{S}_N u_N, v_N)_{2N,\rho} := \sum_{i=0}^{2N} \tilde{S}_N u_N(y_i) v_N(y_i) \rho_i,
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_k = x_k^{-\mu,0}, \quad \varrho_k = \omega_k^{-\mu,0}, \quad y_i = x_i^{\alpha,\alpha+1-\mu}, \quad \rho_i = \omega_i^{\alpha,\alpha+1-\mu}, \\
k = 0,1,\ldots,N; \quad i = 0,1,\ldots,2N.
\end{aligned}
\tag{2.8}
$$

The above approximations lead to the following pseudo-spectral method: find $\bar{u}_N \in \mathcal{P}_N(\Lambda)$ such that

$$
(\bar{u}_N, v_N)_{N,\omega} + (\tilde{S}_N \bar{u}_N, v_N)_{2N,\rho} = (g, v_N)_\omega, \quad \forall v_N \in \mathcal{P}_N(\Lambda), \tag{2.9}
$$

or, equivalently,

$$
\begin{aligned}
&\sum_{i=0}^{N} \bar{u}_N(x_i) v_N(x_i) \omega_i + \sum_{i=0}^{2N} \sum_{k=0}^{N} \bar{K}(y_i,\theta_k) \bar{u}_N\left(s_{y_i}(\theta_k)\right) \rho_k v_N(y_i) \rho_i \\
&\quad = (g, v_N)_\omega, \quad \forall v_N \in \mathcal{P}_N(\Lambda).
\end{aligned}
\tag{2.10}
$$

The remaining sections are mainly devoted to carrying out error analysis for both discrete problems (2.4) and (2.9) under the assumption that the exact solution is smooth enough.

Remark 2.1 The smoothness assumption on the exact solution used in the analysis makes sense for the following reasons: 1) if the known terms take some special forms, such as functions having weakly singularity at the end points, then the corresponding solutions would be sufficiently smooth; 2) more importantly, the convergence analysis carried out in this paper is extendable to some cases where the solution has weak singularity. It has been well known [2] that if $g \in C^m(\bar{\Lambda})$ and $K \in C^m(\bar{\Lambda} \times \bar{\Lambda})$ with $K(s,s) \neq 0$ in $\bar{\Lambda} = [-1, 1]$, then the solution of (2.1) can be expressed as

$$u(x) = \sum_{(j,k)\in G} \gamma_{j,k} (1+x)^{j+k(1-\mu)} + u_r(x), \quad x \in \Lambda, \tag{2.11}$$

where $G := \{(j,k) : \ j, k$ are non-negative integers such that $j + k(1-\mu) < m\}$, $\gamma_{j,k}$ are constants, and $u_r(\cdot) \in C^m(\bar{\Lambda})$. This means the solution for smooth known data is the sum of a smooth function and a singular part of known form. Now we consider the case μ is a rational number, i.e., $\mu = \frac{q}{p}, q < p, q, p \in \mathrm{Z}^+$, then the Eq. (2.1) can be transformed into the following equation:

$$\begin{aligned} \bar{u}(x) = p \int_{-1}^{(} x - s)^{-\mu} \left(\sum_{i=0}^{p-1} (x+1)^{p-1-i} (s+1)^i \right)^{-\mu} K\left((x+1)^p - 1, (s+1)^p - 1 \right) \\ \times (s+1)^{p-1} \bar{u}(s) \mathrm{d}s + \bar{g}(x) \end{aligned} \tag{2.12}$$

by the variable change

$$x \to (x+1)^p - 1, \ s \to (s+1)^p - 1, \tag{2.13}$$

where $\bar{u}(x) = u((x+1)^p - 1)$, $\bar{g}(x) = g((x+1)^p - 1)$. Applying the linear transformation $s_x(\theta) = \frac{1+x}{2}\theta + \frac{x-1}{2}$, it is seen that the new integral operator, denoted by $\bar{S}\bar{u}$, in the right hand side of (2.12) reads:

$$\begin{aligned} &\bar{S}\bar{u}(x) \\ &:= p \int_{-1}^{x} (x-s)^{-\mu} \left(\sum_{i=0}^{p-1} (x+1)^{p-1-i} (s+1)^i \right)^{-\mu} \\ &K\left((x+1)^p - 1, (s+1)^p - 1 \right) (s+1)^{p-1} \bar{u}(s) \mathrm{d}s \\ &= p \int_{-1}^{1} (1-\theta)^{-\mu} \left(\frac{x+1}{2} \right)^{p-\mu p} \left(\sum_{i=0}^{p-1} 2^{p-1-i} (1+\theta)^i \right)^{-\mu} \\ &\cdot K\left((x+1)^p - 1, (s_x(\theta)+1)^p - 1 \right) (1+\theta)^{p-1} \bar{u}(s_x(\theta)) \mathrm{d}\theta \\ &= \int_{-1}^{1} (1-\theta)^{-\mu} \bar{K}(x,\theta) \bar{u}\left(s_x(\theta)\right) \mathrm{d}\theta, \end{aligned}$$

where

$$\begin{aligned} \bar{K}(x,\theta) =: p \left(\frac{x+1}{2} \right)^{p-\mu p} \left(\sum_{i=0}^{p-1} 2^{p-1-i} (1+\theta)^i \right)^{-\mu} \\ K\left((x+1)^p - 1, (s_x(\theta)+1)^p - 1 \right) (1+\theta)^{p-1}. \end{aligned}$$

The new kernel function $\bar{K}(x,\theta)$ now plays the same role as the one in (2.5), and it can be directly verified that the two main conditions for the convergence analysis are satisfied. First, all three terms $\max_{-1\le\theta\le1}\left\|\bar{K}(\cdot,\theta)\right\|_{H_\rho^{m;N}}$, $\max_{-1\le x\le1}\left\|\bar{K}(x,\cdot)\right\|_{H_\varrho^{m;N}}$ and K^* appearing in Theorem 4.4 are bounded. Second, it deduces from (2.11) that the solution $\bar{u}(x)$ to the new Eq. (2.12) is smooth enough. Therefore the above spectral method and convergence analysis presented in the next sections is directly applicable to the transformed Eq. (2.12). The numerical tests given in the last section confirm this point.

In fact, the smoothing strategy has been used by a number of authors for different types of equations. To mention a few most relevant among many, Monegato and Scuderi [14] considered classical Fredholm integral equations of the second kind, and proposed a nonlinear transformation to absorb the singularities of the solution. Baratella and Orsi [1] considered the linear VIEs of the second kind and used a smoothing change of variable so that the solution of the transformed equation is smooth, then solved the resulting equation by standard product integration methods. Chen and Tang [5] proposed a preliminary smoothing transformation to remove a single term singularity like $(1+x)^{\mu+m-1}$. Kolk and Pedas [9] used suitable smoothing techniques to treat the diagonal and end point singularity. It is also worth to mention the work by Sidi [19] and Elliott and Prossdorf [8], which used trigonometric and rational transformations for numerical integrations. The smoothing strategy is specially efficient in the case when μ is a rational $\frac{q}{p}, q<p, q, p\in \mathrm{Z}^+$. One should also be aware from (2.13) that the larger is the transformation exponent p, the sharper is the transformed solution near the left end point. Thus this smoothing is usually used for moderate p.

3 Some Preliminary Results

In this section, we will give some useful lemmas which play a significant role in the convergence analysis later. Let C stand for a generic positive constant independent of any functions and of any discretization parameters. We use the expression $A\lesssim B$ to mean that $A\le CB$, and use the expression $A\simeq B$ to mean that $A\lesssim B\lesssim A$.

First, we define the projection operator Π_N^ω as follows: for all $u\in L^2_\omega(\Lambda)$, $\Pi_N^\omega u\in\mathcal{P}_N(\Lambda)$ is given by

$$(\Pi_N^\omega u, v_N)_\omega=(u,v_N)_\omega,\quad \forall v_N\in\mathcal{P}_N(\Lambda).$$

Then let $I_N^{\alpha,\beta}$ denotes the interpolation operator based on $N+1$ degree Jacobi Gauss or Gauss–Radau points associated to the weight function $\omega^{\alpha,\beta}(x)$, i.e., $\forall u\in C(\Lambda)$, $I_N^{\alpha,\beta}u\in\mathcal{P}_N(\Lambda)$ such that

$$(I_N^{\alpha,\beta}u)(x_i^{\alpha,\beta})=u(x_i^{\alpha,\beta}),\ i=0,1,\ldots,N. \tag{3.1}$$

Define the space

$$W^{m,p}_{\omega^{\alpha,\beta}}(\Lambda)=\left\{v: D^k v\in L^p_{\omega^{\alpha,\beta}}(\Lambda), 0\le k\le m, 1\le p\le+\infty\right\},$$

equipped with the norm

$$\|v\|_{W^{m,p}_{\omega^{\alpha,\beta}}}=\left(\sum_{k=0}^m\left\|D^k v\right\|^p_{L^p_{\omega^{\alpha,\beta}}}\right)^{1/p},$$

where $D^k=\frac{d^k}{dx^k}$.

