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A finite difference scheme for partial integro-differential equations with a weakly singular kernel *

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Abstract

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A finite difference method for the numerical solution of partial integro-differential equations is considered. In the time direction, a Crank–Nicolson time-stepping is used to approximate the differential term and the product trapezoidal method is employed to treat the integral term. An error bound is derived for the numerical scheme. Due to lack of smoothness of the exact solution, the overall numerical procedure does not achieve second-order convergence in time. But the convergence order in time is shown to be greater than one, which is confirmed by a numerical example.

Keywords. Partial integro-differential equations; Crank–Nicolson method; product trapezoidal method.

1. Introduction

This paper is concerned with the Crank–Nicolson method for the partial integro-differential equation

$$u_t = Bu + \int_0^t (t-s)^{-1/2} u_{xx}(x, s)\, ds, \tag{1.1}$$

subject to the boundary condition

$$u(0, t) = u(1, t) = 0, \quad t \geqslant 0, \tag{1.2}$$

and the initial condition

$$u(x, 0) = u_0(x), \quad 0 \leqslant x \leqslant 1, \tag{1.3}$$

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where B in (1.1) is a linear or nonlinear differential operator. In the practical applications, we often have

$$Bu = -uu_x, \tag{1.4}$$

or, with $\mu \geqslant 0$ a constant,

$$Bu = \mu u_{xx}. \tag{1.5}$$

The problem (1.1)–(1.4) (hereafter referred to as Problem I) can be found in the modeling of physical phenomena involving viscoelastic forces, e.g., [5,10]. It has been pointed out by Sanz-Serna [14] that (1.1) and (1.4) give a simple model equation that combines the Eulerian derivative $u_t + uu_x$ with a viscoelastic effect, just as Burgers' equation provides a simple model for the study of more realistic situations involving Eulerian derivatives and viscous forces. Problem I has been recently considered numerically by several researchers [1,3,6,14]. The method implemented by Christie [3] treats the weakly singular integral kernel in (1.1) by means of the product trapezoidal technique (see, e.g., [11, Chapter 8]). The Crank–Nicolson method has been employed by this author for time-stepping but it has been observed that the overall procedure does not achieve second-order convergence in time.

The problem (1.1)–(1.3) and (1.5) (hereafter referred to as Problem II) can be found in the modeling of heat flow in materials with memory, e.g. [4,7], and in linear viscoelastic mechanics, e.g., [2,12]. In linear viscoelastic problems, the integral term in (1.1) represents the viscosity part of the equation, and μ in (1.5) is a Newtonian contribution to the viscosity. It can be seen that in (1.1) the kernel function has a weak singularity at the origin. This is particularly interesting in viscoelasticity, because it might smooth the solution when the boundary data is discontinuous [12]. Numerical investigations for Problem II have been given by several authors (see, e.g., [8,9,15]), but most of them considered smooth integral kernels only.

In this paper, we shall construct a finite-difference scheme for (1.1), based on the Crank–Nicolson method and the product trapezoidal technique, and present convergence analysis for the scheme. Throughout this paper, we assume that u_0 in (1.3) is such that the problem (1.1)–(1.3) has a unique solution in $[0, 1] \times [0, T]$. Furthermore, we suppose that u_{xxxx} and u_t are continuous in $[0, 1] \times [0, T]$. We also assume that u_{tt}, u_{ttt}, and u_{xxtt} are continuous for $0 \leqslant x \leqslant 1$ and $0 < t \leqslant T$, and that there exists a positive constant C_0 such that for $0 \leqslant x \leqslant 1$ and $0 < t \leqslant T$,

$$\begin{aligned} &|u_{tt}(x, t)| \leqslant C_0 t^{-1/2}, \qquad |u_{ttt}(x, t)| \leqslant C_0 t^{-3/2}, \\ &|u_{xxtt}(x, t)| \leqslant C_0 t^{-1/2} \end{aligned} \tag{1.6}$$

(see [6] for these assumptions).

The remainder of the paper is organized as follows. In Section 2 we introduce a numerical scheme for solving Problems I and II, which is based on the Crank–Nicolson method in time and central differences in space. The product trapezoidal technique is used to approximate the integral term. The convergence properties of the numerical scheme are investigated in Section 3. Numerical results are presented in the final section.

