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Convergence Analysis of Spectral Galerkin Methods for Volterra Type Integral Equations

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Abstract This work is to provide spectral and pseudo-spectral Jacobi-Galerkin approaches for the second kind Volterra integral equation. The Gauss-Legendre quadrature formula is used to approximate the integral operator and the inner product based on the Jacobi weight is implemented in the weak formulation in the numerical implementation. For some spectral and pseudo-spectral Jacobi-Galerkin methods, a rigorous error analysis in both the infinity and weighted norms is given provided that both the kernel function and the source function are sufficiently smooth. Numerical experiments validate the theoretical prediction.

Keywords The second kind Volterra integral equations · Spectral Galerkin · Pseudo-spectral Galerkin · Spectral convergence

1 Introduction

This paper is concerned with the second kind Volterra integral equation

$$u(x)+\int_{-1}^{x} k(x,s)u(s)\mathrm{d}s = g(x), \quad x\in I=[-1,1], \tag{1.1}$$

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where the kernel function $k(x,s)$ and the source function $g(x)$ are given smooth functions, $u(x)$ is the unknown function. Actually any second kind Volterra integral equation can be transformed into (1.1) by a simple linear transformation [11]. As a result, our approach can be generalized to the second kind Volterra integral equation defined in any interval with a smooth kernel. We will consider the case that the solutions of (1.1) are sufficiently smooth. Consequently it is natural to implement very high-order numerical methods such as spectral methods for the solutions of (1.1). It is known that there are many numerical approaches for solving (1.1), such as collocation methods, product integration methods, see, e.g., [1] and references therein. Nevertheless, few works touched the spectral approximations to (1.1). In [5], Chebyshev spectral methods were proposed to solve nonlinear Volterra-Hammerstein integral equations. Then Chebyshev spectral methods were investigated in [6] for the first kind Fredholm integral equations under multiple-precision arithmetic. However, no theoretical results were provided to justify the high accuracy numerically obtained. Recently, Tang and Xu [11] developed a novel spectral Legendre-collocation method to solve (1.1). It seems the first spectral approach where the spectral accuracy is justified both theoretically and numerically. Inspired by the work of [11], Chen and Tang [3] implemented the spectral Jacobi-collocation method to solve the second kind Volterra integral equation with weakly singular kernel $(t-s)^{\alpha}k(t,s)$, where $-\frac{1}{2}<\alpha<0$ and $k(t,s)$ is a smooth function. Then they [4] extended the approach in [3] to the second kind Volterra integral equation with more general weakly singular kernel $(t-s)^{\alpha}k(t,s)$, where $-1<\alpha<0$ and $k(t,s)$ is a smooth function, when the solution of the underlying equation has a weak singularity at $t=0$. The spectral accuracy of the approaches is verified both theoretically and numerically in [3, 4]; see also Chap. 5 of the recent book [10].

The purpose of this work is to provide numerical methods for the second kind Volterra integral equations based on spectral and pseudo-spectral Galerkin methods. For some spectral and pseudo-spectral Jacobi-Galerkin approaches, a rigorous error analysis which theoretically justifies the spectral rate of convergence of our approaches is provided. Although [3, 4] are concerned with more difficult kernels, i.e., the weakly singular kernel containing the factor $(t-s)^{\alpha}$, the main approach used there is the spectral-collocation method which is similar to a finite-difference approach. Consequently, the corresponding error analysis is more tedious as it does not fit in a unified framework. However, with a finite-element type approach, as will be performed in this work, it is natural to put the approximation scheme under the general Jacobi-Galerkin type framework. As demonstrated in the recent book of Shen et al. [10], there is a unified theory with Jacobi polynomials to approximate numerical solutions for differential and integral equations. It is also rather straightforward to derive the pseudo-spectral Jacobi-Galerkin method from the corresponding continuous version. The relevant convergence theories under the unified framework, as will be seen from Sects. 4 and 5, are cleaner and more reasonable than those obtained in [3, 4].

This paper is organized as follows. In Sect. 2, we demonstrate the implementation of the spectral and pseudo-spectral Galerkin approaches for the underlying equation. Some lemmas useful for the convergence analysis will be provided in Sect. 3. The convergence analysis for both spectral and pseudo-spectral Jacobi-Galerkin methods in L^{∞} and $L^2_{\omega^{\alpha,\beta}}$ norm, under some assumptions on the weight function $\omega^{\alpha,\beta}(x)$, will be given in Sect. 4 and Sect. 5, respectively. Numerical experiments are carried out in Sect. 6, which will be used to validate the theoretical results in Sect. 4 and Sect. 5. Some concluding remarks will be given in Sect. 7.

2 Spectral and Pseudo-Spectral Galerkin Methods

By introducing the integral operator S defined by

$$Su(x)=\int_{-1}^{x}k(x,s)u(s)\mathrm{d}s,$$

(1.1) can be reformulated as

$$u(x)+Su(x)=g(x),\quad x\in I=[-1,1]. \tag{2.1}$$

We will adopt the spectral and pseudo-spectral Jacobi-Galerkin methods to solve this underlying problem.

Let us demonstrate the numerical implementation of the spectral Jacobi-Galerkin approach first. Denote P_N a space consisting of polynomials defined on $[-1,1]$ with degree at most N, $\phi_j(x)$ is the j-th Jacobi polynomial corresponding to the weight function $\omega^{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$, with $\alpha,\beta>-1$, $j=0,1,\dots,N$. As a result,

$$P_N=\operatorname{span}\{\phi_0(x),\phi_1(x),\dots,\phi_N(x)\}.$$

Our aim is to find $u_N\in P_N$ such that

$$(u_N,v_N)_{\omega^{\alpha,\beta}}+(Su_N,v_N)_{\omega^{\alpha,\beta}}=(g,v_N)_{\omega^{\alpha,\beta}},\quad \forall v_N\in P_N, \tag{2.2}$$

where $(u,v)_{\omega^{\alpha,\beta}}=\int_{-1}^{1}\omega^{\alpha,\beta}(x)u(x)v(x)\mathrm{d}x$ is the continuous inner product. Set $u_N(x)=\sum_{j=0}^{N}\xi_j\phi_j(x)$. Substituting it into (2.2) and taking $v_N=\phi_i(x)$, we obtain

$$\sum_{j=0}^{N}(\phi_i,\phi_j)_{\omega^{\alpha,\beta}}\xi_j+\sum_{j=0}^{N}(\phi_i,S\phi_j)_{\omega^{\alpha,\beta}}\xi_j=(\phi_i,g)_{\omega^{\alpha,\beta}}, \tag{2.3}$$

which leads to an equation of the matrix form

$$(A+B)\boldsymbol{\xi}=g_N, \tag{2.4}$$

where $\boldsymbol{\xi}=[\xi_0,\xi_1,\dots,\xi_N]^T$, $A(i,j)=(\phi_i,\phi_j)_{\omega^{\alpha,\beta}}$, $B(i,j)=(\phi_i,S\phi_j)_{\omega^{\alpha,\beta}}$, $g_N(i)=(\phi_i,g)_{\omega^{\alpha,\beta}}$.

Now we turn to describe the pseudo-spectral Jacobi-Galerkin method. For this purpose, set $s=s(x,\theta)=\frac{x-1}{2}+\frac{x+1}{2}\theta$, $\theta\in[-1,1]$. It is clear that

$$Su(x)=\int_{-1}^{x}k(x,s)u(s)\mathrm{d}s=\int_{-1}^{1}\widetilde{k}\big(x,s(x,\theta)\big)u\big(s(x,\theta)\big)\mathrm{d}\theta \tag{2.5}$$

with $\widetilde{k}(x,s(x,\theta))=\frac{x+1}{2}k(x,s(x,\theta))$. Using $(N+1)$-point Gauss-Legendre quadrature formula to approximate (2.5) yields

$$Su(x)\approx S_Nu(x):=\sum_{n=0}^{N}\widetilde{k}\big(x,s(x,\theta_n)\big)u\big(s(x,\theta_n)\big)\nu_n, \tag{2.6}$$

where $\{\theta_n\}_{n=0}^N$ are the $(N+1)$-degree Legendre-Gauss points, and $\{\nu_n\}_{n=0}^N$ are the corresponding Legendre weights. On the other hand, instead of the continuous inner product, the discrete inner product will be implemented in (2.2) and (2.3), i.e.,

$$(u,v)_{\omega^{\alpha,\beta}} \approx (u,v)_{\omega^{\alpha,\beta},N} = \sum_{m=0}^{N} u(x_m)v(x_m)\omega_m^{\alpha,\beta}, \tag{2.7}$$

where $\{x_m\}_{m=0}^N$ and $\{\omega_m^{\alpha,\beta}\}_{m=0}^N$ are the $(N+1)$-degree Jacobi-Gauss points and their corresponding Jacobi weights, respectively. As a result,

$$(u,v)_{\omega^{\alpha,\beta}} = (u,v)_{\omega^{\alpha,\beta},N}, \quad \text{if } uv \in P_{2N}.$$