Let $H^m_{\omega^{\alpha,\beta}}(\Lambda) = W^{m,2}_{\omega^{\alpha,\beta}}(\Lambda)$. In bounding approximation errors of the projection and interpolation operators, it may be more convenient to use the seminorms as follows:

$$|v|_{H^{m;N}_{\omega^{\alpha,\beta}}(\Lambda)} = \left(\sum_{k=\min(m,N+1)}^{m} \| |D^k v| \|^2_{L^2_{\omega^{\alpha,\beta}}(\Lambda)} \right)^{\frac{1}{2}}.$$

Moreover, we will need to recall some more Sobolev spaces: for real $s \geq 0$, let

$$H^s(R) = \left\{ v(t) | v \in L^2(\mathbb{R});\ (1 + |\omega|^2)^{\frac{s}{2}} \mathcal{F}(v)(\omega) \in L^2(\mathbb{R}) \right\},$$

and the norm:

$$\|v\|_{s,\mathbb{R}} = \left\| (1 + |\omega|^2)^{\frac{s}{2}} \mathcal{F}(v)(\omega) \right\|_{0,\mathbb{R}},$$

where $\mathcal{F}(v)$ denotes the Fourier transform of v. For bounded domain Λ, we define space:

$$H^s(\Lambda) = \left\{ v \in L^2(\Lambda) |\ \exists\, \tilde{v} \in H^s(\mathbb{R}) \text{ such that } \tilde{v}|_\Lambda = v \right\},$$

endowed with the norm:

$$\|v\|_{s,\Lambda} = \inf_{\tilde{v} \in H^s(\mathbb{R}), \tilde{v}|_\Lambda = v} \|\tilde{v}\|_{s,\mathbb{R}}.$$

Let $H^s_0(\Lambda)$ be the closure of $C^\infty_0(\Lambda)$ with respect to norm $\|\cdot\|_{s,\Lambda}$. As in [12], for $s > 0$, we define the seminorm:

$$|v|_{{}^lH^s(\Lambda)} := \|{}_{-1}D^s_x v\|_{L^2(\Lambda)},$$

where left Riemann–Liouville fractional derivative ${}_{-1}D^s_x$ is defined by:

$${}_{-1}D^s_x v(x) = \frac{1}{\Gamma(1-s)} \frac{d}{dx} \int_{-1}^{x} \frac{v(s)ds}{(x-s)^s}, \quad \forall x \in \Lambda, \tag{3.2}$$

equipped with the norm:

$$\|v\|_{{}^lH^s(\Lambda)} := \left(\|v\|^2_{L^2(\Lambda)} + |v|^2_{{}^lH^s(\Lambda)} \right)^{1/2}.$$

Let ${}^lH^s_0(\Lambda)$ be the closure of $C^\infty_0(\Lambda)$ with respect to norm $\|\cdot\|_{{}^lH^s(\Lambda)}$.

Property 3.1 *(see [15]) (Fourier transform) For all real s, $v \in C^\infty_0(\mathbb{R})$, let $\mathcal{F}$ denote the Fourier transform operator, then*

$$\mathcal{F}({}_{-\infty}D^s_x v(x)) = (i\omega)^s \mathcal{F}(v)(\omega),$$
$$\mathcal{F}({}_xD^s_{+\infty} v(x)) = (-i\omega)^s \mathcal{F}(v)(\omega).$$

Lemma 3.1 *[12] For $s > 0$, $s \neq n - 1/2$ with n being an integer, the spaces ${}^lH^s_0(\Lambda)$ and $H^s_0(\Lambda)$ are equal and their seminorms and norms are all equivalent to each other.*

Lemma 3.2 *[3] Given $v \in H^s_{\omega^{\alpha,\beta}}(\Lambda)$, $s \geq 0$.*

(i) If $\alpha, \beta > -1$, then

$$\|v - \Pi^{\alpha,\beta}_N v\|_{0,\omega^{\alpha,\beta}} \lesssim N^{-s} \|v\|_{H^s_{\omega^{\alpha,\beta}}(\Lambda)}, \quad 0 \leq s < 1, \tag{3.3}$$

$$\|v - \Pi^{\alpha,\beta}_N v\|_{0,\omega^{\alpha,\beta}} \lesssim N^{-s} \|v\|_{H^{s;N}_{\omega^{\alpha,\beta}}(\Lambda)}, \quad s \in \mathbb{Z}^+, \tag{3.4}$$

$$\|v - I^{\alpha,\beta}_N v\|_{0,\omega^{\alpha,\beta}} \lesssim N^{-s} |v|_{H^{s;N}_{\omega^{\alpha,\beta}}(\Lambda)}, \quad s \in \mathbb{Z}^+. \tag{3.5}$$

(ii) If $\alpha = \beta = 0$, then

$$\|v - \Pi_N^{\alpha,\beta} v\|_\infty \lesssim N^{\frac{3}{4}-s} |v|_{H^{s;N}(\Lambda)}, \quad s \in \mathbb{Z}^+, \tag{3.6}$$

$$\|v - I_N^{\alpha,\beta} v\|_\infty \lesssim N^{\frac{3}{4}-s} |v|_{H^{s;N}(\Lambda)}, \quad s \in \mathbb{Z}^+. \tag{3.7}$$

(iii) If $\omega^{\alpha,\beta}$ is the Chebyshev weight, i.e., $\alpha = \beta = -\frac{1}{2}$, then

$$\|v - \Pi_N^{\alpha,\beta} v\|_\infty \lesssim N^{\frac{3}{4}-s} |v|_{H^{s;N}_{\omega^{\alpha,\beta}}(\Lambda)}, \quad s \in \mathbb{Z}^+, \tag{3.8}$$

$$\|v - I_N^{\alpha,\beta} v\|_\infty \lesssim N^{\frac{1}{2}-s} |v|_{H^{s;N}_{\omega^{\alpha,\beta}}(\Lambda)}, \quad s \in \mathbb{Z}^+. \tag{3.9}$$

Lemma 3.3 *[3] Let ω be the Chebyshev weight, and $u \in W_\omega^{m,p}(\Lambda)$ for some $m \geq 0$ and $1 \leq p \leq \infty$. Then for all $N \geq 0$,*

$$\|u - \Pi_N^\omega u\|_{W_\omega^{0,p}(\Lambda)} \lesssim \sigma_{N,p} N^{-m} \|u\|_{W_\omega^{m,p}(\Lambda)},$$

where

$$\sigma_{N,p} = \begin{cases} \log N & p = 1, \infty, \\ 1 & otherwise. \end{cases}$$

Lemma 3.4 *[13] Let $\{F_j(x)\}_{j=0}^N$ be the N-th Lagrange interpolation polynomials associated with the $N+1$ Gauss points of the Jacobi polynomials. Then*

$$\left\| |I_N^{\alpha,\beta} \right\| |_\infty := \max_{x \in \Lambda} \sum_{j=0}^N \|F_j(x)\| = \begin{cases} \mathcal{O}(\log N), & -1 < \alpha, \ \beta \leq -\frac{1}{2}, \\ \mathcal{O}\left(N^{\gamma + \frac{1}{2}}\right), \ \gamma = \max(\alpha, \beta), & \text{otherwise.} \end{cases} \tag{3.10}$$

From now on, for $r \geq 0$ and $\kappa \in [0, 1]$, $C^{r,\kappa}(\bar{\Lambda})$ will denote the space of functions whose r-th derivatives are Hölder continuous with exponent κ, endowed with the usual norm:

$$||v||_{r,\kappa} = \max_{0 \leq k \leq r} \max_{x \in \bar{\Lambda}} |\partial_x^k v(x)| + \max_{0 \leq k \leq r} \sup_{\substack{x,\, y \in \bar{\Lambda} \\ x \neq y}} \frac{\left|\partial_x^k v(x) - \partial_x^k v(y)\right|}{|x - y|^\kappa}.$$

When $\kappa = 0$, $C^{r,0}(\bar{\Lambda})$ denotes the space of functions with r continuous derivatives on $\bar{\Lambda}$, which is also commonly denoted by $C^r(\bar{\Lambda})$, endowed with the norm $|| \cdot ||_r$.

We shall make use of a result of Ragozin [16,17], which states that, for non-negative integers r and $\kappa \in (0, 1)$, there exists a constant $C_{r,\kappa} > 0$ such that for any function $v \in C^{r,\kappa}(\bar{\Lambda})$, there exists a polynomial $\mathcal{T}_N v \in \mathcal{P}_N(\Lambda)$ such that

$$||v - \mathcal{T}_N v||_\infty \leq C_{r,\kappa} N^{-(r+\kappa)} ||v||_{r,\kappa}. \tag{3.11}$$

Actually, as stated in [16,17], $\mathcal{T}_N$ is a linear operator from $C^{r,\kappa}(\bar{\Lambda})$ into $\mathcal{P}_N(\Lambda)$.

Let $\kappa \in (0, 1)$ and S be defined in (2.2). It follows from [6] that, for any function $v \in C(\bar{\Lambda})$, and $0 < \kappa < 1 - \mu$,

$$\frac{|Sv(x') - Sv(x'')|}{|x' - x''|^\kappa} \lesssim \max_{x \in \bar{\Lambda}} |v(x)|, \quad \forall x', \ x'' \in \bar{\Lambda}, x' \neq x''. \tag{3.12}$$

This implies that

$$||Sv||_{0,\kappa} \lesssim ||v||_\infty, \quad 0 < \kappa < 1 - \mu, \tag{3.13}$$

where $||\cdot||_\infty$ is the standard norm in $C(\bar{\Lambda})$. That is, S is compact as an operator from $C(\bar{\Lambda})$ to $C^{0,\kappa}(\bar{\Lambda})$.

For any function $u(x)$ on $\bar{\Lambda}$, we define the function

$$\tilde{u}(\theta) = u(\cos\theta), \quad \theta \in [-\pi, 0]. \tag{3.14}$$

Then

$$\int_{-1}^{1} u^2(x)\omega^{-\frac{1}{2},-\frac{1}{2}}(x)\mathrm{d}x = \int_{-\pi}^{0} \tilde{u}^2(\theta)\mathrm{d}\theta.$$

Lemma 3.5 *For any* $\sigma \geq 0$*, let* ω *be the Chebyshev weight. Then*

$$\left\|u - \Pi_N^{\omega} u\right\|_{0,\omega} \lesssim N^{-\sigma} |\tilde{u}|_{H^\sigma[-\pi,0]}, \quad \forall \tilde{u} \in H^\sigma(-\pi, 0),$$

where $\tilde{u} = u(\cos\theta), \theta \in [-\pi, 0]$.