2. Numerical scheme

We introduce a grid $x_j = jh$, $j = 0, 1, \ldots, J$, with $h = 1/J$ and J a positive integer. The steplength in time is denoted by k, $k = T/N$ with N a positive integer, and a subscript n refers to the time level $t_n = nk$, $n = 0, 1, \ldots, N$. Moreover, we let $t_{n+1/2} = (n + \frac{1}{2})k$, $0 \leqslant n \leqslant N-1$. We first consider the approximation of an integral term with a similar type as that in (1.1). For any $f \in C^1([0, T]) \cap C^3((0, T])$ satisfying $f''(t) = O(t^{-1/2})$ and $f'''(t) = O(t^{-3/2})$ as $t \to 0+$, we will approximate $I(f, t) := \int_0^t (t-s)^{-1/2} f(s)\, ds$ numerically. It is easy to see that

$$I(f, t_{n+1/2}) = \tfrac{1}{2}[I(f, t_n) + I(f, t_{n+1})] + O(k^2 t_n^{-3/2}), \quad n \geqslant 1 \tag{2.1}$$

and

$$I(f, t_{1/2}) = \tfrac{1}{2} I(f, t_1) + O(k^{1/2}). \tag{2.2}$$

Since $t_n^{-3/2} \leqslant 2^{3/2} t_{n+1}^{-3/2}$ for $n \geqslant 1$, (2.1) and (2.2) yield that

$$I(f, t_{n+1/2}) = \tfrac{1}{2}[I(f, t_n) + I(f, t_{n+1})] + O(k^2 t_{n+1}^{-3/2}), \quad n \geqslant 0. \tag{2.3}$$

We now use the product trapezoidal technique to approximate the integrals $I(f, t_n)$, $1 \leqslant n \leqslant N$. For $f(t_n - \theta)$ with $\theta \in [t_j, t_{j+1}]$, $0 \leqslant j \leqslant n-2$, we have

$$f(t_n - \theta) = \frac{t_{j+1} - \theta}{k} f(t_{n-j}) + \frac{\theta - t_j}{k} f(t_{n-j-1}) + R_{n1}, \tag{2.4}$$

and with $\theta \in [t_{n-1}, t_n]$, we obtain

$$f(t_n - \theta) = \frac{t_n - \theta}{k} f(t_1) + \frac{\theta - t_{n-1}}{k} f(t_0) + O(k^{3/2}), \tag{2.5}$$

where in (2.4) the remainder term R_{n1} can be bounded by $O(k^2 t_{n-j-1}^{-1/2}) = O(k^{3/2})$, since $0 \leqslant j \leqslant n-2$ implies that $n - j - 1 \geqslant 1$. Further, using (2.4) and (2.5), we have

$$\begin{aligned} I(f, t_n) &= \int_0^{t_n} \theta^{-1/2} f(t_n - \theta)\, d\theta = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \theta^{-1/2} f(t_n - \theta)\, d\theta \\ &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \theta^{-1/2} \left[\frac{t_{j+1} - \theta}{k} f(t_{n-j}) + \frac{\theta - t_j}{k} f(t_{n-j-1}) \right] d\theta + R_{n2}, \end{aligned} \tag{2.6}$$

with

$$|R_{n2}| \leqslant \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \theta^{-1/2} O(k^{3/2})\, d\theta = O(k^{3/2}). \tag{2.7}$$

It follows by using integration by parts that

$$\begin{aligned} &\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \theta^{-1/2} \left[\frac{t_{j+1} - \theta}{k} f(t_{n-j}) + \frac{\theta - t_j}{k} f(t_{n-j-1}) \right] d\theta \\ &\quad = 2 \sum_{j=0}^{n-1} \left[t_{j+1}^{1/2} f(t_{n-j-1}) - t_j^{1/2} f(t_{n-j}) \right] + 2 \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \theta^{1/2} \frac{f(t_{n-j}) - f(t_{n-j-1})}{k}\, d\theta \\ &\quad = A_n f(t_0) + \sum_{p=0}^{n} c_p f(t_{n-p}), \end{aligned} \tag{2.8}$$

where

$$A_n = 2\left[t_n^{1/2} - \frac{1}{k}\int_{t_n}^{t_{n+1}}\theta^{1/2}\,d\theta\right], \tag{2.9}$$

$$c_0 = \frac{2}{k}\int_0^{t_1}\theta^{1/2}\,d\theta, \tag{2.10}$$

$$c_p = \frac{2}{k}\left[\int_{t_p}^{t_{p+1}}\theta^{1/2}\,d\theta - \int_{t_{p-1}}^{t_p}\theta^{1/2}\,d\theta\right], \quad p \geqslant 1. \tag{2.11}$$