Substitute (2.6) and (2.7) into (2.2). The pseudo-spectral Jacobi-Galerkin method is to find

$$\bar{u}_N(x) = \sum_{j=0}^{N} \bar{\xi}_j \phi_j(x) \in P_N, \tag{2.8}$$

such that

$$(\bar{u}_N, v_N)_{\omega^{\alpha,\beta},N} + (S_N \bar{u}_N, v_N)_{\omega^{\alpha,\beta},N} = (g, v_N)_{\omega^{\alpha,\beta},N}, \quad \forall v_N \in P_N, \tag{2.9}$$

where $\{\bar{\xi}_j\}_{j=0}^N$ are determined by

$$\sum_{j=0}^{N} (\phi_i, \phi_j)_{\omega^{\alpha,\beta},N} \bar{\xi}_j + \sum_{j=0}^{N} (\phi_i, S_N \phi_j)_{\omega^{\alpha,\beta},N} \bar{\xi}_j = (\phi_i, g)_{\omega^{\alpha,\beta},N}. \tag{2.10}$$

Denoting $\bar{\boldsymbol{\xi}} = [\bar{\xi}_0, \bar{\xi}_1, \dots, \bar{\xi}_N]^T$, (2.10) yields an equation of the matrix form

$$(\bar{A} + \bar{B})\bar{\boldsymbol{\xi}} = \bar{g}_N, \tag{2.11}$$

where $\bar{A}(i,j) = (\phi_i, \phi_j)_{\omega^{\alpha,\beta},N}$, $\bar{B}(i,j) = (\phi_i, S_N\phi_j)_{\omega^{\alpha,\beta},N}$, $\bar{g}_N(i) = (\phi_i, g)_{\omega^{\alpha,\beta},N}$.

It is worthwhile to point out that the known recurrence formula for Jacobi polynomials can be used to calculate $\phi_i(x)$ in the two approaches mentioned above.

3 Some Useful Lemmas

In this section, we will give some useful lemmas which play a significant role in the convergence analysis later. First we define the projection operator $\Pi_N^{\alpha,\beta} : L^2_{\omega^{\alpha,\beta}} \to P_N$ which satisfies

$$\left(\Pi_N^{\alpha,\beta} u, v_N\right)_{\omega^{\alpha,\beta}} = (u, v_N)_{\omega^{\alpha,\beta}}, \quad \forall u \in L^2_{\omega^{\alpha,\beta}}, v_N \in P_N. \tag{3.1}$$

Secondly, $I_N^{\alpha,\beta}$ denotes the interpolation operator of u based on $(N+1)$-degree Jacobi Gauss points corresponding to the weight function $\omega^{\alpha,\beta}(x)$. Moreover, define a weighted space as

$$L^2_{\omega^{\alpha,\beta}}(I) = \{v : v \text{is measurable and } \|v\|_{\omega^{\alpha,\beta}} < \infty\},$$

where

$$\|v\|_{\omega^{\alpha,\beta}} = \left(\int_I \omega^{\alpha,\beta}(x)v^2(x)\mathrm{d}x\right)^{\frac{1}{2}}.$$

Further, define

$$H^m_{\omega^{\alpha,\beta}}(I) = \left\{v : D^k v \in L^2_{\omega^{\alpha,\beta}}(I), 0 \le k \le m\right\},$$

equipped with the norm

$$\|v\|_{H^m_{\omega^{\alpha,\beta}}(I)} = \left(\sum_{k=0}^{m} \|D^k v\|^2_{\omega^{\alpha,\beta}}\right)^{\frac{1}{2}}$$

with $D^k v = \frac{\mathrm{d}^k v}{\mathrm{d}x^k}$. When $\omega^{\alpha,\beta}(x) = 1$, $L^2_{\omega^{\alpha,\beta}}(I)$, $H^m_{\omega^{\alpha,\beta}}(I)$ and $\|\cdot\|_{\omega^{\alpha,\beta}}$ are denoted simply by $L^2(I)$, $H^m(I)$ and $\|\cdot\|$, respectively.

In bounding the above approximation error, only some of the L^2-norms appearing on the right-hand side of above norm enter into play. Thus, it is convenient to introduce the seminorms

$$|v|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)} = \left(\sum_{k=\min(m,N+1)}^{m} \|D^k v\|^2_{\omega^{\alpha,\beta}}\right)^{\frac{1}{2}}.$$

Lemma 3.1 *Suppose that $v \in H^m_{\omega^{\alpha,\beta}}(I)$ and $m \ge 1$.*

(i) *If $\alpha, \beta > -1$, then*

$$\|v - \Pi_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}} \le CN^{-m}|v|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}, \tag{3.2}$$

$$\|v - \Pi_N^{\alpha,\beta} v\|_{H^l_{\omega^{\alpha,\beta}}(I)} \le CN^{2l-\frac{1}{2}-m}|v|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}, \tag{3.3}$$

for any l such that $1 \le l \le m$.

(ii) *If $-1 < \alpha, \beta \le 0$, then*

$$\|v - \Pi_N^{\alpha,\beta} v\|_{L^\infty(I)} \le CN^{\frac{3}{4}-m}|v|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}. \tag{3.4}$$

Proof The inequalities in (i) can be found in [2, 9]. We only prove (ii). It is straightforward to have, for $-1 < \alpha, \beta \le 0$,

$$\|w\| \le C\|w\|_{\omega^{\alpha,\beta}}, \qquad \|w\|_{H^1(I)} \le C\|w\|_{H^1_{\omega^{\alpha,\beta}}(I)}. \tag{3.5}$$

Consequently, using (3.5) and the Sobolev inequality ([2], p. 490)

$$\|w\|_{L^\infty(I)} \le C\|w\|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}_{H^1(I)} \tag{3.6}$$

gives, for $-1 < \alpha, \beta \le 0$,

$$\|w\|_{L^\infty(I)} \le C\|w\|^{\frac{1}{2}}_{\omega^{\alpha,\beta}}\|w\|^{\frac{1}{2}}_{H^1_{\omega^{\alpha,\beta}}(I)}. \tag{3.7}$$

This result, together with the estimates in (i), yields (3.4). □

Lemma 3.2 *Suppose that $v \in H^m_{\omega^{\alpha,\beta}}(I)$ and $m \geq 1$.*

(i) *If $\alpha, \beta > -1$, then*

$$\|v - I_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}} \leq C N^{-m} |v|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}. \tag{3.8}$$

(ii) *If $\omega^{\alpha,\beta}$ is the Legendre weight, i.e., $\alpha = \beta = 0$, then*

$$\|v - I_N^{\alpha,\beta} v\|_{H^l(I)} \leq C N^{2l-\frac{1}{2}-m} |v|_{H^{m;N}(I)}, \tag{3.9}$$

$$\|v - I_N^{\alpha,\beta} v\|_{L^\infty(I)} \leq C N^{\frac{3}{4}-m} |v|_{H^{m;N}(I)}. \tag{3.10}$$

If $\omega^{\alpha,\beta}$ is the Chebyshev weight, i.e., $\alpha = \beta = -\frac{1}{2}$, then

$$\|v - I_N^{\alpha,\beta} v\|_{L^\infty(I)} \leq C N^{\frac{1}{2}-m} |v|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}. \tag{3.11}$$

Proof The conclusion in (i) is a classical one; see, e.g., [10]. The first estimate in (ii) can be found in ([2], p. 289), which also leads to the second estimate in (ii) by using (i) and the Sobolev inequality (3.6). The estimate in (3.11) can be seen in ([2], p. 297). □