Proof The proof is similar to the one of Theorem 1.1 in [4]. □

The following lemma plays a key role in the error analysis.

Lemma 3.6 *For any* $0 < \mu < 1$*, if* $0 < \sigma < \min\{\frac{1}{2}, 1-\mu\}$*, then*

$$\|Su - \Pi_N^{\omega} Su\|_{0,\omega} \lesssim N^{-\sigma} \|u\|_{0,\omega}. \tag{3.15}$$

Proof 1) We first prove the result for the Chebyshev case: $\omega(x) = \left(1-x^2\right)^{-1/2}$. From Lemma 3.5, we have

$$\|Su - \Pi_N^{\omega} Su\|_{0,\omega} \lesssim N^{-\sigma} |\widetilde{Su}|_{H^\sigma[-\pi,0]}. \tag{3.16}$$

Thus it remains to prove

$$|\widetilde{Su}|_{H^\sigma[-\pi,0]} \lesssim \|u\|_{0,\omega},\ 0 < \sigma < \min\left\{\frac{1}{2}, 1-\mu\right\}. \tag{3.17}$$

In fact, from the definitions (3.14) and (2.2), we have

$$\begin{aligned} \widetilde{Su}(\zeta) &= Su(\cos\zeta) \\ &= \int_{-1}^{\cos\zeta} (\cos\zeta - s)^{-\mu} K(\cos\zeta, s)u(s)\mathrm{d}s \\ &\overset{s=\cos\eta}{=} -\int_{-\pi}^{\zeta} k_\mu(\zeta,\eta)\tilde{u}(\eta)\sin\eta \mathrm{d}\eta, \end{aligned}$$

where $k_\mu(\zeta,\eta) = (\cos\zeta - \cos\eta)^{-\mu} K(\cos\zeta, \cos\eta)$. Combining the above result with Lemma 3.1 yields,

$$|\widetilde{Su}|^2_{H^\sigma[-\pi,0]} = \left\| \frac{d}{\mathrm{d}\theta} \int_{-\pi}^{\theta} (\theta-\zeta)^{-\sigma} \int_{-\pi}^{\zeta} k_\mu(\zeta,\eta)\tilde{u}(\eta)\sin\eta \mathrm{d}\eta\mathrm{d}\zeta \right\|^2_{L^2[-\pi,0]}. \tag{3.18}$$

The right hand side term in (3.18) can be reformulated by changing the order of integration, and making linear transformation $\zeta = \zeta(\xi) = \frac{\theta-\eta}{2}\xi + \frac{\theta+\eta}{2}$:

$$
\begin{aligned}
&\frac{d}{\mathrm{d}\theta}\int_{-\pi}^{\theta}(\theta-\zeta)^{-\sigma}\int_{-\pi}^{\zeta}k_\mu(\zeta,\eta)\tilde{u}(\eta)\sin\eta\mathrm{d}\eta\mathrm{d}\zeta\\
&=\frac{d}{\mathrm{d}\theta}\int_{-\pi}^{\theta}\int_{\eta}^{\theta}(\theta-\zeta)^{-\sigma}k_\mu(\zeta,\eta)\mathrm{d}\zeta\ \tilde{u}(\eta)\sin\eta\mathrm{d}\eta\\
&=\frac{d}{\mathrm{d}\theta}\int_{-\pi}^{\theta}\int_{-1}^{1}(1-\xi)^{-\sigma}k_\mu(\zeta(\xi),\eta)\mathrm{d}\xi\ \left(\frac{\theta-\eta}{2}\right)^{1-\sigma}\tilde{u}(\eta)\sin\eta\mathrm{d}\eta\\
&=A_1+A_2+A_3+A_4,
\end{aligned}\tag{3.19}
$$

where

$$
\begin{aligned}
A_1&=\int_{-1}^{1}(1-\xi)^{-\sigma}k_\mu(\zeta(\xi),\eta)\mathrm{d}\xi\ \left(\frac{\theta-\eta}{2}\right)^{1-\sigma}\tilde{u}(\eta)\sin\eta\Big|_{\eta=\theta},\\
A_2&=-\mu\int_{-\pi}^{\theta}\int_{-1}^{1}(1-\xi)^{-\sigma}\frac{k_\mu(\zeta(\xi),\eta)}{\cos\zeta(\xi)-\cos\eta}\sin(\zeta(\xi))\frac{\xi+1}{2}\mathrm{d}\xi\left(\frac{\theta-\eta}{2}\right)^{1-\sigma}\tilde{u}(\eta)\sin\eta\mathrm{d}\eta,\\
A_3&=\frac{1-\sigma}{2}\int_{-\pi}^{\theta}\int_{-1}^{1}(1-\xi)^{-\sigma}k_\mu(\zeta(\xi),\eta)\mathrm{d}\xi\left(\frac{\theta-\eta}{2}\right)^{-\sigma}\tilde{u}(\eta)\sin\eta\mathrm{d}\eta,\\
A_4&=-\int_{-\pi}^{\theta}\int_{-1}^{1}(1-\xi)^{-\sigma}k_{1,\mu}(\zeta(\xi),\eta)\sin(\zeta(\xi))\frac{\xi+1}{2}\mathrm{d}\xi\left(\frac{\theta-\eta}{2}\right)^{1-\sigma}\tilde{u}(\eta)\sin\eta\mathrm{d}\eta,
\end{aligned}
$$

where $k_{1,\mu}(\zeta,\eta)=(\cos\zeta-\cos\eta)^{-\mu}\,\frac{\partial K(x,y)}{\partial x}$. Next we estimate $\|A_i\|^2_{L^2[-\pi,0]}$, $i=1,\cdots,4$, term by term. For $-\pi\le\eta\le\zeta\le\theta\le 0$, it is observed that

$$(\cos\zeta-\cos\eta)^{-\mu}\lesssim(\zeta-\eta)^{-\mu}(-\zeta)^{-\frac{\mu}{2}}(-\eta)^{-\frac{\mu}{2}}(\zeta+\pi)^{-\frac{\mu}{2}}(\eta+\pi)^{-\frac{\mu}{2}},\tag{3.20}$$

$$|\sin\zeta|\le|\zeta|,\tag{3.21}$$

$$(\theta+\pi)^{-\frac{\mu}{2}}\lesssim(\zeta+\pi)^{-\frac{\mu}{2}}\lesssim(\eta+\pi)^{-\frac{\mu}{2}},\tag{3.22}$$

$$(-\eta)^{-\frac{\mu}{2}}\lesssim(-\zeta)^{-\frac{\mu}{2}}\lesssim(-\theta)^{-\frac{\mu}{2}},\tag{3.23}$$

$$|(\eta+\pi)^{-\mu}(-\eta)^{-\mu}\sin\eta|\lesssim C.\tag{3.24}$$

Using (3.20) and (3.22)–(3.24), we obtain

$$
\begin{aligned}
&\left|\int_{-1}^{1}(1-\xi)^{-\sigma}k_\mu(\xi,\eta)\mathrm{d}\xi\ \left(\frac{\theta-\eta}{2}\right)^{1-\sigma}\sin\eta\tilde{u}(\eta)\right|\\
&\lesssim\Bigg|\int_{-1}^{1}(1-\xi)^{-\sigma}(\zeta-\eta)^{-\mu}(-\zeta)^{-\frac{\mu}{2}}(-\eta)^{-\frac{\mu}{2}}(\zeta+\pi)^{-\frac{\mu}{2}}(\eta+\pi)^{-\frac{\mu}{2}}\mathrm{d}\xi\ \left(\frac{\theta-\eta}{2}\right)^{1-\sigma}\\
&\qquad\sin\eta\tilde{u}(\eta)\Bigg|\\
&\lesssim\left|\int_{-1}^{1}(1-\xi)^{-\sigma}(1+\xi)^{-\mu}\mathrm{d}\xi\ (\theta-\eta)^{1-\sigma-\mu}(\eta+\pi)^{-\mu}(-\theta)^{-\frac{\mu}{2}}(-\eta)^{-\frac{\mu}{2}}\sin\eta\tilde{u}(\eta)\right|\\
&\lesssim\left|(\theta-\eta)^{1-\sigma-\mu}(\eta+\pi)^{-\mu}(-\theta)^{-\mu}\sin\eta\tilde{u}(\eta)\right|.
\end{aligned}
$$

Inserting $\eta=\theta$ into the right hand side of the above inequality, we get

$$A_1=0.\tag{3.25}$$

Using (3.20), (3.23), and (3.24), we get

$$
\begin{aligned}
&|A_2| \\
&\simeq \left| \int_{-1}^{1} (1-\xi)^{-\sigma} \frac{k_\mu(\zeta(\xi),\eta)}{\cos\zeta(\xi)-\cos\eta} \sin(\zeta(\xi)) \frac{\xi+1}{2} \mathrm{d}\xi \right| \left| \left(\frac{\theta-\eta}{2}\right)^{1-\sigma} \sin\eta\tilde{u}(\eta) \right| \\
&\lesssim \left| \int_{-1}^{1} (1-\xi)^{-\sigma}(1+\xi)^{-\mu}(-\zeta)^{-\frac{\mu+1}{2}}(-\eta)^{-\frac{\mu+1}{2}}(\zeta+\pi)^{-\frac{\mu+1}{2}}(\eta+\pi)^{-\frac{\mu+1}{2}} \sin(\zeta(\xi))\mathrm{d}\xi \right| \\
&\quad \cdot \left| (\theta-\eta)^{-\sigma-\mu} \sin\eta\tilde{u}(\eta) \right| \\
&\lesssim \left| \int_{-1}^{1} (1-\xi)^{-\sigma}(1+\xi)^{-\mu}\mathrm{d}\xi \;\; (\theta-\eta)^{-\sigma-\mu}(\eta+\pi)^{-\frac{\mu+1}{2}}(-\eta)^{-\frac{\mu+1}{2}} \sin\eta\tilde{u}(\eta) \right| \\
&\lesssim \left| (\theta-\eta)^{-\sigma-\mu}\, \tilde{u}(\eta) \right|.
\end{aligned}
$$