Combining (2.6), (2.7), and (2.8), we obtain that

$$I(f, t_n) = A_n f(t_0) + \sum_{p=0}^{n} c_p f(t_{n-p}) + O(k^{3/2}), \quad 1 \leqslant n \leqslant N, \tag{2.12}$$

where A_n and c_p are given by (2.9)–(2.11). Let

$$\beta_0 = c_0 + \frac{4\sqrt{k}}{3}\beta; \qquad \beta_1 = c_1 - \frac{4\sqrt{k}}{3}\beta; \qquad \beta_p = c_p, \quad p \geqslant 2, \tag{2.13}$$

where β is a nonnegative constant and is independent of k and h, i.e., $\beta \geqslant 0$ and $\beta = O(1)$. The sequence $\{\beta_p\}_{p=0}^{\infty}$ reduces to $\{c_p\}_{p=0}^{\infty}$ if the parameter $\beta = 0$. Since

$$\frac{4\sqrt{k}}{3}\beta[f(t_n) - f(t_{n-1})] = O(k^{3/2}),$$

we obtain from (2.12) that

$$I(f, t_n) = A_n f(t_0) + \sum_{p=0}^{n} \beta_p f(t_{n-p}) + O(k^{3/2}). \tag{2.14}$$

Now an application of the standard Crank–Nicolson method for (1.1) gives

$$\frac{u_j^{n+1} - u_j^n}{k} = \tilde{B}\left(\frac{u_j^{n+1} + u_j^n}{2}\right) + \int_0^{t_{n+1/2}}\left(t_{n+1/2} - s\right)^{-1/2}\frac{\delta^2 u(x_j, s)}{h^2}\,ds$$
$$+ O\left(k^2 t_{n+1}^{-3/2} + h^2\right), \tag{2.15}$$

for $0 \leqslant n \leqslant N-1$ and $1 \leqslant j \leqslant J-1$, where $\tilde{B}$ is a difference operator which is a discretized approximation for the differential operator B such that

$$\tilde{B}\left(\frac{u_j^{n+1} + u_j^n}{2}\right) = Bu(x_j, t_{n+1/2}) + O\left(k^2 t_{n+1}^{-3/2} + h^2\right). \tag{2.16}$$

Also in (2.15) we have used the notation $\delta^2 V_j = V_{j+1} - 2V_j + V_{j-1}$. The remainder term in (2.15) has been obtained by using Taylor's expansion and in particular we have noticed that

$$\left|\frac{u(x, t_{n+1}) - u(x, t_n)}{k} - u_t(x, t_{n+1/2})\right| = O\left(k^2 \| u_{ttt}(x, \cdot) \|_{\infty,[t_n,t_{n+1}]}\right)$$
$$= O\left(k^2 t_{n+1}^{-3/2}\right),$$

where we have made use of (1.6). Combining (2.3), (2.14), and (2.15), we obtain that

$$\frac{u_j^{n+1}-u_j^n}{k}=\tilde{B}\left(\frac{u_j^{n+1}+u_j^n}{2}\right)+\tilde{A}_n\delta^2u_j^0+\sum_{p=0}^{n}\beta_p\frac{\delta^2\left(u_j^{n-p}+u_j^{n+1-p}\right)}{2h^2}$$
$$+\mathrm{O}\left(k^2t_{n+1}^{-3/2}+k^{3/2}+h^2\right),\quad n\geqslant 0, \tag{2.17}$$

where

$$\tilde{A}_0=\frac{\beta_1+A_1+A_0+4\sqrt{k}\,\beta/3}{2h^2},\qquad \tilde{A}_n=\frac{\beta_{n+1}+A_{n+1}+A_n}{2h^2}\quad(n\geqslant 1). \tag{2.18}$$

Our numerical scheme is based on the formula (2.17) and takes the form

$$\frac{U_j^{n+1}-U_j^n}{k}=\tilde{B}\left(\frac{U_j^{n+1}+U_j^n}{2}\right)+\tilde{A}_n\delta^2U_j^0+\sum_{p=0}^{n}\beta_p\frac{\delta^2\left(U_j^{n-p}+U_j^{n+1-p}\right)}{2h^2}, \tag{2.19}$$

for $0\leqslant n\leqslant N-1$ and $1\leqslant j\leqslant J-1$. The above algebraic system will be solved subject to the following boundary and initial conditions:

$$U_0^n=U_J^n=0,\quad 0\leqslant n\leqslant N, \tag{2.20}$$
$$U_j^0=u_0(x_j),\quad 1\leqslant j\leqslant J-1. \tag{2.21}$$