Lemma 3.3 *Suppose that $v \in H^m_{\omega^{\alpha,\beta}}(I)$ with $\alpha, \beta > -1$, $m \geq 1$ and $\phi \in P_N$. Then we have*

$$|(v, \phi)_{\omega^{\alpha,\beta}} - (v, \phi)_{\omega^{\alpha,\beta},N}| \leq C N^{-m} |v|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)} \|\phi\|_{\omega^{\alpha,\beta}}. \tag{3.12}$$

Proof Note that the discrete inner product is based on the $(N+1)$-degree Jacobi-Gauss points corresponding to the weight function $\omega^{\alpha,\beta}(x)$. We have

$$(v, \phi)_{\omega^{\alpha,\beta},N} = \left(I_N^{\alpha,\beta} v, \phi\right)_{\omega^{\alpha,\beta}}. \tag{3.13}$$

Consequently, we have

$$|(v, \phi)_{\omega^{\alpha,\beta}} - (v, \phi)_{\omega^{\alpha,\beta},N}| = \left|\left(v - I_N^{\alpha,\beta} v, \phi\right)_{\omega^{\alpha,\beta}}\right| \leq \|v - I_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}} \|\phi\|_{\omega^{\alpha,\beta}}, \tag{3.14}$$

which, together with Lemma 3.2, leads to the desired estimate (3.12). □

Lemma 3.4 *For each bounded function $v(x)$, there exists a constant C, independent of v, such that*

$$\sup_N \|I_N^{\alpha,\beta} v\|_{\omega^{\alpha,\beta}} \leq C \|v\|_\infty, \tag{3.15}$$

where $I_N^{\alpha,\beta} v = \sum_{j=0}^N v(x_j) h_j(x)$ is the interpolation of v, with $h_j(x)$, $j = 0, 1, \ldots, N$, the Lagrange interpolation basis functions based on $(N+1)$-degree Jacobi-Gauss points corresponding to the weight function $\omega^{\alpha,\beta}(x)$ with $\alpha, \beta > -1$.

Proof As the $(N+1)$-point Jacobi-Gauss quadrature formulas are accurate for the polynomials with degree no more than $2N$, direct calculation shows that

$$\|I_N^{\alpha,\beta}v\|^2_{\omega^{\alpha,\beta}} = \int_{-1}^{1}\omega^{\alpha,\beta}(x)\big(I_N^{\alpha,\beta}v\big)^2\mathrm{d}x = \sum_{j=0}^{N}v^2(x_j)\omega_j^{\alpha,\beta}$$

$$\le \|v\|^2_\infty\sum_{j=0}^{N}\omega_j^{\alpha,\beta} = \gamma_0\|v\|^2_\infty, \tag{3.16}$$

where $\gamma_0 = (\phi_0,\phi_0)_{\omega^{\alpha,\beta}}$. As a consequence,

$$\sup_N \|I_N^{\alpha,\beta}v\|_{\omega^{\alpha,\beta}} \le C\|v\|_\infty,$$

with $C=\sqrt{\gamma_0}$. □

4 Convergence Analysis for Spectral Jacobi-Galerkin Method

According to (2.2) and the definition of the projection operator $\Pi_N^{\alpha,\beta}$, the spectral Jacobi-Galerkin solution u_N satisfies

$$u_N + \Pi_N^{\alpha,\beta}Su_N = \Pi_N^{\alpha,\beta}g. \tag{4.1}$$

Theorem 4.1 *Suppose that u_N is the spectral Jacobi-Galerkin solution determined by* (2.2) *with α and β satisfying one of the following assumptions*, *i.e.*, (i) $-1<\alpha,\beta<1$; (ii) $\alpha=0,\beta>-1$; (iii) $\alpha>-1,\beta=0$; (iv) $\alpha>-1,-1<\beta\le 0$. *If the solution u of* (2.1) *satisfies $u\in H^{m;N}_{\omega^{\alpha,\beta}}(I)$, then we have the following error estimate*

$$\|u-u_N\|_{\omega^{\alpha,\beta}} \le CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}.$$

Proof When $g=0$, (4.1) can be written as

$$u_N + \Pi_N^{\alpha,\beta}Su_N = 0.$$

In terms of the fact that

$$u_N + \Pi_N^{\alpha,\beta}Su_N = u_N + Su_N - \big(Su_N - \Pi_N^{\alpha,\beta}Su_N\big),$$

it is clear that

$$u_N = -\int_{-1}^{x}k(x,s)u_N(s)\mathrm{d}s + \big(Su_N - \Pi_N^{\alpha,\beta}Su_N\big),$$

which yields

$$|u_N| \le M\int_{-1}^{x}|u_N(s)|\mathrm{d}s + |J|,$$

with $J = Su_N - \Pi_N^{\alpha,\beta}Su_N$. This, together with the standard Gronwall inequality (see, e.g., [3, 4]), gives

$$\|u_N\|_{\omega^{\alpha,\beta}} \le C\|J\|_{\omega^{\alpha,\beta}}. \tag{4.2}$$

In virtue of Lemma 3.1,

$$\begin{aligned}\|J\|_{\omega^{\alpha,\beta}} &\le CN^{-1}\left\|k(x,x)u_N(x)+\int_{-1}^{x}k_x(x,s)u_N(s)\mathrm{d}s\right\|_{\omega^{\alpha,\beta}}\\&\le CN^{-1}\left(\|u_N\|_{\omega^{\alpha,\beta}}+\left\|\int_{-1}^{x}|u_N(s)|\mathrm{d}s\right\|_{\omega^{\alpha,\beta}}\right)\\&\le CN^{-1}\|u_N\|_{\omega^{\alpha,\beta}},\end{aligned}\tag{4.3}$$

in which, we have implemented the fact that

$$\left\|\int_{-1}^{x}u(s)\mathrm{d}s\right\|^2_{\omega^{\alpha,\beta}}\le C\|u\|^2_{\omega^{\alpha,\beta}}\tag{4.4}$$

when α and β satisfy one of the assumptions above. Actually, if α, β satisfy one of (i)–(iii), (4.4) holds according to ([7], p. 239). On the other hand, if α, β satisfy (iv), then

$$\begin{aligned}\left\|\int_{-1}^{x}u(s)\mathrm{d}s\right\|^2_{\omega^{\alpha,\beta}}&=\int_{-1}^{1}\omega^{\alpha,\beta}(x)\left(\int_{-1}^{x}u(s)\mathrm{d}s\right)^2\mathrm{d}x\\&\le C\int_{-1}^{1}\omega^{\alpha,\beta}(x)\int_{-1}^{x}u^2(s)\mathrm{d}s\mathrm{d}x\\&=C\int_{-1}^{1}u^2(s)\int_{s}^{1}(1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x\mathrm{d}s\\&\le C\int_{-1}^{1}u^2(s)(1+s)^{\beta}\int_{s}^{1}(1-x)^{\alpha}\mathrm{d}x\mathrm{d}s\\&\le C\int_{-1}^{1}(1-s)^{\alpha}(1+s)^{\beta}u^2(s)\mathrm{d}s\\&=C\|u\|^2_{\omega^{\alpha,\beta}}.\end{aligned}$$

The combination of (4.2) and (4.3) leads to

$$\|u_N\|_{\omega^{\alpha,\beta}}\le CN^{-1}\|u_N\|_{\omega^{\alpha,\beta}},$$

which implies, when N is large enough, $u_N=0$. Hence, the spectral Galerkin solution u_N is existent and unique as P_N is finite-dimensional.