Then applying the Hardy's inequality [6,10] with $p=q=2, u=v=1$ to the last term above, we obtain

$$
\|A_2\|^2_{L^2[-\pi,0]} \lesssim \left\| \int_{-\pi}^{\theta} (\theta-\eta)^{-\sigma-\mu} |\tilde{u}(\eta)|\mathrm{d}\eta \right\|^2_{L^2[-\pi,0]} \lesssim \|\tilde{u}\|^2_{L^2[-\pi,0]} = \|u\|^2_{0,\omega}. \tag{3.26}
$$

Similarly, we have

$$
\|A_4\|^2_{L^2[-\pi,0]} \lesssim \|u\|^2_{0,\omega}. \tag{3.27}
$$

For A_3, in virtue of (3.20)–(3.24), we have

$$
\begin{aligned}
|A_3| &= \left| \int_{-1}^{1} (1-\xi)^{-\sigma} k_\mu(\zeta(\xi),\eta)\mathrm{d}\xi \left(\frac{\theta-\eta}{2}\right)^{-\sigma} \sin\eta\tilde{u}(\eta) \right| \\
&\lesssim \left| \int_{-1}^{1} (1-\xi)^{-\sigma}(1+\xi)^{-\mu}(-\zeta)^{-\frac{\mu}{2}}(-\eta)^{-\frac{\mu}{2}}(\zeta+\pi)^{-\frac{\mu}{2}}(\eta+\pi)^{-\frac{\mu}{2}}\mathrm{d}\xi \;\; (\theta-\eta)^{-\sigma-\mu} \right. \\
&\qquad \left. \sin\eta\tilde{u}(\eta) \right| \\
&\lesssim \left| \int_{-1}^{1} (1-\xi)^{-\sigma}(1+\xi)^{-\mu}\mathrm{d}\xi \;\; (\theta-\eta)^{-\sigma-\mu}(\eta+\pi)^{-\mu}(-\eta)^{-\frac{\mu}{2}}(-\theta)^{-\frac{\mu}{2}} \sin\eta\tilde{u}(\eta) \right| \\
&\lesssim \left| (\theta-\eta)^{-\sigma-\mu}(-\theta)^{-\frac{\mu}{2}}\tilde{u}(\eta) \right|.
\end{aligned}
$$

Once again, using the Hardy's inequality with $p=q=2,\ u=(-\theta)^{-\mu},\ v=1$ leads to

$$
\|A_3\|^2_{L^2[-\pi,0]} \lesssim \left\| (-\theta)^{-\frac{\mu}{2}} \int_{-\pi}^{\theta} (\theta-\eta)^{-\sigma-\mu} |\tilde{u}(\eta)|\mathrm{d}\eta \right\|^2_{L^2[-\pi,0]} \lesssim \|\tilde{u}\|^2_{L^2[-\pi,0]} = \|u\|^2_{0,\omega}. \tag{3.28}
$$

Finally the estimate (3.15) follows from combining (3.16), (3.17), and (3.25)–(3.28).

2) Now we turn to the case of Legendre weight. It follows from the estimate (3.3) and Lemma 3.1:

$$
\|Su - \Pi_N Su\|_0 \lesssim N^{-\sigma}\|Su\|_{H^\sigma} \simeq N^{-\sigma}\|_{-1}D_x^\sigma Su\|_0. \tag{3.29}
$$

By change of order of integration, we have

$$
\begin{aligned}
{}_{-1}D_x^{\sigma} Su &= \frac{d}{\mathrm{d}x}\int_{-1}^{x}(x-\tau)^{-\sigma}\int_{-1}^{\tau}(\tau-s)^{-\mu}K(\tau,s)u(s)\mathrm{d}s\ \mathrm{d}\tau \\
&= \frac{d}{\mathrm{d}x}\int_{-1}^{x}\int_{s}^{x}(x-\tau)^{-\sigma}(\tau-s)^{-\mu}K(\tau,s)\mathrm{d}\tau\ u(s)\mathrm{d}s. \qquad (3.30)
\end{aligned}
$$

Then applying the variable change $\tau = \frac{x-s}{2}\xi + \frac{x+s}{2}$ gives

$$
\begin{aligned}
{}_{-1}D_x^{\sigma} Su \ &= \ \frac{d}{\mathrm{d}x}\int_{-1}^{x}\int_{-1}^{1}(1-\xi)^{-\sigma}(1+\xi)^{-\mu}K(\tau(\xi),s)\mathrm{d}\xi\left(\frac{x-s}{2}\right)^{1-\sigma-\mu}u(s)\mathrm{d}s \\
&=: B_1 + B_2, \qquad (3.31)
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= \frac{1-\mu-\sigma}{2}\int_{-1}^{x}\int_{-1}^{1}(1-\xi)^{-\sigma}(1+\xi)^{-\mu}K(\tau(\xi),s)\mathrm{d}\xi\ (x-s)^{-\sigma-\mu}u(s)\mathrm{d}s, \\
B_2 &= \int_{-1}^{x}\int_{-1}^{1}(1-\xi)^{-\sigma}(1+\xi)^{-\mu}\frac{\partial K(\tau(\xi),s)}{\partial x}\mathrm{d}\xi\left(\frac{x-s}{2}\right)^{1-\sigma-\mu}u(s)\mathrm{d}s.
\end{aligned}
$$

Consequently,

$$
\|{}_{-1}D_x^{\sigma} Su\|_0 \lesssim \|B_1\|_0 + \|B_2\|_0. \qquad (3.32)
$$

Applying Hardy's inequality to the terms B_1 and B_2 yields

$$
\|B_1\|_0 \lesssim \left\|\int_{-1}^{x}(x-s)^{-\sigma-\mu}u(s)\mathrm{d}s\right\|_0 \lesssim \|u\|_0, \qquad (3.33)
$$

$$
\|B_2\|_0 \lesssim \left\|\int_{-1}^{x}(x-s)^{1-\sigma-\mu}u(s)\mathrm{d}s\right\|_0 \lesssim \|u\|_0. \qquad (3.34)
$$

Obviously, combining (3.29)–(3.34) gives the estimate (3.15) for the Legendre case. □

4 Convergence Analysis

4.1 Error Estimates for the Spectral Galerkin Method

We start with establishing an error estimate for the discrete solution of the Chebyshey/Legendre spectral Galerkin problem (2.4). According to the definition of the L^2-projector Π_N^{ω}, the problem (2.4) can be rewritten under the form: find $u_N \in \mathcal{P}_N(\Lambda)$ such that

$$
u_N + \Pi_N^{\omega} Su_N = \Pi_N^{\omega} g. \qquad (4.1)
$$

The error estimate of the numerical solution u_N is given in the following theorem.

Theorem 4.1 *The problem* (4.1) *admits a unique solution* u_N*, which satisfies*

$$
\|u_N\|_{0,\omega} \lesssim \|g\|_{0,\omega}. \qquad (4.2)
$$

Furthermore, if u *is the solution of* (2.1)*, and* $u \in H_{\omega}^m(\Lambda)$, $m \geq 1$*, then for sufficiently large* N*, the following error estimate holds*

$$
\|u-u_N\|_{0,\omega} \lesssim N^{-m}|u|_{H_{\omega}^{m;N}}. \qquad (4.3)
$$

Proof First, we prove the existence and uniqueness of the solution of (4.1). It suffices to prove that the following homogeneous problem

$$u_N + \Pi_N^{\omega} S u_N = 0$$

has only the trivial solution in $\mathcal{P}_N(\Lambda)$. In fact, we have

$$u_N = -\int_{-1}^{x} (x-s)^{-\mu} K(x,s) u_N(s) \mathrm{d}s + S u_N - \Pi_N^{\omega} S u_N.$$

It then follows by using the Gronwall inequality [5]:

$$\|u_N\|_{0,\omega} \lesssim \|S u_N - \Pi_N^{\omega} S u_N\|_{0,\omega}. \tag{4.4}$$

Using Lemma 3.6 we get, for $0 < \sigma < \min\{\frac{1}{2}, 1-\mu\}$,

$$\|S u_N - \Pi_N^{\omega} S u_N\|_{0,\omega} \lesssim N^{-\sigma} \|u_N\|_{0,\omega}. \tag{4.5}$$

Combining (4.4) with (4.5), we obtain

$$\|u_N\|_{0,\omega} \lesssim N^{-\sigma} \|u_N\|_{0,\omega}. \tag{4.6}$$

This implies that $u_N = 0$ if N is large enough. This proves the well-posedness of the problem (4.1) and estimate (4.2). Next, we prove the estimate (4.3). Subtracting (2.1) from (4.1) gives

$$u_N - u + \Pi_N^{\omega} S u_N - S u = \Pi_N^{\omega} g - g. \tag{4.7}$$

Let $e = u_N - u$. Then we derive from (4.7):

$$e = -Se + (I - \Pi_N^{\omega}) Se + (\Pi_N^{\omega} u - u). \tag{4.8}$$

Applying the Gronwall inequality to the above equation, we get

$$\|e\|_{0,\omega} \lesssim \|u - \Pi_N^{\omega} u\|_{0,\omega} + \|(I - \Pi_N^{\omega}) Se\|_{0,\omega}. \tag{4.9}$$

It remains to bound the RHS of (4.9). For the first term, by using the estimate (3.3), we have

$$\|u - \Pi_N^{\omega} u\|_{0,\omega} \lesssim N^{-m} |u|_{H_{\omega}^{m;N}}. \tag{4.10}$$

For the second term, by using Lemma 3.6 for $0 < \sigma < \min\{\frac{1}{2}, 1-\mu\}$, we obtain

$$\|Se - \Pi_N^{\omega} Se\|_{0,\omega} \lesssim N^{-\sigma} \|e\|_{0,\omega}. \tag{4.11}$$

Finally we get (4.3) by combining (4.9)–(4.11). □

The following theorem provides an error estimate in the L^{∞}-norm for the Chebyshev spectral method.