For Problem I, we choose the difference operator $\tilde{B}$ such that

$$\tilde{B}\left(V_j^n\right)=-\frac{V_{j+1}^n+V_j^n+V_{j-1}^n}{3}\,\frac{\Delta V_j^n}{2h} \tag{2.22}$$

to approximate the differential operator B given in (1.4), where we have used the notation $\Delta W_j=W_{j+1}-W_{j-1}$. Since

$$\left|\frac{u(x,t_{n+1})+u(x,t_n)}{2}-u(x,t_{n+1/2})\right|=\mathrm{O}\left(k^2t_{n+1}^{-1/2}\right)\leqslant\mathrm{O}\left(k^2t_{n+1}^{-3/2}\right),$$

it can be verified that $\tilde{B}$ given by (2.22) and B given by (1.4) satisfy the requirement (2.16).

For Problem II, we choose $\tilde{B}$ such that

$$\tilde{B}\left(V_j^n\right)=\mu\frac{\delta^2V_j^n}{h^2}, \tag{2.23}$$

and it can be shown that this operator and the one given by (1.5) satisfy (2.16). For later use we collect the following notations and results, as given in [6]. Throughout this paper, we shall use the notation U^n, $0\leqslant n\leqslant N$, to refer to the vector in $\mathbb{R}^{J-1}$ comprising the approximations $(U_1^n,U_2^n,\ldots,U_{J-1}^n)$ corresponding to the time level t_n. Whenever symbols such as V_0 or V_J appear in the analysis we shall understand that $V_0=V_J=0$. If $V=(V_1,V_2,\ldots,V_{J-1})$ and

$W=(W_1, W_2, \ldots, W_{J-1})$ are real vectors, then we define

$$\Delta_+ V_j = V_{j+1} - V_j, \qquad \| V \|_\infty = \max_{1 \leqslant j \leqslant J-1} | V_j |, \qquad \| V \|_1 = \sum_{j=1}^{J-1} h | V_j |,$$

$$\langle V, W \rangle = \sum_{j=1}^{J-1} h V_j W_j, \qquad \| V \|^2 = \langle V, V \rangle.$$

Furthermore, the following identities hold (see, e.g., [13]):

$$\left(V_{j+1}^n + V_j^n + V_{j-1}^n\right) \Delta V_j^n = V_j^n \Delta V_j^n + \Delta\left(V_j^n\right)^2, \tag{2.24}$$

$$\langle V \Delta V + \Delta(V)^2, V \rangle = 0, \tag{2.25}$$

$$\langle \delta^2 V, W \rangle = -\langle \Delta_+ V, \Delta_+ W \rangle = -\sum_{j=1}^{J-1} h \Delta_+ V_j \Delta_+ W_j. \tag{2.26}$$

3. Convergence analysis

From now on, C denotes a positive constant which is independent of k, h, j, and n (with $0 \leqslant j \leqslant J$ and $0 \leqslant n \leqslant N$), but possibly with different values at different places. The following result, which is concerned with the nonnegative character of certain real quadratic forms with convolution structure, is due to Lopez-Marcos [6] and will play an important role in our convergence analysis.

Lemma 1. *Let $\{a_n\}_{n=0}^\infty$ be a sequence of real numbers with the properties*

$$a_n \geqslant 0, \qquad a_{n+1} - a_n \leqslant 0, \qquad a_{n+1} - 2a_n + a_{n-1} \geqslant 0. \tag{3.1}$$

Then for any positive integer M, and real vector $(V_1, V_2, \ldots, V_M)$ with M real entries,

$$\sum_{n=0}^{M-1} \left(\sum_{p=0}^{n} a_p V_{n+1-p} \right) V_{n+1} \geqslant 0. \tag{3.2}$$

Now we will check if $\{c_p\}_{p=0}^\infty$ defined by (2.10) and (2.11) satisfies (3.1).

