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Optimal Convergence of an Euler and Finite Difference Method for Nonlinear Partial Integro-Differential Equations

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Abstract—Fully discretized Euler method in time and finite difference method in space are constructed and analyzed for a class of nonlinear partial integro-differential equations emerging from practical applications of a wide range, such as the modeling of physical phenomena associated with non-Newtonian fluids. Though first-order and second-order time discretizations (based on truncation errors) have been investigated recently, due to lack of the smoothness of the exact solutions, the overall numerical procedures do not achieve the optimal convergence rates in time. In this paper, however, by using the energy method, we prove that it is possible for the scheme to obtain the optimal convergence rate $O(\tau)$. Numerical demonstrations are given to illustrate our result.

Keywords—Partial integro-differential equations, Euler method, Finite difference method, Convergence rate.

1. INTRODUCTION

Flows of fluids with complex macrostructures cannot be described by the classical Navier-Stokes equations [1]. Many model problems exist that are based on linear viscoelasticity, nonlinear elasticity, and Newtonian or non-Newtonian fluid mechanics or on molecular considerations. Popular models essentially fall into two categories: the differential models and the integral models. For a general discussion of the mathematical principles governing the formulation of constitutive laws and methods, the reader is referred to [1–3]. In this paper, we only deal with a particular case of the integral model problems.

Given any positive number T, we denote the interval $[0,1]$ and the rectangle $[0,1]\times[0,T]$ as I and I_T, respectively. We consider the nonlinear partial integro-differential equation

$$u_t + uu_x = \int_0^t (t-s)^{-\alpha} u_{xx}(x,s)\,ds, \qquad (x,t)\in I_T, \tag{1.1}$$

where $0<\alpha<1$ is a constant, together with the boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0\le t\le T, \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \qquad x\in I. \tag{1.3}$$

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When the second term of the left-hand side of (1.1) is absent, we obtain the linear counterpart of equation (1.1),

$$u_t = \int_0^t (t-s)^{-\alpha} u_{xx}(x,s)\,ds, \qquad (x,t) \in I_T, \tag{1.4}$$

and it may be considered as an equation intermediate between the standard (parabolic) heat equation and (hyperbolic) wave equation [4]. Equations (1.1), (1.4) arise in a number of practical applications, for example, in the modeling of physical phenomena associated with non-Newtonian fluids [5,6]. In fact, (1.4) can be viewed as a particular case when the Newtonian contribution to the viscosity reduces to null in the Boltzmann equation [7]

$$u_t - \eta u_{xx} = \int_{-\infty}^t G(t-s) u_{xx}(x,s)\,ds.$$

Much effort has been devoted during recent years to the numerical investigations of problems (1.1)–(1.3) and (1.4), (1.2)–(1.3) (cf., [4,8,9]). By setting $\alpha = 1/2$, Sanz-Serna [4] studied a time discretization of the linear model equation (1.4). The method reduces to the backward Euler method if the integral term is absent. The integral term is approximated by means of a convolution quadrature of Lubich [10,11]. Following his approach, Lopéz-Marcos [8] studied a fully discretized scheme in which again the integral term was treated by means of a convolution quadrature of Lubich and the backward Euler approximation. The derivatives in space were approximated by the central difference approximants. It was found that under certain smoothness assumptions, the numerical scheme was of a convergence order $\tau|\ln\tau|^{1/2} + h^2$, where τ and h are the time and space step sizes, respectively.

More recently, Tang [12] further studied the problem (1.1)–(1.3) numerically. Again, $\alpha = 1/2$ was assumed. The time derivative term was approximated by a Crank-Nicolson time-stepping and the integral term was treated by the product trapezoidal method. A convergence rate of $O(\tau^{3/2} + h^2)$ was proved.

In this paper, we shall investigate an Euler and finite difference approach for the nonlinear integro-differential equation problem (1.1)–(1.3). u_t will be replaced by the backward Euler approximation, and the integral term in (1.1) will be calculated by using the Euler product integration technique, while the spatial derivatives are to be approximated by conventional central difference approximants. Throughout this paper, we assume u_0 in (1.3) is such that the problem (1.1)–(1.3) has a unique solution in I_T. Furthermore, we suppose that u_{xxxx} and u_t are continuous in I_T. We also assume that u_{tt} and u_{xxtt} are continuous for $0 \le x \le 1$ and $0 < t < T$, and there exists a positive constant C_0 such that for $x \in I$ and $0 < t \le T$,

$$|u_{tt}(x,t)| \le C_0 t^{-\alpha}, \qquad |u_{ttxx}(x,t)| \le C_0 t^{-\alpha} \tag{1.5}$$

(see [8] for these assumptions).

