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Blowup of Volterra Integro-Differential Equations and Applications to Semi-Linear Volterra Diffusion Equations

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Abstract. In this paper, we discuss the blowup of Volterra integro-differential equations (VIDEs) with a dissipative linear term. To overcome the fluctuation of solutions, we establish a Razumikhin-type theorem to verify the unboundedness of solutions. We also introduce leaving-times and arriving-times for the estimation of the spending-times of solutions to ∞. Based on these two typical techniques, the blowup and global existence of solutions to VIDEs with local and global integrable kernels are presented. As applications, the critical exponents of semi-linear Volterra diffusion equations (SLVDEs) on bounded domains with constant kernel are generalized to SLVDEs on bounded domains and $\mathbb{R}^N$ with some local integrable kernels. Moreover, the critical exponents of SLVDEs on both bounded domains and the unbounded domain $\mathbb{R}^N$ are investigated for global integrable kernels.

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1. Introduction

Volterra integral equations (VIEs) have a lot of applications in physics and experimental sciences: problems in mechanics, scattering theory, spectroscopy, stereology, seismology, elasticity theory, plasma physics (see in [25]). The blowup of solutions to VIEs rises in [25] and a particular example is given in [27]. After that a sequence of blowup results for some special VIEs are done by [22, 26, 32] (see also in the survey papers [17, 33] and the references in [6]).

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In [6], we present two features of blowup solutions : (i) the tendency to ∞, (ii) the finite spending-time for tending to ∞. Based on some assumptions ensuring the strictly monotone increasing of solutions to VIEs, the necessary and sufficient conditions for the blowup of Hammerstein-type nonlinear VIEs are discovered. By transforming into the equivalent form of VIEs, we also present some blowup results of Volterra integro-differential equations (VIDEs)

$$u'(t) = -\lambda u(t) + \int_0^t k(t-s)u^p(s)\mathrm{d}s, \quad t > 0, \tag{1.1a}$$

$$u(0) = u_0 > 0. \tag{1.1b}$$

While the condition $\lambda \leqslant 0$ is imposed in our paper [6] for the strict monotonicity of solutions. The blowup of solutions to non-homogeneous VIDEs is discussed in [21] under some conditions on the non-homogeneous term such that the solution is also strictly increasing. The blowup results of VIDEs with a dissipative linear term (i.e. $\lambda > 0$) and constant kernels are discussed in [35], since the solutions are convex and eventually increasing. Up to now, for (1.1) with $\lambda > 0$ and a general kernel, the blowup of fluctuation solutions is still open.

As important as VIDEs, semi-linear Volterra diffusion equations (SLVDEs)

$$u_t = \Delta u + \int_0^t k(t-s)u^p(s,x)\mathrm{d}s, \quad t > 0, \ x \in \Omega \subseteq \mathbb{R}^N, \tag{1.2a}$$

$$u(0,x) = u_0(x) \gneqq 0, \qquad x \in \Omega, \tag{1.2b}$$

$$u(t,x) \equiv 0, \qquad x \in \partial\Omega \tag{1.2c}$$

are introduced to model the effects of the memory effects in a population dynamics in [38, 39], and widely used in compression of poro-viscoelastic media in [11], the thermodynamics of phase transition in [3], reaction-diffusion problems in [9], and the theory of nuclear reactor kinetics in [15, 28–30]. The finite blowup analysis to (1.2) is begun in [16] and a complete result is obtained in [35] that the critical exponent of SLVDEs on bounded domains with constant kernels is $p^* = \infty$. That is to say, any positive solution to (1.2) blows up in finite time. However, for any $p > 1$, there always exists a global positive solution to

$$u_t = \Delta u + u^p(t,x), \quad t > 0, \ x \in \Omega, \tag{1.3}$$

when Ω is a bounded domain (see in [18, 23, 24]). Hence the critical exponent $p^* = 1$ of (1.3) is totally changed by the non-local time-integration with a constant kernel. Note that the local problem (1.3) can be written as the form of (1.2) with a Dirac delta function $k(z) = \delta(z)$ and the difference between constant kernels and delta functions is the integration on the whole interval $[0,\infty)$. Therefore it is more interesting how the blowup of SLVDEs on bounded domains is influenced by the global integrability of kernels. It is also interesting whether the critical exponent of SLVDEs on $\mathbb{R}^N$ is also influenced by the kernels.

By Kaplan's first eigenvalue and eigenfunction, the blowup results of SLVDEs on bounded domains come from the ones of solutions to (1.1), but the linear coefficient λ corresponding to the first eigenvalue of the Laplacian operator is positive. The blowup analysis of SLVDEs on $\mathbb{R}^N$ is also related to (1.1) by Fujita's approach. However, the coefficient λ is not only positive but also couples with the initial condition. Hence for both two cases, the corresponding solutions to (1.1) have the fluctuation although solutions are increasing after a long time periodic.