Lemma 2. *The sequence $\{c_p\}_{p=0}^\infty$ defined by* (2.10) *and* (2.11) *satisfies $c_p \geqslant 0$ and $c_{p+1} \leqslant c_p$.*

Proof. It is obvious that $c_0 \geqslant 0$. It can be shown from (2.11) that

$$c_p = \int_{-k}^{k} \left(t_p - \theta\right)^{-1/2} \left(1 - \frac{|\theta|}{k}\right) \mathrm{d}\theta, \quad p \geqslant 1, \tag{3.3}$$

which implies that $c_p \geqslant 0$ for $p \geqslant 1$. Further, a direct calculation shows that $c_1 < c_0$. Using the fact that

$$\frac{\mathrm{d}}{\mathrm{d}r}(rk-\theta)^{-1/2} < 0 \quad \text{for } r \geqslant 1 \text{ and } \theta \in (-k, k)$$

we can obtain that $c_{p+1} \leqslant c_p$ for $p \geqslant 1$. □

Lemma 3. *The sequence* $\{c_p\}_{p=0}^{\infty}$ *satisfies* $c_2 - 2c_1 + c_0 <$ *and* $c_{p+1} - 2c_p + c_{p-1} \geqslant 0$ *for* $p \geqslant 2$.

Proof. The first part of this lemma follows from

$$c_2 - 2c_1 + c_0 = \frac{4\sqrt{k}}{3}\left[3\sqrt{3} - 8\sqrt{2} + 6\right] < 0. \tag{3.4}$$

The second part follows from (3.3) and the fact that

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(kr-\theta)^{-1/2} > 0 \quad \text{for } r \geqslant 1 \text{ and } \theta \in (-k, k). \qquad \square$$

It can be seen from Lemma 3 that $\{c_p\}_{p=0}^{\infty}$ defined by (2.10) and (2.11) does not satisfy (3.1). In order to give a convergence result for the numerical scheme (2.19)–(2.21) we need to choose the parameter $\beta \geqslant 0$ such that the sequence $\{\beta_p\}_{p=0}^{\infty}$ in (2.13) satisfies the property (3.1). By recalling the proofs of Lemmas 2 and 3, we require that

$$\beta_1 \geqslant 0, \qquad \beta_1 \geqslant \beta_2, \qquad \beta_2 - 2\beta_1 + \beta_0 \geqslant 0, \qquad \beta_3 - 2\beta_2 + \beta_1 \geqslant 0, \tag{3.5}$$

which are equivalent to

$$z \leqslant c_1, \qquad z \leqslant c_1 - c_2, \qquad z \geqslant -\frac{c_2 - 2c_1 + c_0}{3}, \qquad z \leqslant c_3 - 2c_2 + c_1, \tag{3.6}$$

where $z = (\frac{4}{3}\sqrt{k})\beta$. Since

$$c_2 - 2c_1 + c_0 < 0, \qquad c_1 \geqslant c_1 - c_2 \geqslant c_3 - 2c_2 + c_1 \geqslant 0, \tag{3.7}$$

the inequalities (3.6) lead to

$$-\frac{c_2 - 2c_1 + c_0}{3} \leqslant z \leqslant c_3 - 2c_2 + c_1, \tag{3.8}$$

which is equivalent to

$$\frac{-3\sqrt{3} + 8\sqrt{2} - 6}{3} \leqslant \beta \leqslant 4 - 12\sqrt{3} + 12\sqrt{2}. \tag{3.9}$$

The above discussion yields the following result.

Lemma 4. *If* β *satisfies* (3.9), *then the sequence* $\{\beta_p\}_{p=0}^{\infty}$ *defined by* (2.13) *satisfies* (3.1).

Now we begin to derive the error bounds for the numerical scheme (2.19)–(2.21). Let $e_j^n = U_j^n - u_j^n$, with $u_j^n = u(x_j, t_n)$ and U_j^n is the solution of (2.19)–(2.21). Subtraction of (2.17) from (2.19) gives

$$\frac{e_j^{n+1}-e_j^n}{k}=\tilde{B}\left(V_j^n\right)-\tilde{B}\left(v_j^n\right)+\sum_{p=0}^{n}\beta_p\frac{\delta^2\left(e^{n+1-p}+e_j^{n-p}\right)}{2h^2}+r_j^n, \tag{3.10}$$

where $V_j^n:=(U_j^{n+1}+U_j^n)/2$, $v_j^n:=(u_j^{n+1}+u_j^n)/2$ and r_j^n is the residual term which can be bounded by

$$\left|r_j^n\right|\leqslant \mathrm{O}\left(k^2t_{n+1}^{-3/2}+k^{3/2}+h^2\right). \tag{3.11}$$