Theorem 4.2 *(L^{∞}-convergence for the Chebyshev method) Suppose u_N is the solution of the problem (4.1) with ω being the Chebyshev weight, u is the solution of (2.1), for sufficiently large N, then the following estimates hold.*
If $u \in H_{\omega}^{m}(\Lambda)$, $m \geq 1$, then

$$\|u - u_N\|_{\infty} \lesssim N^{3/4-m} |u|_{H_{\omega}^{m;N}}. \tag{4.12}$$

If $u \in W^{m,\infty}(\Lambda)$, then

$$\|u - u_N\|_{\infty} \lesssim \log N N^{-m} \|u\|_{W^{m,\infty}}. \tag{4.13}$$

Proof Let $e = u_N - u$. Applying the Gronwall inequality to (4.8), we have

$$\|e\|_\infty \lesssim \|(I - \Pi_N^\omega)Se\|_\infty + \|u - \Pi_N^\omega u\|_\infty. \tag{4.14}$$

The terms in RHS can be bounded by employing Lemma 3.3, inequalities (3.11) and (3.13):

$$\begin{aligned}\|(I - \Pi_N^\omega)Se\|_\infty &= \|(I - \Pi_N^\omega)(Sv - \mathcal{T}_N Se)\|_\infty \lesssim \log N \|Sv - \mathcal{T}_N Se\|_\infty \\ &\lesssim \log N N^{-\kappa} \|Se\|_{0,\kappa} \lesssim \log N N^{-\kappa} \|e\|_\infty,\end{aligned} \tag{4.15}$$

where $0 < \kappa < 1 - \mu$. Thus for large enough N we have

$$\|e\|_\infty \lesssim \|u - \Pi_N^\omega u\|_\infty.$$

Then we conclude by simply applying the estimate (3.8) and Lemma 3.3. □

We give in the following theorem an error estimate in the L^∞-norm for the Legendre spectral method.

Theorem 4.3 *(L^∞-convergence for Legendre method) If u_N is the solution of the problem* (4.1) *with ω being the Legendre weight, u is the solution of* (2.1)*, and $u \in H^m(\Lambda)$, $m \geq 1$, then for sufficiently large N, the following error estimate holds*

$$\|u - u_N\|_\infty \lesssim N^{1-m} |u|_{H^{m;N}}. \tag{4.16}$$

Proof Applying the projector Π_N^ω to the both sides of (2.1), we obtain

$$\Pi_N^\omega u + \Pi_N^\omega Su = \Pi_N^\omega g. \tag{4.17}$$

Subtracting (4.17) from (4.1) gives

$$u_N - \Pi_N^\omega u + \Pi_N^\omega Su_N - \Pi_N^\omega Su = 0. \tag{4.18}$$

Let $e := u_N - \Pi_N^\omega u$. Then we derive from (4.18):

$$e = -Se + (I - \Pi_N^\omega)Se + \Pi_N^\omega S(u - \Pi_N^\omega u). \tag{4.19}$$

Applying the Gronwall inequality to the above equation, we get

$$\|e\|_\infty \lesssim \| \left(I - \Pi_N^\omega\right) Se\|_\infty + \|\Pi_N^\omega S \left(u - \Pi_N^\omega u\right) \|_\infty. \tag{4.20}$$

Applying the estimates (3.3)–(3.4), Hardy's inequality, and the well known inverse inequality, we have

$$\begin{aligned}\| \left(I - \Pi_N^\omega\right) Se\|_\infty &\lesssim \|Se\|_\infty + \|\Pi_N^\omega Se\|_\infty \lesssim \|e\|_\infty + N\|\Pi_N^\omega Se\|_0 \\ &\lesssim N\|e\|_0 + N\|Se\|_0 \lesssim N\|e\|_0,\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\|\Pi_N^\omega S \left(u - \Pi_N^\omega u\right) \|_\infty &\lesssim N\|\Pi_N^\omega S \left(u - \Pi_N^\omega u\right) \|_0 \lesssim N\|S \left(u - \Pi_N^\omega u\right) \|_0 \\ &\lesssim N\|u - \Pi_N^\omega u\|_0 \lesssim N^{1-m} |u|_{H^{m;N}}.\end{aligned} \tag{4.22}$$

Using all these estimates and the one for $\|e\|_0$, we obtain

$$\|u_N - \Pi_N^\omega u\|_\infty \lesssim N^{1-m} |u|_{H^{m;N}}. \tag{4.23}$$

Finally we conclude by using the triangular inequality. □

4.2 Error Estimates for the Pseudo-Spectral Method

In virtue of the exactitude (2.3) of the Gauss quadrature, the solution $\bar{u}_N$ of the pseudo-spectral problem (2.9) satisfies

$$(\bar{u}_N, v_N)_\omega + \left(I_{2N}^{\rho}\tilde{S}_N\bar{u}_N, v_N\right)_\rho = (g, v_N)_\omega, \ \forall v_N \in \mathcal{P}_N(\Lambda), \tag{4.24}$$

where ρ is defined in (2.7), I_{2N}^{ρ} is the interpolation operator based on $2N+1$ degree Jacobi Gauss points associated to the weight function ρ. Before carrying out the error analysis for the problem (4.24), we first reformulate the term involving the discrete integral operator $\tilde{S}_N$, which is the most difficult to treat.

By using (2.3), and let ϱ_k being defined in (2.8), we have

$$\begin{aligned}
&\left(I_{2N}^{\rho}\tilde{S}_N\bar{u}_N, v_N(x)\right)_\rho \\
&= \left(I_{2N}^{\rho}\sum_{k=0}^{N}\bar{K}(x,\theta_k)\bar{u}_N(s_x(\theta_k))\varrho_k, v_N(x)\right)_\rho \\
&= \left(I_{2N}^{\rho}\int_{-1}^{1}(1-\theta)^{-\mu}I_N^{\varrho,(\theta)}\bar{K}(x,\theta)\bar{u}_N(s_x(\theta))\mathrm{d}\theta, v_N(x)\right)_\rho \\
&= \left(\int_{-1}^{1}(1-\theta)^{-\mu}I_{2N}^{\rho}\left[I_N^{\varrho,(\theta)}\bar{K}(x,\theta)\bar{u}_N(s_x(\theta))\right]\mathrm{d}\theta, v_N(x)\right)_\rho \\
&= \int_{-1}^{1}v_N(x)(1+x)^{\alpha+1-\mu}(1-x)^{\alpha}\int_{-1}^{1}(1-\theta)^{-\mu}I_{2N}^{\rho}\left[I_N^{\varrho,(\theta)}\bar{K}(x,\theta)\bar{u}_N(s_x(\theta))\right]\mathrm{d}\theta\mathrm{d}x \\
&= \int_{-1}^{1}(1-\theta)^{-\mu}\int_{-1}^{1}v_N(x)(1+x)^{\alpha+1-\mu}(1-x)^{\alpha}I_{2N}^{\rho}\left[I_N^{\varrho,(\theta)}\bar{K}(x,\theta)\bar{u}_N(s_x(\theta))\right]\mathrm{d}x\mathrm{d}\theta \\
&= \int_{-1}^{1}(1-\theta)^{-\mu}\left(I_N^{\varrho,(\theta)}\bar{K}(x,\theta)\bar{u}_N(s_x(\theta)), v_N(x)\right)_{2N,\rho}\mathrm{d}\theta \\
&= \int_{-1}^{1}(1-\theta)^{-\mu}\left(I_N^{\varrho,(\theta)}\bar{K}(x,\theta), \bar{u}_N(s_x(\theta))v_N(x)\right)_{2N,\rho}\mathrm{d}\theta \\
&= \int_{-1}^{1}(1-\theta)^{-\mu}\left(I_{2N}^{\rho}I_N^{\varrho,(\theta)}\bar{K}(x,\theta), \bar{u}_N(s_x(\theta))v_N(x)\right)_{\rho}\mathrm{d}\theta \\
&= \int_{-1}^{1}v_N(x)(1+x)^{\alpha+1-\mu}(1-x)^{\alpha}\int_{-1}^{1}(1-\theta)^{-\mu}\bar{u}_N(s_x(\theta))I_{2N}^{\rho}I_N^{\varrho,(\theta)}\bar{K}(x,\theta)\mathrm{d}\theta\mathrm{d}x \\
&= (\tilde{K}_N\bar{u}_N, v_N)_\rho,
\end{aligned}$$

where $I_N^{\varrho,(\theta)}$ denotes the interpolation with respect to variable θ, and

$$\tilde{K}_N\bar{u}_N(x) = \int_{-1}^{1}(1-\theta)^{-\mu}\bar{u}_N(s_x(\theta))I_{2N}^{\rho}I_N^{\varrho,(\theta)}\bar{K}(x,\theta)\mathrm{d}\theta.$$