Subtracting (4.1) from (2.1), yields

$$u-u_N+Su-\Pi_N^{\alpha,\beta}Su_N=g-\Pi_N^{\alpha,\beta}g.\tag{4.5}$$

Set $e=u-u_N$. Direct computation shows that

$$\begin{aligned}&Su-\Pi_N^{\alpha,\beta}Su_N\\&\quad=Su-\Pi_N^{\alpha,\beta}Su+\Pi_N^{\alpha,\beta}S(u-u_N)\\&\quad=Su-\Pi_N^{\alpha,\beta}Su+S(u-u_N)-\left[S(u-u_N)-\Pi_N^{\alpha,\beta}S(u-u_N)\right]\end{aligned}$$

$$
\begin{aligned}
&= (g-u) - \Pi_N^{\alpha,\beta}(g-u) + S(u-u_N) - \left[S(u-u_N) - \Pi_N^{\alpha,\beta} S(u-u_N)\right] \\
&= g - \Pi_N^{\alpha,\beta} g - u + \Pi_N^{\alpha,\beta} u + Se - \left(Se - \Pi_N^{\alpha,\beta} Se\right). \qquad (4.6)
\end{aligned}
$$

The insertion of (4.6) into (4.5) yields

$$
e(x) = -\int_{-1}^{x} k(x,s)e(s)\mathrm{d}s + u - \Pi_N^{\alpha,\beta} u + \left(Se - \Pi_N^{\alpha,\beta} Se\right),
$$

which implies that

$$
|e(x)| \le M \int_{-1}^{x} |e(s)|\mathrm{d}s + |J_1| + |J_2|, \tag{4.7}
$$

where

$$
J_1 = u - \Pi_N^{\alpha,\beta} u, \qquad J_2 = Se - \Pi_N^{\alpha,\beta} Se.
$$

By (4.7) and the standard Gronwall inequality (see, e.g., [10]), we have

$$
\|e\|_{\omega^{\alpha,\beta}} \le C(\|J_1\|_{\omega^{\alpha,\beta}} + \|J_2\|_{\omega^{\alpha,\beta}}). \tag{4.8}
$$

By Lemma 3.1,

$$
\|J_1\|_{\omega^{\alpha,\beta}} \le C N^{-m} |u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}, \tag{4.9}
$$

$$
\begin{aligned}
\|J_2\|_{\omega^{\alpha,\beta}} &\le C N^{-1} \left\| k(x,x)e(x) + \int_{-1}^{x} k_x(x,s)e(s)\mathrm{d}s \right\|_{\omega^{\alpha,\beta}} \\
&\le C N^{-1} \left(\|e\|_{\omega^{\alpha,\beta}} + \left\| \int_{-1}^{x} |e(s)|\mathrm{d}s \right\|_{\omega^{\alpha,\beta}} \right) \\
&\le C N^{-1} \|e\|_{\omega^{\alpha,\beta}}, \qquad (4.10)
\end{aligned}
$$

in which (4.4) is used under the assumptions on α and β above. Combing (4.8), (4.9), and (4.10), we obtain, when N is large enough,

$$
\|u - u_N\|_{\omega^{\alpha,\beta}} = \|e\|_{\omega^{\alpha,\beta}} \le C N^{-m} |u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}. \qquad \square
$$

Now we investigate the L^∞-error estimate.

Theorem 4.2 *Suppose that u_N is the spectral Jacobi-Galerkin solution satisfying* (2.2) *with $-1 < \alpha, \beta \le 0$. If the solution u of* (2.1) *satisfies $u \in H^{m;N}_{\omega^{\alpha,\beta}}(I) \cap L^\infty(I)$, then we have the following error estimate*

$$
\|u - u_N\|_{L^\infty(I)} \le C N^{\frac{3}{4}-m} |u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}.
$$

Proof It follows from (4.7) and the standard Gronwall inequality (see, e.g., [10]) that

$$
\|e\|_{L^\infty(I)} \le C(\|J_1\|_{L^\infty(I)} + \|J_2\|_{L^\infty(I)}). \tag{4.11}
$$

As $-1 < \alpha, \beta \le 0$, by Lemma 3.1, we have

$$
\|J_1\|_{L^\infty(I)} \le C N^{\frac{3}{4}-m} |u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}, \tag{4.12}
$$

$$\|J_2\|_{L^\infty(I)} \le CN^{-\frac{1}{4}} \left\| k(x,x)e(x) + \int_{-1}^{x} k_x(x,s)e(s)\mathrm{d}s \right\|_{\omega^{\alpha,\beta}} \le CN^{-\frac{1}{4}} \|e\|_{L^\infty(I)}. \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.11), when N is large enough, we obtain

$$\|u - u_N\|_{L^\infty(I)} = \|e\|_{L^\infty(I)} \le CN^{\frac{3}{4}-m} |u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}. \qquad \square$$

5 Convergence for Pseudo-spectral Jacobi-Galerkin Method

As $I_N^{\alpha,\beta}$ is the interpolation operator which is based on the $(N+1)$-degree Jacobi-Gauss points, in terms of (2.9), the pseudo-spectral Galerkin solution $\bar{u}_N$ satisfies

$$(\bar{u}_N, v_N)_{\omega^{\alpha,\beta}} + \left(I_N^{\alpha,\beta} S_N \bar{u}_N, v_N\right)_{\omega^{\alpha,\beta}} = \left(I_N^{\alpha,\beta} g, v_N\right)_{\omega^{\alpha,\beta}}, \tag{5.1}$$

where

$$S_N \bar{u}_N = S\bar{u}_N - (S\bar{u}_N - S_N \bar{u}_N) = S\bar{u}_N - Q(x), \tag{5.2}$$

with

$$\begin{aligned} Q(x) &= S\bar{u}_N - S_N \bar{u}_N \\ &= \int_{-1}^{1} \widetilde{k}\big(x, s(x,\theta)\big) \bar{u}_N\big(s(x,\theta)\big)\mathrm{d}\theta - \sum_{j=0}^{N} \widetilde{k}\big(x, s(x,\theta_j)\big) \bar{u}_N\big(s(x,\theta_j)\big) \nu_j \\ &= \Big(\widetilde{k}\big(x, s(x,\cdot)\big), \bar{u}_N\big(s(x,\cdot)\big)\Big) - \Big(\widetilde{k}\big(x, s(x,\cdot)\big), \bar{u}_N\big(s(x,\cdot)\big)\Big)_N, \end{aligned} \tag{5.3}$$

in which $(\cdot,\cdot)$ represents the continuous inner product with respect to θ, and $(\cdot,\cdot)_N$ is the corresponding discrete inner product defined by the Gauss-Legendre quadrature formula. The combination of (5.1) and (5.2), yields

$$(\bar{u}_N, v_N)_{\omega^{\alpha,\beta}} + \left(I_N^{\alpha,\beta} S\bar{u}_N - I_N^{\alpha,\beta} Q(x), v_N\right)_{\omega^{\alpha,\beta}} = \left(I_N^{\alpha,\beta} g, v_N\right)_{\omega^{\alpha,\beta}},$$

which gives rise to

$$\bar{u}_N + I_N^{\alpha,\beta} S\bar{u}_N - I_N^{\alpha,\beta} Q(x) = I_N^{\alpha,\beta} g. \tag{5.4}$$

By the discussion above, (2.9), (5.1) and (5.4) are equivalent.

We first consider an auxiliary problem, i.e., we want to find $\hat{u}_N \in P_N$, such that

$$(\hat{u}_N, v_N)_{\omega^{\alpha,\beta},N} + (S\hat{u}_N, v_N)_{\omega^{\alpha,\beta},N} = (g, v_N)_{\omega^{\alpha,\beta},N}, \quad \forall v_N \in P_N, \tag{5.5}$$

where S is the integral operator defined in Sect. 2, and $(\cdot,\cdot)_{\omega^{\alpha,\beta},N}$ is still the discrete inner product based on the $(N+1)$-degree Jacobi-Gauss points. In terms of the definition of $I_N^{\alpha,\beta}$, (5.5) can be written as

$$(\hat{u}_N, v_N)_{\omega^{\alpha,\beta}} + \left(I_N^{\alpha,\beta} S\hat{u}_N, v_N\right)_{\omega^{\alpha,\beta}} = \left(I_N^{\alpha,\beta} g, v_N\right)_{\omega^{\alpha,\beta}}, \quad \forall v_N \in P_N, \tag{5.6}$$

which is equivalent to

$$\hat{u}_N + I_N^{\alpha,\beta} S\hat{u}_N = I_N^{\alpha,\beta} g. \tag{5.7}$$

Denote

$$L_N(\alpha,\beta) = \max_{x\in I} \sum_{j=0}^{N} \left|h_j(x)\right|,$$

the well-known Lebesgue constant corresponding to the Jacobi polynomial $\phi_N(x) = P_N^{(\alpha,\beta)}(x)$, where $h_j(x)$, $j = 0, 1, \ldots, N$ are the Lagrange interpolation basis functions associated with the $(N+1)$-degree Jacobi-Gauss points.