As mentioned before, first-order and second-order time discretizations (based on truncation errors) have been investigated in [4,8,12]. Due to lack of smoothness of the exact solution, (1.5), the optimal convergence rate in each case cannot be attained. However, it will be shown that the present scheme achieves the optimal convergence rate in time, which holds for all $\alpha \in (0,1)$.

2. NUMERICAL SCHEME

Given positive integers M, N. We define the grid $\tilde I_T = \{(x_m, t_n) : x_m = mh,\ t_n = n\tau;\ m = 0,1,\dots,M,\ n = 0,1,\dots,N\}$, where $h = 1/M$ and $\tau = T/N$ are the spatial and time steps, respectively. Denote w_m^n as a lattice function defined on $\tilde I_T$. We introduce the finite differences, say, with respect to the variable x,

$$\Delta_x^+ w_m^n = w_{m+1}^n - w_m^n, \qquad \Delta_x^- w_m^n = w_m^n - w_{m-1}^n, \qquad \Delta_x w_m^n = w_{m+1}^n - w_{m-1}^n; \tag{2.1}$$

$$\delta_x^2 w_m^n = \Delta_x^+ \Delta_x^- w_m^n = w_{m+1}^n - 2w_m^n + w_{m-1}^n. \tag{2.2}$$

Let $v^n = (v_1^n, v_2^n, \dots, v_{M-1}^n)^T$, $w^n = (w_1^n, w_2^n, \dots, w_{M-1}^n)^T \in \mathbb{R}^{M-1}$. Throughout this paper, whenever symbols such as w_0^n, w_M^n appear in the discussion, we shall understand that $w_0^n = w_M^n = 0$. We further define

$$v^p w^q = \left(v_1^p w_1^q, v_2^p w_2^q, \dots, v_{M-1}^p w_{M-1}^q\right)^T; \tag{2.3}$$

$$\langle v^p, w^q \rangle = \sum_{m=1}^{M-1} h v_m^p w_m^q, \qquad \|w^n\| = \sqrt{\langle w^n, w^n \rangle}, \tag{2.4}$$

where $h > 0$ is the spatial step of the grid. It can be proved [13] that

$$\left(w_{m+1}^n + w_m^n + w_{m-1}^n\right) \Delta_x w_m^n = w_m^n \Delta_x w_m^n + \Delta_x \left(w_m^n\right)^2; \tag{2.5}$$

$$\left\langle w^n \Delta w^n + \Delta (w^n)^2, w^n \right\rangle = 0; \tag{2.6}$$

$$\left\langle \delta_x^2 v^n, w^n \right\rangle = -\left\langle \Delta_x^+ v^n, \Delta_x^+ w^n \right\rangle = -\sum_{m=1}^{M-1} h \left(\Delta_x^+ v_m^n\right)\left(\Delta_x^+ w_m^n\right). \tag{2.7}$$

To study the stability and convergence of the finite difference schemes for problem (1.1)–(1.3) using the energy method, one of the key ingredients is to obtain the nonnegative property of the associated quadratic forms. Lopéz-Marcos [8] shows that the quadratic form, based on the method of Lubich, is nonnegative. His result is obtained by employing the frequency domain techniques and can be viewed as a discrete analog of the result due to Nohel and Shea [14]. However, the kernel function appeared on the right-hand side of (1.1) is no longer of a convolution form, and therefore, the nonnegative character of the corresponding quadratic form needs to be reestablished.

LEMMA 2.1. *Given any $n \in \mathbb{Z}^+$. For any $v \in \mathbb{R}^n$, we have*

$$\sum_{i=1}^{n} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - s)^{-\alpha} v_{j+1} v_i \, ds \geq 0. \tag{2.8}$$

PROOF. Note that

$$(1-t)^{-\alpha} = \sum_{k=0}^{\infty} (-1)^k \binom{-\alpha}{k} t^k.$$

We have

$$(t_i - s)^{-\alpha} = t_i^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{-\alpha}{k} \left(\frac{s}{t_i}\right)^k.$$

Due to the fact that $(-1)^k \binom{-\alpha}{k} > 0$, $k = 0, 1, \dots$, (2.8) is obvious if

$$\sum_{i=1}^{n} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} t_i^{-\alpha-k} s^k v_{j+1} v_i \, ds \geq 0, \qquad k \in \{0\} \cup \mathbb{Z}^+.$$

To prove the above inequalities, we define

$$y_i = \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} s^k v_{j+1} \, ds = \sum_{j=0}^{i-1} \omega_{j+1} v_{j+1}, \qquad i = 1, 2, \dots, n,$$

where

$$\omega_{j+1} = \int_{t_j}^{t_{j+1}} s^k \, ds = \frac{t_{j+1}^{k+1} - t_j^{k+1}}{k+1} > 0, \qquad j = 0, 1, \dots, i-1.$$

Let $y_0 = 0$; from the above equations, we obtain the recurrence $y_i = \omega_i v_i + y_{i-1}$, $i = 1, 2, \ldots$. It follows that