For a strictly increasing global solution, the tendency is either ∞ or a fixed positive finite number and the spending-time is estimated somehow by a "discrete" inverse formula. These typical techniques in [6] are not available to a fluctuating solution. Therefore, we introduce two techniques : a Razumikhin-type theorem for the unboundedness and an introduction of the arriving-time and leaving-time at given level values (see the definitions in Section 2). The Razumikhin-type technique has been used for the stability and boundedness analysis for many kinds of delay differential equations and delay difference equations (see in [12, 20]), while it is the first version for the unboundedness up to the best of our knowledge. For an increasing solution, the arriving-time coincides with the leaving-time and solutions in the periodic are bounded by the level values. Whereas for a fluctuation solution, the upper bounds between two leaving-times may be extremely large. Indeed, a part of estimation is followed from our previous work, but the rest estimation is our key contribution for the fluctuation solutions.

Throughout this paper, the kernels in (1.1) and (1.2) are assumed to be the form of $k(z) = z^{\beta-1}k_1(z)$ with $\beta > 0$ and a continuous function $k_1(z)$. Since the blowup of solutions to (1.1) and (1.2) is strongly influenced by the global integrability of the kernel in the whole interval $[0,\infty)$ (see [10, 35]), we separate the kernels into two types:

Type I : $0 < k_\infty e^{-z} \le k_1(z) \le k^\infty e^{-z}$, i.e., $k(z)$ is a global integrable kernel with

$$I(t) := \int_0^t k(z)\mathrm{d}z \le k^\infty \Gamma(\beta) \quad \text{for all } t \ge 0.$$

Type II : $k_1(z) \ge k_\infty > 0$ for all $z \ge 0$, i.e., $k(z)$ is a local integrable kernel and $I(\infty) = \lim_{t\to\infty} I(t) = \infty$.

This paper is organized as follows. In Section 2, we provide a comparison theorem and establish a Razumikhin-type theorem for the unboundedness of global solutions. Moreover, from the unboundedness of global solutions, the blowup of solutions to VIDEs is analyzed by the estimations of leaving and arriving times. As an application, we consider the blowup of SLVDEs on bounded spatial domains in Section 3. The critical exponent $p^* = 1$ of local parabolic problems are generalized to Type I kernels and the critical exponent $p^* = \infty$ of SLVDEs with constant kernels is extended to some kernels of Type II. As another application, the critical exponent of SLVDEs on $\mathbb{R}^N$ is studied in Section 4. For Type I kernels, the critical exponent p^* is estimated by β. Furthermore, it is shown that for some kernels of Type II all positive solutions to SLVDEs on $\mathbb{R}^N$ blow up in finite time, i.e., the critical exponent $p^* = \infty$ is same as the results of SLVDEs on bounded domains. This is a beginning of blowup analysis for VIDEs and SLVDEs and some interesting and future works are presented in Section 5.

2. Volterra integro-differential equations

In this section, the blowup analysis of VIDEs is followed by the framework in [6], i.e,

(i) suppose that a solution exists globally and investigate the tendency to ∞, then

(ii) the blowup result is yielded by estimating the spending-time of the solution from finite to ∞.

First of all, we introduce some notations, definitions that are used hereafter.

Definition 2.1. *For a function $w:\mathbb{R}_+\to\mathbb{R}_+$, we define*

$$\text{a nondecreasing function } \underline{w}(t):=\inf_{s\geqslant t} w(s),$$
$$\text{a nonincreasing function } \bar{w}(t):=\inf_{s\in[0,t]} w(s).$$

Definition 2.2. *Let $w:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function and $d<D$ be two positive numbers. Then*

(i) the leaving-time $t_d(w)$ is defined by $t_d(w)=\inf\{t:\underline{w}(t)>d\}$;

(ii) the arriving-time $t_d^D(w)$ after $t_d(w)$ is defined by $t_d^D(w)=\inf\{t\ge t_d(w):w(t)\ge D\}$.

In the above definition, we have that the leaving-time and arriving-time of a empty set are infinity, i.e., $\inf\{\emptyset\}=\infty$.

Remark 2.1. Let $w:\mathbb{R}_+\to\mathbb{R}_+$ be a continuous function. It is obvious that the following statements are true.

(i) $\underline{w}(t)\le w(t)$ and $\bar{w}(t)\le w(t)$ for all $t\in\mathbb{R}_+$.

(ii) $\lim_{t\to\infty} w(t)=\infty$ if and only if for any given $d>0$, the leaving-time $t_d(w)<\infty$ is well-defined.

(iii) $w(t)$ is nondecreasing if and only if $\underline{w}(t)=w(t)$ for all $t\in\mathbb{R}_+$.