Lemma 5.1 *Suppose $\hat{u}_N$ is determined by (5.5).*

(i) *If α and β satisfy one of the following assumptions, i.e., $-1 < \alpha, \beta < 1$, or $\alpha = 0$, $\beta > -1$, or $\alpha > -1$, $\beta = 0$, or $\alpha > -1$, $-1 < \beta \le 0$, we have*

$$\|u - \hat{u}_N\|_{\omega^{\alpha,\beta}} \le CN^{-m} |u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}. \tag{5.8}$$

(ii) *If $\omega^{\alpha,\beta}(x)$ is the Legendre weight, i.e., $\alpha = \beta = 0$, then we have*

$$\|u - \hat{u}_N\|_{L^\infty(I)} \le CN^{\frac{3}{4}-m} |u|_{H^{m;N}(I)}. \tag{5.9}$$

If $\omega^{\alpha,\beta}(x)$ is the Chebyshev weight, i.e., $\alpha = \beta = -\frac{1}{2}$, then we have

$$\|u - \hat{u}_N\|_{L^\infty(I)} \le CN^{\frac{1}{2}-m} |u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}. \tag{5.10}$$

(iii) *If $\omega^{\alpha,\beta}(x)$ is the Jacobi weight with $-1 < \alpha, \beta < -\frac{1}{2}$, then*

$$\|u - \hat{u}_N\|_{L^\infty(I)} \le CN^{\frac{1}{2}-m} \log N |u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}. \tag{5.11}$$

If $\omega^{\alpha,\beta}(x)$ is the Jacobi weight with $-\frac{1}{2} < \alpha, \beta < 0$ and set $\gamma = \max(\alpha, \beta)$, then

$$\|u - \hat{u}_N\|_{L^\infty(I)} \le CN^{\gamma+1-m} |u|_{H^{m;N}_{\omega^{-1/2,-1/2}}(I)}. \tag{5.12}$$

Proof (i) The existence and uniqueness of $\hat{u}_N$ and the $L^2_{\omega^{\alpha,\beta}}$ error estimate of $u - \hat{u}_N$ can be established in a similar way as those for the spectral Jacobi Galerkin solution u_N in the proof of Theorem 4.1, with $\Pi_N^{\alpha,\beta}$ replaced by $I_N^{\alpha,\beta}$. For simplicity, we omit it here.

(ii) Subtracting (5.7) from (2.1) yields

$$u - \hat{u}_N + Su - I_N^{\alpha,\beta} S\hat{u}_N = g - I_N^{\alpha,\beta} g. \tag{5.13}$$

Set $\epsilon = u - \hat{u}_N$. Direct computation shows that

$$\begin{aligned}
&Su - I_N^{\alpha,\beta} S\hat{u}_N \\
&\quad = Su - I_N^{\alpha,\beta} Su + I_N^{\alpha,\beta} S(u - \hat{u}_N) \\
&\quad = Su - I_N^{\alpha,\beta} Su + S(u - \hat{u}_N) - \left[S(u - \hat{u}_N) - I_N^{\alpha,\beta} S(u - \hat{u}_N)\right] \\
&\quad = (g - u) - I_N^{\alpha,\beta}(g - u) + S(u - \hat{u}_N) - \left[S(u - \hat{u}_N) - I_N^{\alpha,\beta} S(u - \hat{u}_N)\right] \\
&\quad = g - I_N^{\alpha,\beta} g - u + I_N^{\alpha,\beta} u + S\epsilon - \left(S\epsilon - I_N^{\alpha,\beta} S\epsilon\right).
\end{aligned} \tag{5.14}$$

The insertion of (5.14) into (5.13) yields

$$\epsilon(x) = -\int_{-1}^{x} k(x,s)\epsilon(s)\mathrm{d}s + u - I_N^{\alpha,\beta}u + \big(S\epsilon - I_N^{\alpha,\beta}S\epsilon\big),$$

which implies that

$$|\epsilon(x)| \le M\int_{-1}^{x} |\epsilon(s)|\mathrm{d}s + |J_3| + |J_4|, \tag{5.15}$$

where

$$J_3 = u - I_N^{\alpha,\beta}u, \qquad J_4 = S\epsilon - I_N^{\alpha,\beta}S\epsilon.$$

Using the standard Gronwall inequality (see, e.g., [3, 4]) gives

$$\|\epsilon\|_{L^\infty(I)} \le C(\|J_3\|_{L^\infty(I)} + \|J_4\|_{L^\infty(I)}). \tag{5.16}$$

Actually, by Lemma 3.2,

$$\|J_3\|_{L^\infty(I)} \le CN^{\theta-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)}, \tag{5.17}$$

where $\theta = \frac{3}{4}$ when $\omega^{\alpha,\beta}(x)$ is the Legendre weight, and $\theta = \frac{1}{2}$ when $\omega^{\alpha,\beta}(x)$ is the Chebyshev weight. On the other hand, using Lemma 3.2 again gives

$$\begin{aligned}\|J_4\|_{L^\infty(I)} &\le CN^{-\eta}\left\|k(x,x)\epsilon(x) + \int_{-1}^{x} k_x(x,s)\epsilon(s)\mathrm{d}s\right\|_{\omega^{\alpha,\beta}}\\ &\le CN^{-\eta}\|\epsilon\|_{L^\infty(I)},\end{aligned} \tag{5.18}$$

where $\eta = \frac{1}{4}$ when $\omega^{\alpha,\beta}(x)$ is the Legendre weight, and $\eta = \frac{1}{2}$ when $\omega^{\alpha,\beta}(x)$ is the Chebyshev weight. Substituting (5.17) and (5.18) into (5.16) gives, as N is sufficiently large,

$$\|\epsilon\|_{L^\infty(I)} \le CN^{\theta-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)},$$

where $\theta = \frac{3}{4}$ for the Legendre weight, and $\theta = \frac{1}{2}$ for the Chebyshev weight.

(iii) According to [8] and Lemma 3.2,

$$\begin{aligned}\|J_3\|_{L^\infty(I)} &\le \big(1 + L_N(\alpha,\beta)\big)\|u - u_N^*\|_{L^\infty(I)}\\ &\le CL_N(\alpha,\beta)\big\|u - I_N^{-1/2,-1/2}u\big\|_{L^\infty(I)}\\ &\le CL_N(\alpha,\beta)N^{\frac{1}{2}-m}|u|_{H^{m;N}_{\omega^{-1/2,-1/2}}},\end{aligned} \tag{5.19}$$

where u_N^* is the best approximation of $u(x)$ among the polynomials of degree no more than N. In a similar way,

$$\begin{aligned}\|J_4\|_{L^\infty(I)} &\le CL_N(\alpha,\beta)\|S\epsilon - I_N^{-1/2,-1/2}S\epsilon\|_{L^\infty(I)}\\ &\le CL_N(\alpha,\beta)N^{-\frac{1}{2}}\left\|k(x,x)\epsilon(x) + \int_{-1}^{x} k_x(x,s)\epsilon(s)\mathrm{d}s\right\|_{\omega^{-1/2,-1/2}}\\ &\le CL_N(\alpha,\beta)N^{-\frac{1}{2}}\|\epsilon\|_{L^\infty(I)}.\end{aligned} \tag{5.20}$$

According to ([12], p. 335), if $-1 < \alpha, \beta \le -\frac{1}{2}$, $L_N(\alpha,\beta) = O(\log N)$; otherwise $L_N(\alpha,\beta) = N^{\gamma+\frac{1}{2}}$ with $\gamma = \max(\alpha,\beta)$. As a consequence, the combination of (5.11), (5.19) and (5.20) implies that, if $-1 < \alpha, \beta < -\frac{1}{2}$,

$$\begin{aligned}\|\epsilon\|_{L^\infty(I)} &\le CN^{\frac{1}{2}-m}\log N|u|_{H^{m;N}_{\omega^{-1/2,-1/2}}}\\ &\le CN^{\frac{1}{2}-m}\log N|u|_{H^{m;N}_{\omega^{\alpha,\beta}}},\end{aligned} \tag{5.21}$$

which is the desired (5.11); similarly, if $-\frac{1}{2} < \alpha, \beta < 0$, then we have (5.12). In obtaining the last step of (5.21), we used the fact that the conditions $-1 < \alpha, \beta < -\frac{1}{2}$ imply $\frac{1}{2} > -\frac{1}{2} - \alpha > 0$ and $\frac{1}{2} > -\frac{1}{2} - \beta > 0$; consequently,

$$\begin{aligned}|u|_{H^{m;N}_{\omega^{-1/2,-1/2}}} &= \sum_{k=\min(m,N+1)}^{m}\int_{-1}^{1}\left(u^{(k)}\right)^2(1-t)^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}\mathrm{d}t\\ &= \sum_{k=\min(m,N+1)}^{m}\int_{-1}^{1}\left(u^{(k)}\right)^2\omega^{\alpha,\beta}(1-t)^{-\frac{1}{2}-\alpha}(1+t)^{-\frac{1}{2}-\beta}\mathrm{d}t\\ &\le 2\sum_{k=\min(m,N+1)}^{m}\int_{-1}^{1}\left(u^{(k)}\right)^2\omega^{\alpha,\beta}\mathrm{d}t = 2|u|_{H^{m;N}_{\omega^{\alpha,\beta}}}.\end{aligned} \tag{5.22}$$