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} t_i^{-\alpha-k} s^k v_{j+1} v_i \, ds &= \sum_{i=1}^{n} t_i^{-\alpha-k} y_i v_i \\
&= \sum_{i=1}^{n} t_i^{-\alpha-k} \frac{y_i^2 - y_i y_{i-1}}{\omega_i} \geq \sum_{i=1}^{n} t_i^{-\alpha-k} \frac{y_i^2 - y_{i-1}^2}{2\omega_i} \\
&= t_n^{-\alpha-k} \frac{y_n^2}{2\omega_n} + \frac{1}{2}\sum_{i=1}^{n-1} y_i^2 \left(\frac{t_i^{-\alpha-k}}{\omega_i} - \frac{t_{i+1}^{-\alpha-k}}{\omega_{i+1}} \right). \qquad (2.9)
\end{aligned}$$

Let

$$f(\xi) = \xi^{k+1} - (\xi - 1)^{k+1}, \qquad \xi \geq 1, \quad k \geq 0.$$

It is easy to show that f is a positive valued increasing function since that $f(1) = 1$ and $f'(\xi) = (k+1)(\xi^k - (\xi - 1)^k) > 0$ for $\xi > 1$, $k \geq 0$. Thus, we find that

$$\begin{aligned}
\frac{t_i^{-\alpha-k}}{\omega_i} &= \frac{(i\tau)^{-\alpha-k}}{\frac{1}{k+1}\left((i\tau)^{k+1} - ((i-1)\tau)^{k+1}\right)} = \frac{k+1}{\tau^{\alpha+2k+1}} \frac{1}{(i^{k+1} - (i-1)^{k+1}) i^{\alpha+k}} \\
&= \frac{k+1}{\tau^{\alpha+2k+1}} \frac{1}{i^{\alpha+k} f(i)}, \qquad i = 1, 2, \ldots,
\end{aligned}$$

and this indicates that $t_i^{-\alpha-k}/\omega_i$ is a nonincreasing function of i, $i \in \mathbb{Z}^+$. Subsequently, from (2.9) we obtain

$$\sum_{i=1}^{n}\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} t_i^{-\alpha-k} s^k v_{j+1} v_i \, ds \geq 0, \qquad k = 0, 1, \ldots.$$

The proof is then completed. ∎

Let the approximation to $u_m^n = u(x_m, t_n)$, where u is the exact solution of (1.1)–(1.3), be denoted by U_m^n which is computed from the difference equation

$$\begin{gathered}
\Delta_t^- U_m^n + \frac{\tau}{6h}\left(U_{m-1}^n + U_m^n + U_{m+1}^n\right)\Delta_x U_m^n - \frac{\tau}{h^2}\sum_{i=0}^{n-1} c_{n,i}\, \delta_x^2\, U_m^{i+1} = 0, \\
m = 1, 2, \ldots, M-1; \quad n = 1, 2, \ldots, N,
\end{gathered} \qquad (2.10)$$

together with the boundary and initial conditions

$$U_0^n = U_0^n = 0, \qquad 0 \leq n \leq N, \qquad (2.11)$$

$$U^0 = \left(U_1^0, U_2^0, \ldots, U_{M-1}^0\right)^T, \qquad (2.12)$$

where

$$c_{n,i} = \int_{t_i}^{t_{i+1}} (t_n - s)^{-\alpha}\, ds = \frac{1}{1-\alpha}\left((t_n - t_i)^{1-\alpha} - (t_n - t_{i+1})^{1-\alpha}\right). \qquad (2.13)$$

Let the local truncation error σ_m^n be defined by

$$\begin{gathered}
\sigma_m^n = \frac{1}{\tau}\Delta_t^- u_m^n + \frac{1}{6h}\left(u_m^n \Delta_x u_m^n + \Delta_x (u_m^n)^2\right) - \frac{1}{h^2}\sum_{i=0}^{n-1} c_{n,i}\, \delta_x^2\, u_m^{i+1}, \\
1 \leq m \leq M-1, \quad 1 \leq n \leq N.
\end{gathered} \qquad (2.14)$$

We have the following lemma.

LEMMA 2.2. *Given any $n \in \mathbb{Z}^+$, $n \le N$. For any fixed α, $0 < \alpha < 1$, we have*

$$\|\sigma^n\|_\infty = O\left(\tau \cdot t_n^{-\alpha} + h^2\right), \tag{2.15}$$

$$\sum_{n=1}^{N} \tau \|\sigma^n\|_\infty = O\left(\tau + h^2\right), \tag{2.16}$$

as τ, $h \to 0$.