Let $\hat{K}_N\bar{u}_N(x) =: (1+x)^{1-\mu}\tilde{K}_N\bar{u}_N(x)$. Then the variable change $\theta_x(s) = \frac{2s-x+1}{x+1}$ gives

$$\hat{K}_N\bar{u}_N(x) = 2^{1-\mu}\int_{-1}^{x}(x-s)^{-\mu}\bar{u}_N(s)I_{2N}^{\rho}I_N^{\varrho,(\theta)}\bar{K}(x,\theta_x(s))\mathrm{d}s.$$

And (4.24) becomes

$$(\bar{u}_N, v_N)_\omega + (\hat{K}_N \bar{u}_N, v_N)_\omega = (g, v_N)_\omega, \ \forall v_N \in \mathcal{P}_N(\Lambda), \tag{4.25}$$

or, equivalently,

$$\bar{u}_N + \Pi_N^\omega \hat{K}_N \bar{u}_N = \Pi_N^\omega g. \tag{4.26}$$

Theorem 4.4 *For large enough N, the problem (4.26) admits a unique solution $\bar{u}_N$ in $\mathcal{P}_N(\Lambda)$. Furthermore, if u is the solution of (2.1), and $u \in W^{m,\infty}(\Lambda)$. Then we have*

$$\begin{aligned} &\|u - \bar{u}_N\|_{0,\omega} \\ &\lesssim N^{-m} |u|_{H_\omega^{m;N}} + N^{-m} \Big(\max_{-1\le\theta\le1} \big\|\bar{K}(\cdot,\theta)\big\|_{H_\rho^{m;N}} + \max_{-1\le x\le1} \big\|\bar{K}(x,\cdot)\big\|_{H_\varrho^{m;N}} \Big) \|g\|_{0,\omega}. \end{aligned} \tag{4.27}$$

If ω is the Chebyshev weight, then

$$\begin{aligned} \|u - \bar{u}_N\|_\infty &\lesssim N^{-m} \log N \|u\|_{W^{m,\infty}} \\ &\quad + \log^2 N \Big(N^{2-\mu-m} \max_{-1\le\theta\le1} \big\|\bar{K}(\cdot,\theta)\big\|_{H_\rho^{m;N}} + N^{-m} K^* \Big) \|g\|_\infty, \end{aligned} \tag{4.28}$$

where

$$K^* = \left\| \sqrt{\int_0^T \Big(\frac{\partial}{\partial x} \bar{K}(x,\theta)\Big)^2 \mathrm{d}x + \int_0^T \bar{K}^2(x,\theta) \mathrm{d}x} \right\|_{H_\varrho^{m;N}}. \tag{4.29}$$

If ω is the Legendre weight, then

$$\begin{aligned} &\|u - \bar{u}_N\|_\infty \\ &\lesssim N^{1-m} |u|_{H^{m;N}} + N^{1-m} \Big(\max_{-1\le\theta\le1} \big\|\bar{K}(\cdot,\theta)\big\|_{H_\rho^{m;N}} + \max_{-1\le x\le1} \big\|\bar{K}(x,\cdot)\big\|_{H_\varrho^{m;N}} \Big) \|g\|_0. \end{aligned} \tag{4.30}$$

Proof First, we prove the existence and uniqueness of the solution to (4.26). Obviously, we just need to prove that the zero function is the only solution of (4.26) if $g = 0$. To this end, let $g = 0$. Then we have

$$\bar{u}_N = -2^{1-\mu} \int_{-1}^x (x-s)^{-\mu} \bar{u}_N(s) I_{2N}^\rho I_N^{\varrho,(\theta)} \bar{K}(x, \theta_x(s)) \mathrm{d}s + \hat{K}_N \bar{u}_N - \Pi_N^w \hat{K}_N \bar{u}_N.$$

Applying the Gronwall inequality to the above equation and using Lemma 3.6, we get

$$\|\bar{u}_N\|_{0,\omega} \lesssim \|\hat{K}_N \bar{u}_N - \Pi_N^w \hat{K}_N \bar{u}_N\|_{0,\omega} \lesssim N^{-\sigma} \|\bar{u}_N\|_{0,\omega}.$$

Thus $\bar{u}_N \equiv 0$ for large enough N. This proves the well-posedness of (4.26).
Similarly, applying Gronwall inequality to (4.26) and using (4.15), Lemma 3.3, we get

$$\|\bar{u}_N\|_{0,\omega} \lesssim \|g\|_{0,\omega}, \quad \|\bar{u}_N\|_\infty \lesssim \log N \|g\|_\infty. \tag{4.31}$$

Now we derive the estimates (4.27), (4.28), and (4.30). Subtracting (4.26) from (4.1), we obtain

$$\bar{u}_N - u_N + \Pi_N^\omega \hat{K}_N \bar{u}_N - \Pi_N^\omega S u_N = 0. \tag{4.32}$$

A direct computation shows that

$$\begin{aligned}
&\Pi_N^\omega \hat{K}_N \bar{u}_N - \Pi_N^\omega S u_N \\
&\quad = \Pi_N^\omega \hat{K}_N \bar{u}_N - \Pi_N^\omega S \bar{u}_N + \Pi_N^\omega S \bar{u}_N - \Pi_N^\omega S u_N \\
&\quad = \Pi_N^\omega \hat{K}_N \bar{u}_N - \Pi_N^\omega S \bar{u}_N + S(\bar{u}_N - u_N) - \big[(I - \Pi_N^\omega) S(\bar{u}_N - u_N)\big].
\end{aligned} \tag{4.33}$$

Inserting (4.33) into (4.32) and let $e = \bar{u}_N - u_N$, we get

$$e(x) = -\int_{-1}^{x} (x-s)^{-\mu} K(x,s) e(s) \mathrm{d}s + \Pi_N^\omega (S - \hat{K}_N)\bar{u}_N + (Se - \Pi_N^\omega Se).$$

It follows from using Gronwall inequality that

$$\|e\|_{0,\omega} \lesssim \|\Pi_N^\omega (S - \hat{K}_N)\bar{u}_N\|_{0,\omega} + \|Se - \Pi_N^\omega Se\|_{0,\omega}, \tag{4.34}$$

and

$$\|e\|_\infty \lesssim \|\Pi_N^\omega (S - \hat{K}_N)\bar{u}_N\|_\infty + \|Se - \Pi_N^\omega Se\|_\infty. \tag{4.35}$$

For the first term on the right hand side of (4.34), by Hardy's inequality, Lemma 3.2, inequality (4.31), and the inequality [21]

$$\sup_N \|I_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}} \le C\|v\|_\infty,$$

we have

$$\begin{aligned}
&\|\Pi_N^\omega (S - \hat{K}_N)\bar{u}_N\|_{0,\omega} \\
&\quad \lesssim \|(S - \hat{K}_N)\bar{u}_N\|_{0,\omega} \\
&\quad \lesssim \left\|(1+x)^{1-\mu} \int_{-1}^{1} (1-\theta)^{-\mu} \big(\bar{K}(x,\theta) - I_{2N}^\rho I_N^{\varrho,(\theta)} \bar{K}(x,\theta)\big) \bar{u}_N(s_x(\theta)) \mathrm{d}\theta \right\|_{0,\omega} \\
&\quad \lesssim \left\| (1+x)^{1-\mu} \int_{-1}^{1} (1-\theta)^{-\mu} \left(\bar{K}(x,\theta) - I_{2N}^\rho \bar{K}(x,\theta)\right) \bar{u}_N(s_x(\theta)) \mathrm{d}\theta \right\|_{0,\omega} \\
&\qquad + \left\| (1+x)^{1-\mu} \int_{-1}^{1} (1-\theta)^{-\mu} \big(I_{2N}^\rho \bar{K}(x,\theta) - I_{2N}^\rho I_N^{\varrho,(\theta)} \bar{K}(x,\theta)\big) \bar{u}_N(s_x(\theta)) \mathrm{d}\theta \right\|_{0,\omega} \\
&\quad \lesssim \bigg(\max_{-1\le\theta\le 1} \big\|\bar{K}(x,\theta) - I_{2N}^\rho \bar{K}(x,\theta)\big\|_{0,\rho} \\
&\qquad + N^{-m} \big\|(1+x)^{1-\mu} \big\|I_{2N}^\rho \bar{K}(x,\cdot)\big\|_{H_\varrho^{m;N}} \big\|_{0,\omega}\bigg) \|\bar{u}_N\|_{0,\omega} \\
&\quad \lesssim \bigg(N^{-m} \max_{-1\le\theta\le 1} \big\|\bar{K}(\cdot,\theta)\big\|_{H_\rho^{m;N}} + N^{-m} \max_{-1\le x\le 1} \|\bar{K}(x,\cdot)\|_{H_\varrho^{m;N}}\bigg) \|g\|_{0,\omega}.
\end{aligned} \tag{4.36}$$

For the second term, use of Lemma 3.6 gives

$$\|Se - \Pi_N^\omega Se\|_{0,\omega} \lesssim N^{-\sigma} \|e\|_{0,\omega}.$$

Inserting above two inequalities into (4.34), we obtain, for big enough N,

$$\|e\|_{0,\omega} \lesssim N^{-m} \bigg(\max_{-1\le\theta\le 1} \big\|\bar{K}(\cdot,\theta)\big\|_{H_\rho^{m;N}} + \max_{-1\le x\le 1} \big\|\bar{K}(x,\cdot)\big\|_{H_\varrho^{m;N}} \bigg) \|g\|_{0,\omega}. \tag{4.37}$$