This completes the proof of the lemma. □

Now subtracting (5.4) from (5.7) leads to

$$\hat{u}_N - \bar{u}_N + I_N^{\alpha,\beta}S(\hat{u}_N - \bar{u}_N) + I_N^{\alpha,\beta}Q(x) = 0,$$

which can be simplified as, by setting $E = \hat{u}_N - \bar{u}_N$,

$$E + I_N^{\alpha,\beta}SE + I_N^{\alpha,\beta}Q(x) = 0. \tag{5.23}$$

Theorem 5.1 *Suppose that the solution of* (2.1) *is sufficiently smooth. For the pseudo-spectral Jacobi-Galerkin solution $\bar{u}_N$, such that* (2.9) *holds,*

(i) *if $-1 < \alpha, \beta \le 0$, we have*

$$\|u - \bar{u}_N\|_{\omega^{\alpha,\beta}} \le CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)} + CM_mN^{-m}\|u\|; \tag{5.24}$$

(ii) *if $0 < \alpha = \beta < 1$, we have*

$$\|u - \bar{u}_N\|_{\omega^{\alpha,\beta}} \le CN^{-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)} + CM_mN^{-m+\alpha}\|u\|_{\omega^{\alpha,\beta}}, \tag{5.25}$$

where

$$M_m = \max_{x\in I}\sqrt{\frac{1+x}{2}}\left|k\left(x, s(x,\cdot)\right)\right|_{H^{m;N}(I)}. \tag{5.26}$$

Proof We first prove the existence and uniqueness of the pseudo-spectral Jacobi-Galerkin solution $\bar{u}_N$. As the dimension of P_N is finite and (2.9) and (5.4) are equivalent, we only need to prove that the solution of (5.4) is $\bar{u}_N = 0$ when $g = 0$. For this purpose, we consider the equation

$$\bar{u}_N + I_N^{\alpha,\beta} S\bar{u}_N - I_N^{\alpha,\beta} Q(x) = 0. \tag{5.27}$$

Obviously (5.27) can be written as

$$\bar{u}_N + S\bar{u}_N = S\bar{u}_N - I_N^{\alpha,\beta} S\bar{u}_N + I_N^{\alpha,\beta} Q(x) = J_5 + J_6,$$

i.e.,

$$\bar{u}_N = -\int_{-1}^{x} k(x,s)\bar{u}_N(s)\mathrm{d}s + J_5 + J_6,$$

which yields

$$|\bar{u}_N| \le M\int_{-1}^{x} |\bar{u}_N(s)|\mathrm{d}s + |J_5| + |J_6|,$$

with $J_5 = S\bar{u}_N - I_N^{\alpha,\beta} S\bar{u}_N$, $J_6 = I_N^{\alpha,\beta} Q(x)$. Using the standard Gronwall inequality (see, e.g., [3, 4]) yields

$$\|\bar{u}_N\|_{\omega^{\alpha,\beta}} \le C(\|J_5\|_{\omega^{\alpha,\beta}} + \|J_6\|_{\omega^{\alpha,\beta}}). \tag{5.28}$$

The implementation of Lemma 3.2 implies

$$\begin{aligned}
\|J_5\|_{\omega^{\alpha,\beta}} &= \|S\bar{u}_N - I_N^{\alpha,\beta} S\bar{u}_N\|_{\omega^{\alpha,\beta}} \\
&\le CN^{-1}\left\| k(x,x)\bar{u}_N(x) + \int_{-1}^{x} k_x(x,s)\bar{u}_N(s)\mathrm{d}s \right\|_{\omega^{\alpha,\beta}} \\
&\le CN^{-1}\left(\|\bar{u}_N\|_{\omega^{\alpha,\beta}} + \left\| \int_{-1}^{x} |\bar{u}_N(s)|\mathrm{d}s \right\|_{\omega^{\alpha,\beta}} \right) \\
&\le CN^{-1}\|\bar{u}_N\|_{\omega^{\alpha,\beta}},
\end{aligned} \tag{5.29}$$

here in the last inequality, (4.4) is used in terms of the assumption on α and β. On the other hand, according to Lemma 3.4,

$$\|J_6\|_{\omega^{\alpha,\beta}} = \|I_N^{\alpha,\beta} Q(x)\|_{\omega^{\alpha,\beta}} \le C\|Q(x)\|_{L^\infty(I)}. \tag{5.30}$$

By the expression of $Q(x)$ in (5.3) and Lemma 3.3, we have

$$\begin{aligned}
|Q(x)| &\le CN^{-m}\big|\tilde{k}\big(x, s(x,\cdot)\big)\big|_{H^{m;N}(I)} \big\|\bar{u}_N\big(s(x,\cdot)\big)\big\| \\
&\le CN^{-m}\sqrt{\frac{1+x}{2}}\big|k\big(x, s(x,\cdot)\big)\big|_{H^{m;N}(I)} \|\bar{u}_N\|.
\end{aligned}$$

Combining the above result and the definition (5.26) for M_m yields

$$\|Q(x)\|_{L^\infty(I)} \le CM_m N^{-m}\|\bar{u}_N\|, \tag{5.31}$$

which, together with (5.30), gives

$$\|J_6\|_{\omega^{\alpha,\beta}} \le CM_m N^{-m}\|\bar{u}_N\|. \tag{5.32}$$

If $-1 < \alpha, \beta \le 0$, obviously we have

$$\|\bar{u}_N\| \le C\|\bar{u}_N\|_{\omega^{\alpha,\beta}};$$

consequently,

$$\|J_6\|_{\omega^{\alpha,\beta}} \le CM_m N^{-m}\|\bar{u}_N\|_{\omega^{\alpha,\beta}}. \tag{5.33}$$

This, together with (5.28) and (5.29), leads to

$$\|\bar{u}_N\|_{\omega^{\alpha,\beta}} \le C\big(N^{-1} + M_m N^{-m}\big)\|\bar{u}_N\|_{\omega^{\alpha,\beta}}. \tag{5.34}$$

On the other hand, according to ([2], p. 282),

$$\|\phi\| \le CN^{\alpha}\|\phi\|_{\omega^{\alpha,\alpha}}, \forall\phi \in P_N, \tag{5.35}$$

where $\omega^{\alpha,\alpha}(x) = (1-x^2)^{\alpha}$, with $\alpha \ge 0$ and C is a positive constant independent of N. Hence, when $0 < \alpha = \beta < 1$, (5.32) implies

$$\|J_6\|_{\omega^{\alpha,\beta}} \le CM_m N^{-m+\alpha}\|\bar{u}_N\|_{\omega^{\alpha,\beta}}. \tag{5.36}$$

The combination of (5.36), (5.28) and (5.29) yields

$$\|\bar{u}_N\|_{\omega^{\alpha,\beta}} \le C\big(N^{-1} + M_m N^{-m+\alpha}\big)\|\bar{u}_N\|_{\omega^{\alpha,\beta}}. \tag{5.37}$$

Based on (5.34) and (5.37), when $-1 < \alpha, \beta \le 0$ or $0 < \alpha = \beta < 1$ and N is large enough, $\bar{u}_N = 0$. As a result, the existence and uniqueness of the pseudo-spectral Jacobi-Galerkin solution $\bar{u}_N$ is proved.