PROOF. Recall condition (1.5). We have, by using Taylor expansion, that

$$\frac{1}{\tau}\left(u_m^n - u_m^{n-1}\right) - u_t\left(x_m, t_n\right) = O(\tau) \cdot \left|u_{tt}\left(x_m, \xi\right)\right| = O(\tau \cdot t_n^{-\alpha}), \qquad \xi \in \left(t_{n-1}, t_n\right), \tag{2.17}$$

$$\frac{1}{6h}\left(u_m^n \Delta_x u_m^n + \Delta_x \left(u_m^n\right)^2\right) - u\left(x_m, t_n\right) u_x\left(x_m, t_n\right) = O\left(h^2\right). \tag{2.18}$$

On the other hand, we have, for any $s \in [t_i, t_{i+1}]$, $0 \le i \le N-1$, that

$$\begin{aligned}\frac{1}{h^2}\,\delta_x^2\, u_m^{i+1} - u_{xx}(x_m, s) &= \frac{1}{h^2}\,\delta_x^2\, u_m^{i+1} - u_{xx}\left(x_m, t_{i+1}\right) + u_{xx}\left(x_m, t_{i+1}\right) - u_{xx}\left(x_m, s\right) \\ &= O\left(\tau + h^2\right).\end{aligned}$$

Further, we have

$$\begin{aligned}\frac{1}{h^2}\sum_{i=0}^{n-1} c_{n,i}\, \delta_x^2\, u_m^{i+1} &- \int_0^{t_n} \left(t_n - s\right)^{-\alpha} u_{xx}\left(x_m, s\right)\, ds \\ &= \frac{1}{h^2}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \left(t_n - s\right)^{-\alpha}\left(\delta_x^2 u_m^{i+1} - h^2 u_{xx}\left(x_m, s\right)\right)\, ds \\ &= O\left(\tau + h^2\right)\int_0^{t_n} \left(t_n - s\right)^{-\alpha}\, ds = O\left(\tau + h^2\right).\end{aligned} \tag{2.19}$$

Combining (2.17), (2.18) and (2.19), we obtain immediately that

$$\sigma_m^n = O\left(\tau \cdot t_n^{-\alpha} + h^2\right),$$

which gives (2.15). For (2.16), we observe that for any n, $N \in \mathbb{Z}^+$, $n \le N$,

$$\tau \sum_{i=1}^{n} t_i^{-\alpha} = \frac{1}{N}\sum_{i=1}^{n}\left(\frac{i}{N}\right)^{-\alpha} \le \int_0^{n/N} \xi^{-\alpha}\, d\xi \le \int_0^1 \xi^{-\alpha}\, d\xi = \frac{1}{1-\alpha}$$

due to the fact that the function $\xi^{-\alpha}$ is strictly decreasing for $\xi > 0$. This, together with (2.15), yields (2.16). The proof of the lemma is thus completed. ∎

3. CONVERGENCE ANALYSIS

Let C_1, C_2, C_3 and C be positive constants which are independent of τ, h, m and n. We have the following theorem.

THEOREM 3.1. *Suppose that the solution of (1.1)–(1.3) satisfies the smoothness requirements stated in Section 1, and $(U^0, U^1, \dots, U^N)$ are solutions of (2.10)–(2.12). As τ, h tend to zero independently, for any fixed α, $0 < \alpha < 1$, we have*

$$\max_{0 \le n \le N} \|U^n - u^n\| = O\left(\left\|U^0 - u^0\right\| + \tau + h^2\right). \tag{3.1}$$

PROOF. Let $e_m^n = U_m^n - u_m^n$, with $u_m^n = u(x_m, t_n)$ and U_m^n being the solution of (2.10)–(2.12). Subtraction of (2.14) from (2.10) yields

$$\frac{e_m^n - e_m^{n-1}}{\tau} = -\frac{1}{6}\left(B\left(U_m^n\right) - B\left(u_m^n\right)\right) + \frac{1}{h^2}\sum_{i=0}^{n-1} c_{n,i}\,\delta_x^2\, e_m^{i+1} + \sigma_m^n, \tag{3.2}$$
$$1 \le m \le M-1, \quad 1 \le n \le N,$$

where

$$B\left(w_m^n\right) = h^{-1}\left(w_{m-1}^n + w_m^n + w_{m+1}^n\right)\Delta_x w_m^n, \qquad 1 \le m \le M-1, \quad 1 \le n \le N.$$