Then the estimate (4.27) follows from combining (4.37) with (4.2).
Now we estimate the errors in the L^∞-norm. If ω is the Chebyshev weight, the first term on the right hand side of (4.35) can be bounded by using Hardy's inequality, Lemma 3.2, Lemma 3.4, and inequality (4.31),

$$
\begin{aligned}
&\|\Pi_N^{\omega}(S-\hat{K}_N)\bar{u}_N\|_\infty \\
&\lesssim \log N\|(S-\hat{K}_N)\bar{u}_N\|_\infty \\
&\lesssim \log N\left\| (1+x)^{1-\mu}\int_{-1}^{1}(1-\theta)^{-\mu}\left(\bar{K}(x,\theta)-I_{2N}^{\rho}I_N^{\varrho,(\theta)}\bar{K}(x,\theta)\right)\bar{u}_N(s_x(\theta))\mathrm{d}\theta\right\|_\infty \\
&\lesssim \log N\left\| (1+x)^{1-\mu}\int_{-1}^{1}(1-\theta)^{-\mu}\left(\bar{K}(x,\theta)-I_{2N}^{\rho}\bar{K}(x,\theta)\right)\bar{u}_N(s_x(\theta))\mathrm{d}\theta\right\|_\infty \\
&\quad+\log N\left\| (1+x)^{1-\mu}\int_{-1}^{1}(1-\theta)^{-\mu}\left(I_{2N}^{\rho}\bar{K}(x,\theta)-I_{2N}^{\rho}I_N^{\varrho,(\theta)}\bar{K}(x,\theta)\right)\bar{u}_N(s_x(\theta))\mathrm{d}\theta\right\|_\infty \\
&\lesssim \log N\Big(\max_{-1\le\theta\le1}\left\|\bar{K}(\cdot,\theta)-I_{2N}^{\rho}\bar{K}(\cdot,\theta)\right\|_\infty+N^{-m}\max_{-1\le x\le1}\left\|I_{2N}^{\rho}\bar{K}(x,\cdot)\right\|_{H_\varrho^{m;N}}\Big)\|\bar{u}_N\|_\infty \\
&\lesssim \log^2 N\Big(N^{2-\mu-m}\max_{-1\le\theta\le1}\left\|\bar{K}(\cdot,\theta)\right\|_{H_\omega^{m;N}}+N^{-m}K^*\Big)\|g\|_\infty,
\end{aligned}
$$

where K^* is defined in (4.29).
If ω is the Legendre weight, then by combining the inverse inequality with inequality (4.36), we get

$$
\begin{aligned}
&\|\Pi_N^{\omega}(S-\hat{K}_N)\bar{u}_N\|_\infty \\
&\lesssim N\|(S-\hat{K}_N)\bar{u}_N\|_0 \\
&\lesssim \Big(N^{1-m}\max_{-1\le\theta\le1}\left\|\bar{K}(\cdot,\theta)\right\|_{H_\rho^{m;N}}+N^{1-m}\max_{-1\le x\le1}\left\|\bar{K}(x,\cdot)\right\|_{H_\varrho^{m;N}}\Big)\|g\|_0.
\end{aligned}
\tag{4.38}
$$

For the second term, it follows from (4.15) and (4.21):

$$
\|(I-\Pi_N^{\omega})Se\|_\infty \lesssim \begin{cases} \log N N^{-\kappa}\|e\|_\infty,\ 0<\kappa<1-\mu, & \text{if } \omega \text{ is the Chebyshev weight,}\\ N\|e\|_0, & \text{if } \omega \text{ is the Legendre weight.}\end{cases}
\tag{4.39}
$$

By combining the inequalities (4.35), (4.37)–(4.39), it holds, for N big enough,

$$
\|e\|_\infty \lesssim \log^2 N\Big(N^{2-\mu-m}\max_{-1\le\theta\le1}\left\|\bar{K}(\cdot,\theta)\right\|_{H_\omega^{m;N}}+N^{-m}K^*\Big)\|g\|_\infty,
\tag{4.40}
$$

if ω is the Chebyshev weight, and

$$
\|e\|_\infty \lesssim \Big(N^{1-m}\max_{-1\le\theta\le1}\left\|\bar{K}(\cdot,\theta)\right\|_{H_\rho^{m;N}}+N^{1-m}\max_{-1\le x\le1}\left\|\bar{K}(x,\cdot)\right\|_{H_\varrho^{m;N}}\Big)\|g\|_0,
\tag{4.41}
$$

if ω is the Legendre weight. Here the error estimates given in (4.37) has been used to bound $\|e\|_0$.
Finally, the estimates (4.28) and (4.30) are direct consequences of (4.40)-(4.41), the triangular inequality, Theorem 4.2, and Theorem 4.3. □

5 Numerical Results

In this section, we present numerical results to validate the error estimates obtained in Theorem 4.4 for the proposed Chebyshev and Legendre pseudo-spectral methods. Theorem 4.4 indicates that the convergence of numerical solutions would be exponential with respect to the polynomial degree if the exact solution and the kernel function K were smooth.

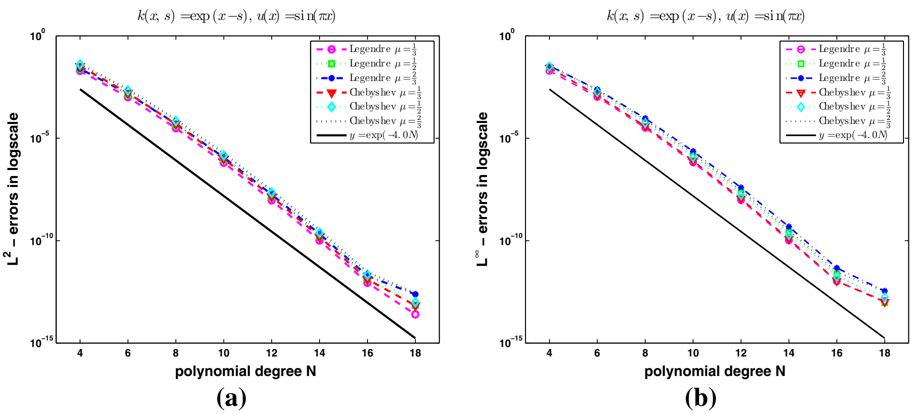


Fig. 1 Example 5.1. **a** L^2-errors versus N; **b** L^∞-errors versus N

Example 5.1 Consider the Volterra integral equation having smooth kernel and smooth solution:

$$u(x) = g(x) - \int_0^x (x-s)^{-\mu} \exp(x-s) u(s) \mathrm{d}s, \quad 0 \le x \le 2,$$

where $g(x)$ is chosen such that the exact solution is $u(x) = \sin(\pi x)$.

In Fig. 1, we plot the L^2-errors (a) and L^∞-errors (b) respectively in semi-log scale as a function of N, for $\mu = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$. As expected, the error variations are linear versus the degrees of polynomial N, which means that the error decays exponentially. More precisely, a careful verification shows that the error decays as e^{-cN} with the constant c close to 5.5.

Next test is for the purpose to examine the sharpness of the estimate given in Theorem 4.4. To this end, we consider the following two examples with γ being no integer.

Example 5.2 Consider the equation (2.1)–(2.2) with the smooth kernel function $K(x,s) = \exp(x-s)$, and limited regular solution $u(x) = |x-0.5|^\gamma$.

Example 5.3 Consider the equation (2.1)–(2.2) with the smooth solution $u(x) = \sin(\pi x)$ and limited regular kernel function $K(x,s) = |x+s|^\gamma + 1$.

For the example 5.2, it can be verified that the solution $u(x)$ belongs to the space $H^{\gamma+\frac{1}{2}-\epsilon}(I)$ or $W^{[\gamma],\infty}(I)$ if γ is not an integer. In Fig. 2a, we plot the L^2-errors as functions of the polynomial degrees N with $\mu = \frac{1}{4}$ for two different $\gamma = \frac{5}{3}, \frac{16}{3}$. Since $K(x,s)$ is a smooth function, the second term in the error estimates (4.27), (4.28) and (4.30) is expected to be negligible as compared with the first term in these estimates for sufficiently large N. As a consequence, the error behavior shown in this figure should reflect the impact of the regularity of the exact solution. To precisely observe the error decay rates, the $N^{-\frac{13}{6}}$ and $N^{-\frac{35}{6}}$ decay rates are also plotted in the figure. Two points can be drawn from the observation: (1) all the error curves are straight lines in this log-log representation, which indicates the algebraic convergence due to the limited regularity of the solutions; (2) the errors decrease with rates approximately conform to the estimate (4.27), i.e., $N^{-\frac{13}{6}}$ decay rate for $\gamma = \frac{5}{3}$ and $N^{-\frac{35}{6}}$ decay rate for $\gamma = \frac{16}{3}$. The L^∞-error behavior is plotted in Fig. 2b with $\mu = \frac{3}{4}$ for two different $\gamma = \frac{5}{3}, \frac{16}{3}$. The $N^{-\frac{5}{3}}$ and $N^{-\frac{16}{3}}$ decay rates are also shown for comparison