Now we turn to the $L^2_{\omega^{\alpha,\beta}}$ error estimate of $u - \bar{u}_N$. Actually (5.23) can be transformed into

$$E = -\int_{-1}^{x} k(x,s)E(s)\mathrm{d}s + SE - I_N^{\alpha,\beta}SE - I_N^{\alpha,\beta}Q(x),$$

which yields

$$|E| \le M\int_{-1}^{x} |E(s)|\mathrm{d}s + |J_6| + |J_7|, \tag{5.38}$$

with $J_6 = I_N^{\alpha,\beta}Q(x)$, $J_7 = SE - I_N^{\alpha,\beta}SE$. It follows from (5.38) and the standard Gronwall inequality that

$$\|E\|_{\omega^{\alpha,\beta}} \le C(\|J_6\|_{\omega^{\alpha,\beta}} + \|J_7\|_{\omega^{\alpha,\beta}}). \tag{5.39}$$

Similar to the estimate of $\|J_2\|_{\omega^{\alpha,\beta}}$, we obtain

$$\begin{aligned}\|J_7\|_{\omega^{\alpha,\beta}} &\le CN^{-1}\left\|k(x,x)E(x) + \int_{-1}^{x} k_x(x,s)E(s)\mathrm{d}s\right\|_{\omega^{\alpha,\beta}} \\ &\le CN^{-1}\left(\|E\|_{\omega^{\alpha,\beta}} + \left\|\int_{-1}^{x}|E(s)|\mathrm{d}s\right\|_{\omega^{\alpha,\beta}}\right) \\ &\le CN^{-1}\|E\|_{\omega^{\alpha,\beta}}, \end{aligned} \tag{5.40}$$

where (4.4) is used in the last inequality under the assumptions on α and β. In terms of (5.32), (5.39) and (5.40), when $-1 < \alpha, \beta \le 0$, we have

$$\|E\|_{\omega^{\alpha,\beta}} \le CM_m N^{-m}\|\bar{u}_N\| \le CM_m N^{-m}(\|u-\bar{u}_N\|_{\omega^{\alpha,\beta}} + \|u\|). \tag{5.41}$$

When $0 < \alpha = \beta < 1$, in terms of (5.36), (5.39) and (5.40), we have

$$\|E\|_{\omega^{\alpha,\beta}} \le CM_m N^{-m+\alpha}\|\bar{u}_N\|_{\omega^{\alpha,\beta}} \le CM_m N^{-m+\alpha}(\|u-\bar{u}_N\|_{\omega^{\alpha,\beta}} + \|u\|_{\omega^{\alpha,\beta}}). \tag{5.42}$$

By the triangular inequality,

$$\|u-\bar{u}_N\|_{\omega^{\alpha,\beta}} \le \|u-\hat{u}_N\|_{\omega^{\alpha,\beta}} + \|\hat{u}_N-\bar{u}_N\|_{\omega^{\alpha,\beta}}, \tag{5.43}$$

as well as Lemma 5.1, (5.41), and (5.42), we can obtain the desired estimated (5.24)–(5.25) provided N is sufficiently large. □

Theorem 5.2 *Suppose that the solution of* (2.1) *is sufficiently smooth. For the pseudo-spectral Jacobi-Galerkin solution defined in* (2.9), *we have the following estimates*

(i) *If $\omega^{\alpha,\beta}(x)$ is the Legendre weight, then*

$$\|u-\bar{u}_N\|_{L^\infty(I)} \le CN^{\frac{3}{4}-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)} + CM_m N^{\frac{1}{2}-m}\|u\|. \tag{5.44}$$

If $\omega^{\alpha,\beta}(x)$ is the Chebyshev weight, then

$$\|u-\bar{u}_N\|_{L^\infty(I)} \le CN^{\frac{1}{2}-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}(I)} + CM_m N^{-m}\log N\|u\|. \tag{5.45}$$

(ii) *If $\omega^{\alpha,\beta}(x)$ is the Jacobi weight with $-1 < \alpha, \beta < -\frac{1}{2}$, then*

$$\|u-\bar{u}_N\|_{L^\infty(I)} \le C\log N\big(N^{\frac{1}{2}-m}|u|_{H^{m;N}_{\omega^{\alpha,\beta}}} + M_m N^{-m}\|u\|\big). \tag{5.46}$$

If $\omega^{\alpha,\beta}(x)$ is the Jacobi weight with $-\frac{1}{2} < \alpha, \beta < 0$, then

$$\|u-\bar{u}_N\|_{L^\infty(I)} \le CN^{1+\gamma-m}|u|_{H^{m;N}_{\omega^{-1/2,-1/2}}} + CM_m N^{\frac{1}{2}+\gamma-m}\|u\|. \tag{5.47}$$

Proof Using (5.38) and the standard Gronwall inequality gives

$$\|E\|_{L^\infty(I)} \le C(\|J_6\|_{L^\infty(I)} + \|J_7\|_{L^\infty(I)}). \tag{5.48}$$

It follows from Lemma 3.2 that

$$\begin{aligned}\|J_7\|_{L^\infty(I)} &\le CN^{-\eta}\left\|k(x,x)E(x) + \int_{-1}^{x} k_x(x,s)E(s)\mathrm{d}s\right\|_{\omega^{\alpha,\beta}}\\ &\le CN^{-\eta}\|E\|_{L^\infty(I)},\end{aligned} \tag{5.49}$$

with $\eta = \frac{1}{4}$ when $\omega^{\alpha,\beta}(x)$ is the Legendre weight, and $\eta = \frac{1}{2}$ when $\omega^{\alpha,\beta}(x)$ is the Chebyshev weight. On the other hand, similar to (5.20), we have

$$\begin{aligned}\|J_7\|_{L^\infty(I)} &= \|SE - I_N^{\alpha,\beta}SE\|_{L^\infty(I)}\\ &\le CL_N(\alpha,\beta)N^{-\frac{1}{2}}\|E\|_{L^\infty(I)}.\end{aligned}$$

If $-1 < \alpha, \beta < -\frac{1}{2}$, then

$$\|J_7\|_{L^\infty(I)} \leq CN^{-\frac{1}{2}} \log N \|E\|_{L^\infty(I)}; \tag{5.50}$$

and if $-\frac{1}{2} < \alpha, \beta < 0$, then

$$\|J_7\|_{L^\infty(I)} \leq CN^{\gamma} \|E\|_{L^\infty(I)}, \tag{5.51}$$

with $\gamma = \max(\alpha, \beta) < 0$. Furthermore,

$$\begin{aligned} \|J_6\|_{L^\infty(I)} &= \|I_N^{\alpha,\beta} Q(x)\|_{L^\infty(I)} \\ &\leq \max_{0\leq j\leq N} |Q(x_j)| \max_I \sum_{j=0}^{N} |h_j(x)| \\ &\leq CM_m N^{-m} L_N(\alpha, \beta)\|\bar{u}_N\|, \end{aligned} \tag{5.52}$$

where (5.31) is used. Combining (5.48)–(5.52) yields

$$\begin{aligned} \|E\|_{L^\infty(I)} &\leq CM_m N^{-m} L_N(\alpha, \beta)\|\bar{u}_N\| \\ &\leq CM_m N^{-m} L_N(\alpha, \beta)(\|u\| + \|u - \bar{u}_N\|_{L^\infty(I)}). \end{aligned} \tag{5.53}$$

It follows from triangular inequality and (5.53) that

$$\begin{aligned} &\|u - \bar{u}_N\|_{L^\infty(I)} \\ &\quad\leq \|u - \hat{u}_N\|_{L^\infty(I)} + \|\hat{u}_N - \bar{u}_N\|_{L^\infty(I)} \\ &\quad\leq \|u - \hat{u}_N\|_{L^\infty(I)} + CM_m N^{-m} L_N(\alpha, \beta)(\|u\| + \|u - \bar{u}_N\|_{L^\infty(I)}). \end{aligned} \tag{5.54}$$

By Lemma 5.1 and (5.54), we obtain the desired estimated (5.44)–(5.47). □

We close this section by pointing out that the equivalence of the pseudo-spectral Jacobi-Galerkin and the spectral Jacobi-collocation methods can be verified by following a standard process if the uniqueness of the pseudo-spectral Jacobi-Galerkin approach can be obtained. Consequently, the convergence analysis for the pseudo-spectral Jacobi-Galerkin approach in this section also holds for the spectral Jacobi-collocation methods. Actually it is an extension of L^2 and L^∞ error estimates for the spectral Legendre-collocation method in [11].

6 Numerical Experiments

The efficiency of spectral or pseudo-spectral Legendre-Galerkin methods and Chebyshev-Galerkin methods will be demonstrated in the following as two special cases of the spectral or pseudo-spectral Jacobi-Galerkin approaches.

We consider the second kind Volterra integral equation (1.1) with

$$k(x, s) = e^{xs}, \qquad g(x) = e^{2x} + \frac{e^{x(x+2)} - e^{-(x+2)}}{x + 2}.$$

The corresponding exact solution is given by $u(x) = e^{2x}$.