Multiplying both sides of (3.2) by $\tau h e_m^n$ and summing in m, we have

$$\begin{aligned}\langle e^n, e^n\rangle - \left\langle e^{n-1}, e^n\right\rangle &= \frac{\tau}{6}\left\langle B(U^n) - B(u^n), e^n\right\rangle \\ &\quad + \frac{\tau}{h^2}\sum_{i=0}^{n-1} c_{n,i}\left\langle \delta_x^2 e^{i+1}, e^n\right\rangle + \tau\left\langle \sigma^n, e^n\right\rangle \\ &= \frac{\tau}{6}\left\langle B(U^n) - B(u^n), e^n\right\rangle - \frac{\tau}{h^2}\sum_{i=0}^{n-1} c_{n,i}\left\langle \Delta_x^+ e^{i+1}, \Delta_x^+ e^n\right\rangle \\ &\quad + \tau\left\langle \sigma^n, e^n\right\rangle, \qquad 1 \le n \le N,\end{aligned} \tag{3.3}$$

in which relation (2.7) is used to get the second term of the right-hand side of (3.3). It follows then from (3.3) that

$$\begin{aligned}\|e^n\|^2 &= \left\langle e^{n-1}, e^n\right\rangle + \frac{\tau}{6}\left\langle B(U^n) - B(u^n), e^n\right\rangle \\ &\quad - \frac{\tau}{h^2}\sum_{i=0}^{n-1} c_{n,i}\left\langle \Delta_x^+ e^{i+1}, \Delta_x^+ e^n\right\rangle + \tau\left\langle \sigma^n, e^n\right\rangle \\ &\le \frac{1}{2}\left(\|e^{n-1}\|^2 + \|e^n\|^2\right) + \frac{\tau}{6}\left\langle B\left(U^n\right) - B\left(u^n\right), e^n\right\rangle \\ &\quad - \frac{\tau}{h^2}\sum_{i=0}^{n-1} c_{n,i}\left\langle \Delta_x^+ e^{i+1}, \Delta_x^+ e^n\right\rangle + \tau\left\langle \sigma^n, e^n\right\rangle, \qquad 1 \le n \le N.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|e^n\|^2 - \left\|e^{n-1}\right\|^2 &\le \frac{\tau}{3}\left\langle B\left(U^n\right) - B\left(u^n\right), e^n\right\rangle - \frac{2\tau}{h^2}\sum_{i=0}^{n-1} c_{n,i}\left\langle \Delta_x^+ e^{i+1}, \Delta_x^+ e^n\right\rangle \\ &\quad + 2\tau\left\langle \sigma^n, e^n\right\rangle, \qquad 1 \le n \le N.\end{aligned} \tag{3.4}$$

Following the discussion of Lopéz-Marcos [8], we can show that the modulus of the first term of the right-hand side of the above inequality can be bounded by $C_1\tau\|e^n\|$. For the last term of (3.4), making use of the fact that $\|w\|_1 \le \sqrt{Mh}\|w\| = \|w\|$, we find that

$$\left\langle \sigma^n, e^n\right\rangle \le \|\sigma^n\|_\infty \|e^n\|_1 \le \|\sigma^n\|_\infty \Lambda, \tag{3.5}$$

where $\Lambda = \max_{0\le n\le N}\|e^n\|$. Substituting the above estimates into (3.4) and summing up over n, we obtain that

$$\begin{aligned}\|e^n\|^2 &\le \left\|e^0\right\|^2 + C_1\tau\sum_{i=0}^{n}\left\|e^i\right\|^2 + \tau\Lambda\sum_{i=1}^{n}\left\|\sigma^i\right\|_\infty \\ &\quad - \frac{2\tau}{h^2}\sum_{i=1}^{n}\sum_{j=0}^{i-1} c_{i,j}\left\langle \Delta_x^+ e^{j+1}, \Delta_x^+ e^i\right\rangle \\ &= \left\|e^0\right\|^2 + C_1\tau\sum_{i=1}^{n}\left\|e^i\right\|^2 + \tau\Lambda\sum_{i=1}^{n}\left\|\sigma^i\right\|_\infty \\ &\quad - \frac{2\tau}{h^2}\sum_{i=1}^{n}\sum_{j=0}^{i-1} c_{i,j}\sum_{m=1}^{M-1}\left(\Delta_x^+ e_m^{j+1}\right)\left(\Delta_x^+ e_m^i\right), \qquad 1 \le n \le N.\end{aligned}$$

We observe that, due to Lemma 2.1,

$$\sum_{i=1}^{n}\sum_{j=0}^{i-1} c_{i,j} \sum_{m=1}^{M-1} \left(\Delta_x^+ e_m^{j+1}\right)\left(\Delta_x^+ e_m^i\right)$$
$$= \sum_{m=1}^{M-1} \left\{ \sum_{i=1}^{n}\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} (t_i - s)^{-\alpha} \left(\Delta_x^+ e_m^{j+1}\right)\left(\Delta_x^+ e_m^i\right) ds \right\} \geq 0.$$