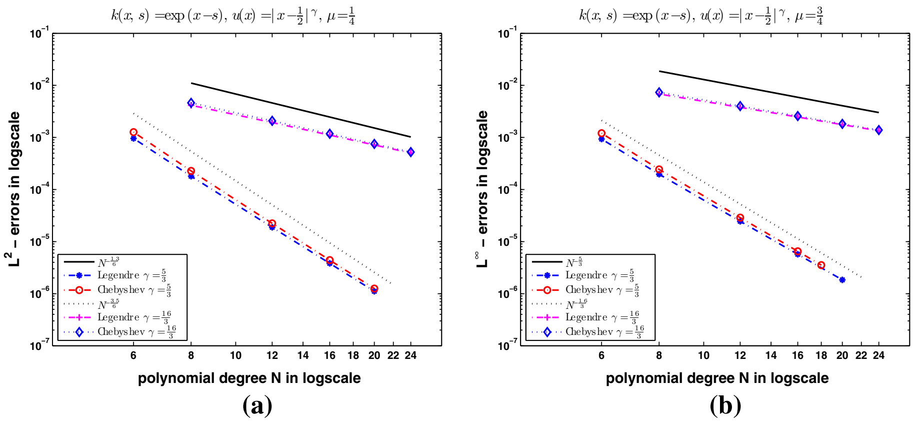


Fig. 2 Example 5.2. **a** L^2-errors versus N; **b** L^{∞}-errors versus N

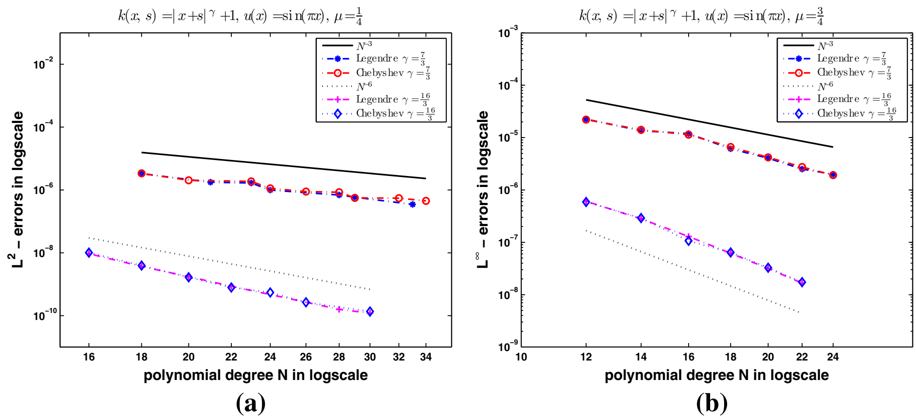


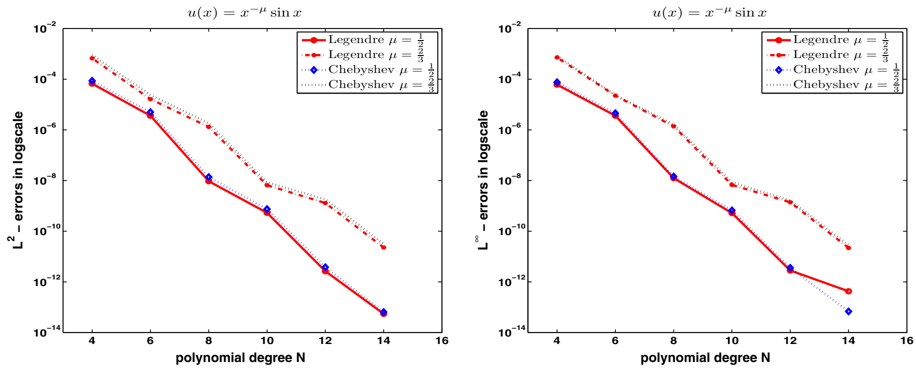
Fig. 3 Example 5.3. **a** L^2-errors versus N; **b** L^{∞}-errors versus N

reason. Similar to the L^2-errors, the L^{∞}-error curves are also straight lines, and the observed error decay rates are in good agreement with the estimates (4.28) and (4.30), i.e., $N^{-\frac{5}{3}}$ rate for $\gamma = \frac{5}{3}$ and $N^{-\frac{6}{3}}$ rate for $\gamma = \frac{16}{3}$.

In the example 5.3, we investigate the impact of the regularity of the kernel function on the convergence rate. We plot in Fig. 3 the L^2-errors and L^{∞}-errors respectively as functions of the polynomial degrees N with $\mu = \frac{1}{4}$ and $\mu = \frac{3}{4}$ for two different $\gamma = \frac{7}{3}, \frac{16}{3}$. N^{-3} and N^{-6} decay lines are shown for comparison. Once again, we observe the algebraic convergence from the error straight lines in the log–log plot, and that the convergence rates are closely related to the regularity, i.e., γ, of the kernel functions.

Example 5.4 Consider the Volterra integral equation:

$$u(x) = g(x) - \int_0^x (x-s)^{-\mu} u(s) \mathrm{d}s, \quad 0 \le x \le 1,$$



(a) **(b)**

Fig. 4 Example 5.4. **a** L^2-errors versus N; **b** L^∞-errors versus N

with the solution $u(x) = \frac{\sin x}{x^\mu}$, μ being the rational number $\frac{q}{p}, q < p$, and the source function

$$g(x) = \frac{\sin(x)}{x^\mu} + \sqrt{\pi}\Gamma(1-\mu)x^{\frac{1}{2}-\mu} \sin\frac{x}{2} B\left(\frac{1}{2}-\mu, \frac{x}{2}\right),$$

where $B(\cdot, \cdot)$ is the Bessel function, i.e.,

$$B(\mu, x) = \left(\frac{x}{2}\right)^\mu \sum_{k=0}^{+\infty} \frac{(-x^2)^k}{k!\Gamma(\mu+k+1)4^k}.$$

Clearly this solution has singularity at the left end point, i.e., $u'(x) \sim \frac{1}{x^\mu}$ at $x = 0$. By applying the smoothing transformation proposed in Remark 2.1, we get the transformed Eq. (2.12) with the exact solution $\bar{u}(x) = u(x^p) = \frac{\sin x^p}{x^{p\mu}}$, which is now regular at the left end point $x = 0$. Thus the proposed spectral method remains applicable to the Eq. (2.12) with efficiency. In Fig. 4, we plot the L^2-errors (a) and L^∞-errors (b) respectively in semi-log scale as a function of N for $\frac{1}{2}, \frac{2}{3}$. As expected, the error variations are nearly linear versus the degrees of polynomial N, which indicates the exponential convergence of the proposed method.

References

1. Baratella, P., Palamara Orsi, A.: A new approach to the numerical solution of weakly singular Volterra integral equations. J. Comp. Appl. Math. **163**, 401–418 (2004)
2. Brunner, H.: Collocation methods for Volterra integral and related functional differential equations. Cambridge University Press, Cambridge (2004)
3. Canuto, C., Hussaini, M.Y., Quarteroni, A., Zang, T.A.: Spectral methods: fundamentals in single domains. Springer, Berlin (2006)
4. Canuto, C., Quarteroni, A.: Approximation results for orthogonal polynomials in Sobolev spaces. Math. Comput. **38**(157), 67–86 (1982)
5. Chen, Y., Tang, T.: Spectral methods for weakly singular Volterra integral equations with smooth solutions. J. Comput. Appl. Math. **233**, 938–950 (2009)
6. Chen, Y., Tang, T.: Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel. Math. Comput. **79**(269), 147–167 (2010)
7. Chen, Y.P., Li, X.J., Tang, T.: A note on Jacobi spectral-collocation methods for weakly singular Volterra integral equations with smooth solutions. J. Comput. Math. **31**, 47–56 (2013)

8. Elliott, D., Prossdorf, S.: An algorithm for the approximate solution of integral equations of Mellin type. Numer. Math. **70**, 427452 (1995)
9. Kolk, M., Pedas, A.: Numerical solution of Volterra integral equations with singularities. Front. Math. China **8**(2), 239–259 (2013)
10. Kufner, A., Persson, L.E.: Weighted inequalities of hardy types. World Scientific, Singapore (2003)
11. Li, X.J., Tang, T.: Convergence analysis of Jacobi spectral collocation methods for Abel–Volterra integral equations of second kind. Front. Math. China **7**(1), 69–84 (2012)
12. Li, X.J., Xu, C.J.: Existence and uniqueness of the weak solution of the space-time fractional diffusion equation and a spectral method approximation. Commun. Comput. Phys. **8**(5), 1016–1051 (2011)
13. Mastroianni, G., Occorsio, D.: Optimal systems of nodes for Lagrange interpolation on bounded intervals. A survey. J. Comput. Appl. Math. **134**(1–2), 325–341 (2001)
14. Monegato, G., Scuderi, L.: High order methods for weakly singular integral equations with nonsmooth input functions. Math. Comput. **224**(67), 1493–1515 (1989)
15. Podlubny, I.: Fractional differential equations. Academic Press, San Diego, London (1999)
16. Ragozin, D.L.: Polynomial approximation on compact manifolds and homogeneous spaces. Trans. Am. Math. Soc. **150**(1), 41–53 (1970)
17. Ragozin, D.L.: Constructive polynomial approximation on spheres and projective spaces. Trans. Am. Math. Soc. **162**, 157–170 (1971)
18. Sedaghat, S., Ordokhani, Y., Dehghan, M.: On spectral method for Volterra functional integro-differential equations of neutral type. Numer. Func. Anal. Optim. **35**(2), 223–239 (2014)
19. Sidi, A.: A new variable transformation for numerical integration. In: H. Brass, G. Hammerlin (eds.) Numerical Integration IV, International Series of Numerical Mathematics, vol. 112, pp. 359–373. Birkhäuser, Basel (1993)
20. Tao, X., Xie, Z.Q., Zhou, X.J.: Spectral Petrov–Galerkin methods for the second kind Volterra type integro-differential equations. Numer. Math. Theor. Meth. Appl. **4**(2), 216–236 (2011)
21. Xie, Z.Q., Li, X.J., Tang, T.: Convergence analysis of spectral Galerkin methods for Volterra type integral equations. J. Sci. Comput. **53**(2), 414–434 (2012)
22. Zhang, R., Zhu, B., Xie, H.: Spectral methods for weakly singular Volterra integral equations with pantograph delays. Front. Math. China **8**(2), 281–299 (2013)