Multiplying both sides of (3.10) by $hk(e_j^{n+1}+e_j^n)$ and summing in j, we obtain

$$\begin{aligned}&\|e^{n+1}\|^2-\|e^n\|^2\\&\quad=k\langle\tilde{B}(V^n)-\tilde{B}(v^n),\,e^{n+1}+e^n\rangle\\&\qquad-\frac{k}{2h^2}\sum_{p=0}^{n}\beta_p\big\langle\Delta_+(e^{n+1-p}+e^{n-p}),\,\Delta_+(e^{n+1}+e^n)\big\rangle+k\langle r^n,\,e^{n+1}+e^n\rangle,\end{aligned} \tag{3.12}$$

where we have used (2.25) to get the second term of the right-hand side of (3.12). For Problem I, following Lopez-Marcos [6, p. 128], we can show that the modulus of the first term of the right-hand side of (3.12) can be bounded by

$$Ck\,\|V^n-v^n\|^2\leqslant Ck\left(\|e^{n+1}\|^2+\|e^n\|^2\right). \tag{3.13}$$

For Problem II, it follows from (2.26) that

$$\begin{aligned}k\langle\tilde{B}(V^n)-\tilde{B}(v^n),\,e^{n+1}+e^n\rangle&=2k\langle\tilde{B}(V^n)-\tilde{B}(v^n),\,V^n-v^n\rangle\\&=-\frac{2\mu k}{h^2}\left\|\Delta_+(V^n-v^n)\right\|^2\leqslant 0.\end{aligned} \tag{3.14}$$

For the last term of (3.12) we have

$$|k\langle r^n,\,e^{n+1}+e^n\rangle|\leqslant k\,\|r^n\|_\infty\left(\|e^{n+1}\|_1+\|e^n\|_1\right)\leqslant Ck\,\|r^n\|_\infty\Lambda, \tag{3.15}$$

where $\Lambda=\max_{0\leqslant n\leqslant N}\|e^n\|$, and we have made use of the fact that $\|W\|_1\leqslant\sqrt{Jh}\,\|W\|=\|W\|$. If we use the estimates (3.13)–(3.15) to (3.12) and sum over n (and note that $\|e^0\|=0$), then we obtain, for both Problems I and II, that

$$\begin{aligned}\|e^{n+1}\|^2&\leqslant Ck\sum_{m=0}^{n}\left(\|e^{m+1}\|^2+\|e^m\|^2\right)+C\Lambda\sum_{m=0}^{n}k\,\|r^m\|_\infty\\&\qquad+\frac{-k}{2h}\sum_{j=1}^{J-1}\sum_{m=0}^{n}\left(\sum_{p=0}^{m}\beta_p\Delta_+\left(e_j^{m+1-p}+e_j^{m-p}\right)\right)\Delta_+\left(e_j^{n+1}+e_j^n\right),\end{aligned} \tag{3.16}$$

for $0\leqslant n\leqslant N-1$. It follows from Lemmas 1 and 4 that the last term of (3.16) is nonpositive. Moreover, it follows from (3.11) that

$$\begin{aligned}\sum_{m=0}^{n}k\,\|r^m\|_\infty&\leqslant C\sum_{m=0}^{n}\left(k^3t_{m+1}^{-3/2}+k^{5/2}+kh^2\right)\\&\leqslant Ck^{3/2}\sum_{m=0}^{n}(m+1)^{-3/2}+Ck^{3/2}+Ch^2+\mathrm{O}(k^{3/2}+h^2).\end{aligned} \tag{3.17}$$

Therefore, (3.16) leads to

$$\| e^{n+1} \|^2 \leqslant Ck \sum_{m=0}^{n+1} \| e^m \|^2 + C(k^{3/2} + h^2)\Lambda. \tag{3.18}$$

An application of the discrete Gronwall lemma for (3.18) yields that

$$\| e^{n+1} \|^2 \leqslant C(k^{3/2} + h^2)\Lambda, \quad 0 \leqslant n \leqslant N-1. \tag{3.19}$$

The above inequality implies that $\Lambda^2 \leqslant C(k^{3/2} + h^2)\Lambda$, which is equivalent to $\Lambda = O(k^{3/2} + h^2)$. Hence, we have obtained the following convergence result.