It follows then

$$\begin{aligned}
\|e^n\|^2 &\leq \left\|e^0\right\|^2 + C_1\tau \sum_{i=1}^{n} \left\|e^i\right\|^2 + \tau\Lambda \sum_{i=1}^{n} \left\|\sigma^i\right\|_\infty \\
&\leq \left\|e^0\right\| \Lambda + C_1\tau \sum_{i=1}^{n} \left\|e^i\right\|^2 + C_2\left(\tau + h^2\right)\Lambda \\
&= C_1\tau \sum_{i=1}^{n} \left\|e^i\right\|^2 + C_3\left(\left\|e^0\right\| + \tau + h^2\right)\Lambda, \qquad 1 \leq n \leq N.
\end{aligned}$$

An application of the Gronwall lemma to the above inequality yields that

$$\|e^n\|^2 \leq C\left(\left\|e^0\right\| + \tau + h^2\right)\Lambda, \qquad 1 \leq n \leq N.$$

The above relation implies that $\Lambda^2 \leq C(\|e^0\| + \tau + h^2)\Lambda$, which is equivalent to $\Lambda = O(\|U^0 - u^0\| + \tau + h^2)$. Hence, the convergence result is proved. ∎

REMARK 3.1. A graded mesh, in which

$$t_n = \left(\frac{n}{N}\right)^\gamma T, \qquad 0 \leq n \leq N,$$

and $\gamma \geq 1$ is the grading exponent, can be also introduced to overcome the difficulties in achieving the optimal convergence rate of the numerical method in practical computations. Following [15], it can be proved that for suitable values of γ, the optimal order of convergence of the numerical scheme can be achieved.

4. NUMERICAL EXAMPLES

Viscoelastic fluids have intermediate properties, and this is reflected in the mathematical nature of their governing equations. To illustrate, we consider the Boltzmann's problem

$$u_t = \int_0^t (t-s)^{-\alpha} u_{xx}(x,s)\, ds, \qquad (x,t) \in I_T, \tag{4.1}$$
$$u(0,t) = u(1,t) = 0, \qquad 0 \leq t \leq T, \tag{4.2}$$
$$u(x,0) = \phi(x), \qquad x \in I, \tag{4.3}$$

where $\phi(0) = \phi(1) = 0$ and the stress-relaxation modulus $G(t-s) = (t-s)^{-\alpha}$, $0 \leq \alpha < 1$, has a singularity at the origin. Alternatively, equation (4.1) can be written in terms of the displacement w,

$$w_{tt} = \int_0^t g(t-s)\left[w_{xx}(x,t) - w_{xx}(x,s)\right] ds, \qquad (x,t) \in I_T,$$

rather than the velocity u. Here, $g = -G'$ is the memory function. If g is smooth, the term containing w_{xx} would be viewed as that of the highest order. Since that we are integrating with respect to s, the term involving w_{xx} is considered a lower order term. The equation would therefore be classified as hyperbolic. Intermediate possibilities arise in the equation due to the singularity of our memory function at the origin. One has to differentiate between an integrable singularity and a nonintegrable singularity in order to determine the types of the equations. The linear problem is no longer simple in the sense.

Linear problems with singular memory functions not only lead to interesting mathematical analysis, but they can also be motivated physically. Historically, Boltzmann was among the pioneers to attempt fitting data with a singular memory function. Modern measurements of oscillatory shear moduli at high frequency and of wave speeds suggest that certain fluids have singular or almost-singular memory functions. The molecular theories in which molecules as chains of beads and springs or rods lead to singular memory functions in the limit of an infinite number of beads in the molecule [1]. A nonlinear extension of (4.1) is the well-known K-BKZ equation.

CASE A. Let $\alpha = 1/2$ and let $\phi(x) = \sin(\pi x)$. The solution of the problem (4.1)–(4.3) can be written as

$$u(x,t) = E\left(\pi^{5/2}t^{3/2}\right)\sin(\pi x), \qquad (x,t) \in I_T, \tag{4.4}$$

where $E(\xi) = \sum_{k=0}^{\infty}(-1)^k\Gamma(3k/2+1)^{-1}\xi^k$ is the entire function [4]. The estimate of Theorem 3.1 is the best possible in space since central difference approximations yield a convergence order of at most two. The purpose of the examples is to verify that the error bound for the time discretization is the optimal one.

From the numerical scheme (2.10)–(2.12), we obtain the system of linear equations of the form

$$AU^n = b^{n-1}, \qquad 1 \le n \le N, \tag{4.5}$$

where $b^{n-1} \in \mathbb{R}^{M-1}$ is independent of U^n and $A \in \mathbb{R}^{(M-1)\times(M-1)}$ is of a tridiagonal form, in which the diagonal entries are $1 + 2\tau c_{n,n-1}/h^2$ and the subdiagonal entries are $-\tau c_{n,n-1}/h^2$. $U^0 = (\sin(h\pi), \sin(2h\pi)\dots, \sin((M-1)h\pi)^T$ is the initial condition. This linear system can be solved by a procedure using the standard decomposition method. However, on the other hand, we observe that in our case, the inverse of A can be computed in an explicit way; that is,

$$A^{-1} = PHP, \tag{4.6}$$

where $P = (\sqrt{2M}\,\sin(ij\pi/M))_{i,j=1}^{M-1}$ is a $(M-1)\times(M-1)$ symmetric matrix and H is a diagonal matrix which takes the form

$$H = \operatorname{diag}\left(\left(1 + \frac{4\tau c_{n,n-1}}{h^2}\sin^2\frac{k\pi}{2M}\right)^{-1}\right)_{k=1}^{M-1}.$$

The equality (4.6) provides an explicit way to compute the numerical solution of (4.5).