Table 1 The errors of spectral Legendre-Galerkin method

N	4	6	8	10	12	14
L^∞-error	5.243e−02	1.262e−03	1.753e−05	1.572e−07	9.779e−10	4.618e−12
L^2-error	2.413e−03	3.942e−05	4.144e−07	3.028e−09	1.622e−11	6.631e−14

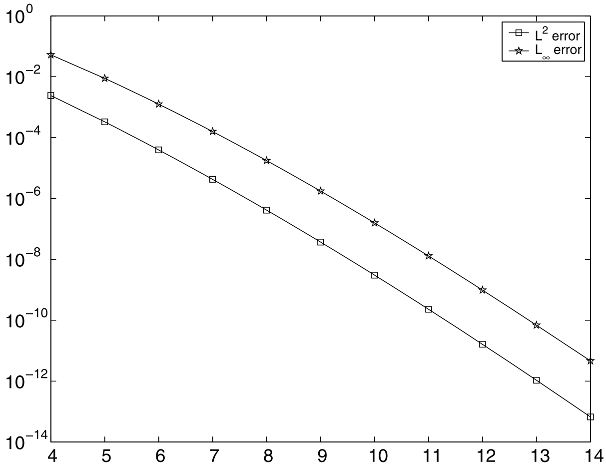

Fig. 1 L^2 and L^∞ errors of spectral Legendre-Galerkin method versus N

First we implement the numerical scheme (2.3) based on the spectral Legendre-Galerkin and Chebyshev-Galerkin methods to solve this example. Table 1 illustrates the L^∞ and L^2 errors of the spectral Legendre-Galerkin method which are also shown in Fig. 1. Next the L^∞ and $L^2_{\omega^{\alpha,\beta}}$ errors of the spectral Chebyshev-Galerkin method are demonstrated in Table 2 and Fig. 2. Clearly the desired spectral accuracy is obtained in these approaches.

Next we turn to the numerical scheme (2.10) based on the pseudo-spectral Legendre-Galerkin and Chebyshev-Galerkin methods to solve the example above. Table 3 illustrates the L^∞ and L^2 errors of the pseudo-spectral Legendre-Galerkin method which are also shown in Fig. 3. Next the L^∞ and $L^2_{\omega^{\alpha,\beta}}$ errors of the pseudo-spectral Chebyshev-Galerkin method are demonstrated in Table 4 and Fig. 4. Once again the desired spectral accuracy is obtained.

7 Conclusions

This paper proposes spectral and pseudo-spectral Jacobi-Galerkin methods for the second kind Volterra integral equations with smooth kernel. The discretization schemes for the general spectral and pseudo-spectral Jacobi-Galerkin approaches are provided. The spectral accuracy associated with L^∞ and $L^2_{\omega^{\alpha,\beta}}$ error estimates are demonstrated theoretically for some spectral and pseudo-spectral Jacobi-Galerkin methods. These results are confirmed by some numerical experiments.

It is a natural question that the simple Legendre-Galerkin method rather than the complicated Jacobi-Galerkin method should be used to approximate the numerical solutions of (1.1) as the kernel and the solutions are both smooth. In fact, the main purpose of this paper is to solve the underlying problem in a *general framework*. Therefore, we adopt both

Table 2 The errors of spectral Chebyshev-Galerkin method

N	4	6	8	10	12	14
L^∞-error	2.915e−02	5.696e−04	7.276e−06	5.751e−08	3.950e−10	1.737e−12
$L^2_{\omega^{\alpha,\beta}}$-error	4.864e−03	7.116e−05	7.051e−07	4.968e−09	2.596e−11	1.054e−13

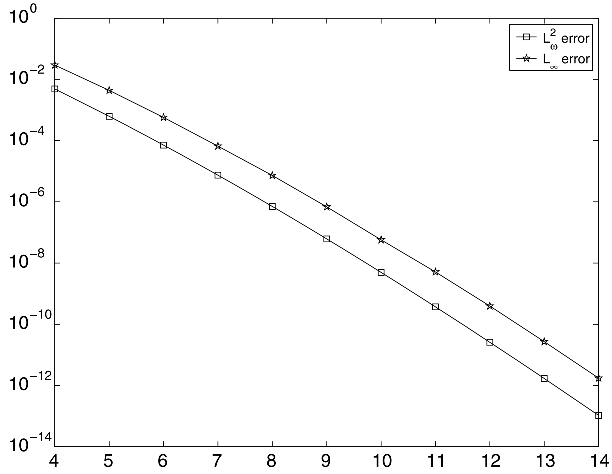

Fig. 2 $L^2_{\omega^{\alpha,\beta}}$ and L^∞ errors of spectral Chebyshev-Galerkin method versus N

Table 3 The errors of pseudo-spectral Legendre-Galerkin method

N	4	6	8	10	12	14
L^∞-error	6.007e−03	9.386e−05	8.710e−07	6.378e−09	3.322e−11	1.323e−13
L^2-error	4.443e−03	7.409e−05	7.874e−07	5.797e−09	3.123e−11	1.289e−13

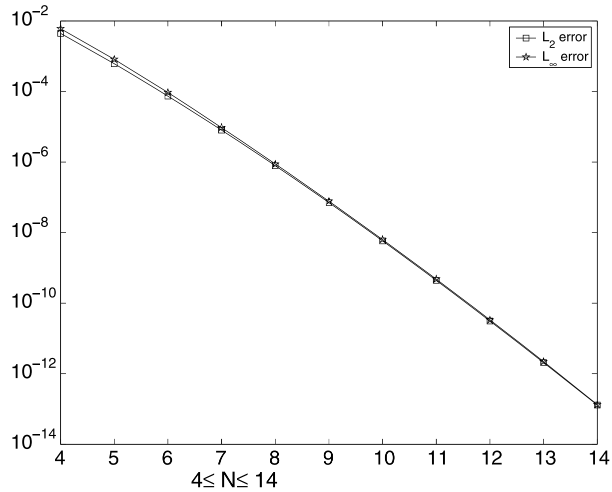

Fig. 3 L^2 and L^∞ errors of pseudo-spectral Legendre-Galerkin method versus N

Table 4 The errors of pseudo-spectral Chebyshev-Galerkin method

N	4	6	8	10	12	14
L^∞-error	7.113e−03	1.003e−04	9.958e−07	6.995e−09	3.638e−11	1.492e−13
$L^2_{\omega^{\alpha,\beta}}$-error	8.050e−03	1.203e−04	1.224e−06	8.814e−09	4.685e−11	1.909e−13

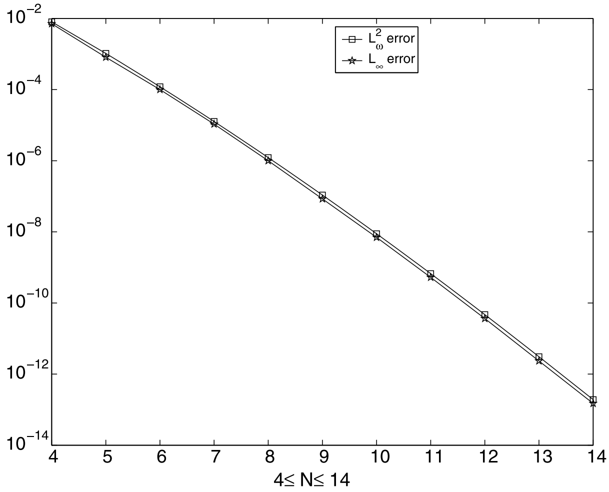

Fig. 4 $L^2_{\omega^{\alpha,\beta}}$ and L^∞ errors of pseudo-spectral Chebyshev-Galerkin method versus N

the spectral and pseudo-spectral Jacobi-Galerkin methods, which also include the spectral and pseudo-spectral Legendre-Galerkin methods and the Chebyshev-Galerkin methods. On the other hand, for pseudo-spectral Jacobi-Galerkin methods with some specially chosen α and β, say $-1 < \alpha, \beta < -\frac{1}{2}$, our theoretical analysis shows that the convergence rate in L^∞ norm is better than the pseudo-spectral Legendre-Galerkin approach, as shown in Theorem 5.2. Further, it is expected that our approaches in this work can provide inspiration for our future work on the second kind Volterra integral equations with weakly singular kernels.

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