Theorem 1. *Assume that the solution of Problem I (Problem II) satisfies the smoothness requirements stated in the Introduction, and that* $(U^1, \ldots, U^N)$ *are solutions of* (2.19)–(2.21) *with the discretized operator given by* (2.22) ((2.23) *for Problem II). If* β *in* (2.13) *satisfies* (3.9), *then as h and k tend to zero independently*,

$$\max_{1 \leqslant n \leqslant N} \| U^n - u^n \| = O(k^{3/2} + h^2). \tag{3.20}$$

4. Numerical experiment

For numerical verification of the above theorem we consider the following example,

$$u_t = \int_0^t (t-s)^{-1/2} u_{xx}(x, s)\, ds, \tag{4.1}$$

$$u(0, t) = u(1, t) = 0, \quad 0 \leqslant t \leqslant T, \tag{4.2}$$

$$u(x, 0) = \sin(\pi x), \quad 0 \leqslant x \leqslant 1. \tag{4.3}$$

The solution of this problem is [14] $u(x, t) = M(\pi^{5/2} t^{3/2}) \sin(\pi x)$, where M denotes the entire function

$$M(z) = \sum_{n=0}^{\infty} (-1)^n \Gamma\left(\tfrac{3}{2}n + 1\right)^{-1} z^n. \tag{4.4}$$

It is well known that the central difference approximations yield a convergence order of (at most) 2, so the estimate of Theorem 1 for the space discretization is the best possible. The main purpose of this section is to verify that the error bound for the time discretization is the optimal one. In other words, we shall show that the predicted convergence order $\frac{3}{2}$ in time is the best possible.

For the numerical scheme (2.19)–(2.21) we obtain a linear algebraic system which takes the form

$$MU^{n+1} = F^n, \quad 0 \leqslant n \leqslant N-1, \tag{4.5}$$

Table 1
Errors and convergence rates

N	Error	Rate
5	7.01D−2 (7.89D−2)	
10	2.49D−2 (2.71D−2)	1.49 (1.54)
20	8.66D−3 (9.80D−3)	1.52 (1.47)
40	3.05D−3 (3.52D−3)	1.51 (1.48)

where $F^n \in \mathbb{R}^{J-1}$ is independent of U^{n+1} and $\boldsymbol{M}$ is an $(J-1)\times(J-1)$ matrix given by

$$\boldsymbol{M}=\begin{pmatrix} a & b & 0 & 0 & 0 & \cdots \\ b & a & b & 0 & 0 & \cdots \\ 0 & b & a & b & 0 & \cdots \\ 0 & 0 & b & a & b & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix},$$

where

$$a = 1 + k\beta_0/h^2, \qquad b = -\tfrac{1}{2}k\beta_0/h^2. \tag{4.6}$$

It can be shown that the solution U^{n+1} of (4.5) satisfies

$$U^{n+1} = \boldsymbol{PHPF}^n, \quad n \geqslant 0, \tag{4.7}$$

where $\boldsymbol{P} = (p_{ij})$ is a $(J-1)\times(J-1)$ symmetrical matrix with $p_{ij} = \sqrt{2/J}\sin(ij\pi/J)$ and $\boldsymbol{H}$ is a diagonal matrix which takes the form

$$\boldsymbol{H} = \operatorname{diag}\left(\left(a + 2b\cos\frac{\pi}{J}\right)^{-1}, \left(a + 2b\cos\frac{2\pi}{J}\right)^{-1}, \ldots, \left(a + 2b\cos\frac{(J-1)\pi}{J}\right)^{-1}\right).$$

Equation (4.7) provides an explicit form for the numerical solutions of problem (4.1)–(4.3).

In Table 1 we list the errors ($\max_{1\leqslant n\leqslant N}\|e^n\|$) and computed rates of convergence when uniform stepsizes $h = k = T/N$ are used. In the calculations we have set $\beta = 0.1$ (see (3.9)) and $T = 0.5$. The numerical results reflect a convergence rate $\approx \frac{3}{2}$ in time, which is in good agreement with the theoretical prediction of Theorem 1. Also in Table 1 we list the errors and rates for numerical solutions with $\beta = 0$; the values appear between parentheses. Although the convergence rate in time is seen to be about $\frac{3}{2}$, it is unclear whether or not Theorem 1 can be extended to the case when $\beta = 0$.

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