Note that $\|w\| \le \|w\|_\infty\sqrt{M-1}$ for $w \in \mathbb{R}^{M-1}$. In Table 1, we list the errors in the infinite-norm and computed rates of convergence when algorithms employing (4.6) and uniform stepsizes $h = \tau = 1/N = 1/M$ are adopted. In the calculation, we take the solutions at $T = 0.6$. Figure 1 plots the result of the computed rates of convergence against N. The numerical results reflect a convergence rate ≈ 1 in time, which is in good agreement with the theoretical prediction in Section 3.

CASE B. Let α be 0.1π and 0.2π, respectively. Note that (4.4) is no longer valid in the case; we choose the numerical solution when $h = \tau = 1/N = 1/M = 1/320$ are used as the "exact solution" for computing the rate of convergence.

Let $\phi(x) = \sin(2\pi x - \frac{\pi}{2}) + 1$ and let $N = M = 10, 20, 40, 80$ and 160, respectively. We compute the corresponding numerical solutions of (2.10)–(2.12) using the uniform stepsizes. The

Table 1. The numerical error and estimated rate of convergence.

N	Error	Rate
5	0.120627	—
10	0.663920×10^{-1}	0.861472
20	0.335817×10^{-1}	0.983335
40	0.167036×10^{-1}	1.007513
80	0.823511×10^{-2}	1.020303
160	0.404303×10^{-2}	1.026352
320	0.198471×10^{-2}	1.026507

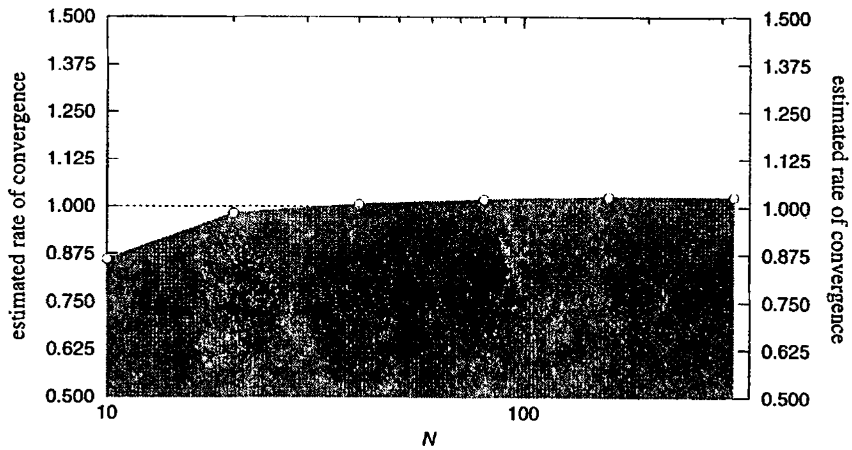

Figure 1. Estimated rate of convergence ($\alpha = 0.5$, $T = 0.6$).

numerical solutions are taken at $T = 1/N, 0.1, 0.2, \dots, 1$ and the estimated rates of convergence are computed through the formula

$$r = \frac{1}{\ln 2} \ln \frac{\|u_\tau - u\|}{\|u_{\tau/2} - u\|}, \tag{4.7}$$

where $u_\tau, u_{\tau/2}$ are numerical solutions using time steps τ and $\tau/2$, respectively, and u is the exact solution of the problem. The solutions are plotted in Figures 2 and 3, where the curves marked with circles, squares, diamonds and triangles are for estimated rates of convergence when $N = M = 20, 40, 80$ and 160, respectively. Because the numerical solution when $N = 160$ can be a "too good" approximation to the "exact solution," the heights of the curves marked with triangles in both figures are higher than 1. Note that the maximum rate of convergence in time is at most 1 due to the forward difference approximation adopted, so the actual rate of convergence should be close to 1 in the case. Thus, we again conclude that the rate of the numerical scheme is approximately 1, though slight oscillations are observed in Figure 2.

CASE C. We now choose different values of the parameter α in the computation. Again, in order to estimate the rate of convergence, we choose $\tau = h = 1/N = 1/M = 1/320$ and compute the numerical solutions with $\alpha = 0.01, 0.1, 0.2, \dots, 0.9, 0.99$, as the "exact solutions" for comparison. Next, for $\tau = h = 1/N$, $N = M = 20, 40, 80$ and 160, we compute solutions of the numerical scheme (2.10)–(2.12) with the same values of α mentioned above, respectively. Finally, we compare the solutions obtained with the "exact solutions" and compute the rate of convergence using (4.7). The initial function ϕ is the same as in Case B, and the solutions are taken at $T = 0.5$. The computed results are plotted in Figure 4, where the curves marked with squares, diamonds and triangles are for computed rates of convergence when $N = 40, 80$ and 160 are used, respectively. The computed result when $N = 160$ again seems to be better than expected due to the fact that the numerical solution when $h = \tau = 1/160$ is "too good" in approximating

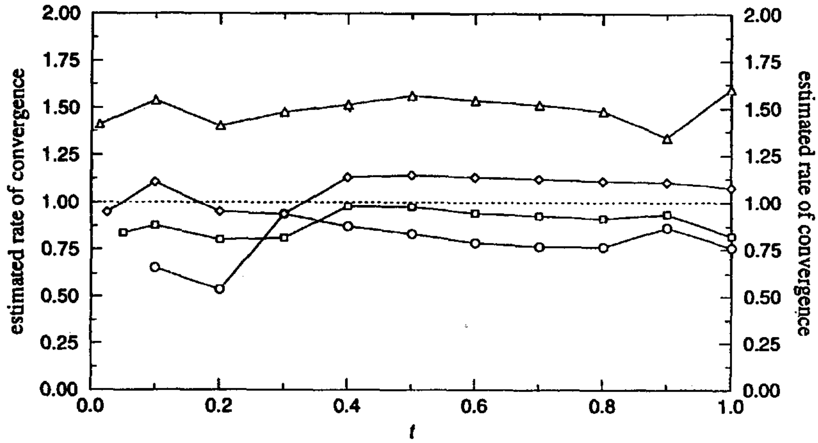

Figure 2. Estimated rate of convergence ($\alpha = 0.1\pi$).

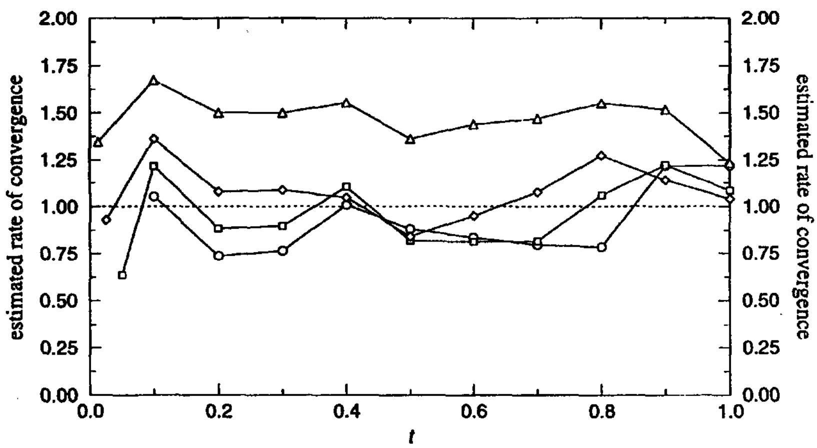

Figure 3. Estimated rate of convergence ($\alpha = 0.2\pi$).

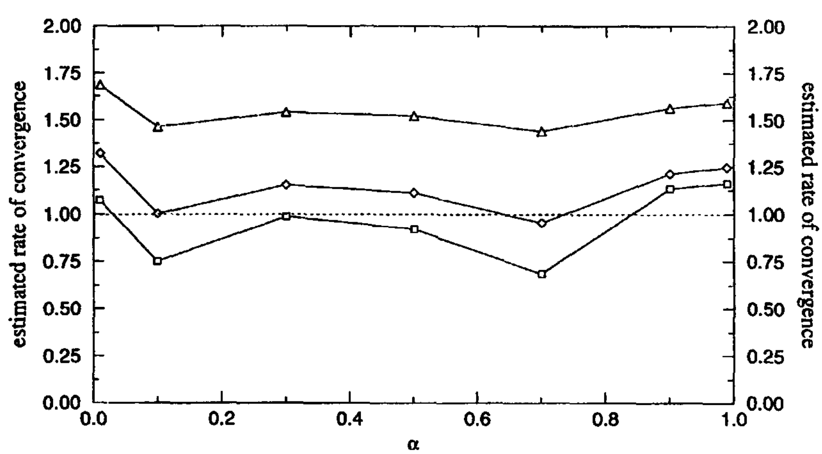

Figure 4. Estimated rate of convergence ($T = 0.5$).

the "exact solution." Hence, the rate of convergence should be 1, and this again leads to the conclusion that the rate of convergence of the method is approximately 1.

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