(iv) $w(t)$ is nondecreasing if and only if the arriving-times coincide with the leaving-times, i.e., $t_d^D(w)=t_D(w)$ for all $d<D$.

2.1. Comparison theorem

Before the detailed discussions, we present the comparison theorem of the supersolutions and subsolutions of general VIDEs

$$u'(t)=-\lambda u(t)+\int_0^t K(t,s)G(u(s))\mathrm{d}s,\quad t>0, \tag{2.1a}$$
$$u(0)=u_0, \tag{2.1b}$$

where $u_0 > 0$ and $\lambda \geq 0$ are constants, $G(u)$ is a C^1-smoothing function and $K(t,s)$ is a locally integrable function with respect to $s \in (0,t)$ for each $t \geq 0$ and satisfies

$$\int_0^T \int_0^t |K(t,s)| \mathrm{d}s\mathrm{d}t < \infty \quad \text{for all } T > 0.$$

Definition 2.3. *A continuously differentiable function $u^*(t)$ is called a supersolution to* (2.1) *on the interval $[t_0, T)$ for $0 \leq t_0 < T$, if it holds for all $t \in [t_0, T)$*

$$\frac{\mathrm{d}}{\mathrm{d}t}u^*(t) + \lambda u^*(t) - \int_0^t K(t,s)G(u^*(s))\mathrm{d}s \geq 0.$$

A supersolution is said to exist globally if it is defined in the whole interval $[t_0, \infty)$.

Definition 2.4. *A continuously differentiable function $u_*(t)$ is called a subsolution to* (2.1) *on the interval $[t_0, T)$ for $0 \leq t_0 < T$, if it holds for all $t \in [t_0, T)$*

$$\frac{\mathrm{d}}{\mathrm{d}t}u_*(t) + \lambda u_*(t) - \int_0^t K(t,s)G(u_*(s))\mathrm{d}s \leq 0.$$

A subsolution is said to exist globally if it is defined in the whole interval $[t_0, \infty)$.

Theorem 2.1. (Comparison theorem) *Assume that $K(t,s) \geq 0$ for all $0 < s < t < \infty$, that $G(u)$ is increasing and that $u^*(t)$ and $u_*(t)$ are supersolution and subsolution to* (2.1) *in the interval $[t_0, T]$, respectively. Then $u_*(t) \leq u^*(t)$ for all $t \in [0, T]$ whenever $u_*(t) \leq u^*(t)$ for $t \in [0, t_0]$.*

Proof. Since $u^*(t)$ and $u_*(t)$ are continuous functions on the closed interval $[0, T]$, there exists an $M > 0$ such that

$$\max\left\{\max_{t\in[0,T]} |u^*(t)|, \max_{t\in[0,T]} |u_*(t)|\right\} \leq M,$$

which implies by the C^1-smoothing of G that for all $t \in [0, T]$

$$|G(u^*(t)) - G(u_*(t))| \leqslant L|u^*(t) - u_*(t)|,$$

where $L = \sup_{|u|\leq M} |G'(u)|$. Suppose that

$$t_1 = \sup\{t : u^*(s) \geq u_*(s) \quad \text{for} \quad s \in [0, t]\} < T.$$

Then there exists a $T_1 \in (t_1, T]$ such that

$$e^{-|\lambda|(T_1 - t_1)} \int_{t_1}^{T_1} \int_{t_1}^{t} K(t,s)\mathrm{d}s\mathrm{d}t < \frac{1}{2L}.$$

Let $t^* \in [t_1, T_1]$ be such that

$$u^*(t^*) - u_*(t^*) = \min_{t\in[t_1,t^*]} (u^*(t) - u_*(t)).$$

Then $u^*(t^*) - u_*(t^*) < 0$. By the definitions 2.3 and 2.4, it follows from $K(t,s) > 0$ and $G(u^*(t)) \geq G(u_*(t))$ for $t \in [0, t_1]$ that for $t \in [t_1, t^*]$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t}(u^*(t) - u_*(t))\right) = e^{\lambda t}\left[\lambda(u^*(t) - u_*(t)) + \frac{\mathrm{d}}{\mathrm{d}t}(u^*(t) - u_*(t))\right] \\
\geq& e^{\lambda t}\int_0^t K(t,s)(G(u^*(s)) - G(u_*(s))\mathrm{d}s \geq e^{\lambda t}\int_{t_1}^t K(t,s)(G(u^*(s)) - G(u_*(s)))\mathrm{d}s \\
\geq& -Le^{\lambda t}\int_{t_1}^t K(t,s)|u^*(s) - u_*(s)|\mathrm{d}s \geq -Le^{\lambda t}|u^*(t^*) - u_*(t^*)|\int_{t_1}^t K(t,s)\mathrm{d}s,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(u^*(t^*) - u_*(t^*)) \geq -L|u^*(t^*) - u_*(t^*)|\int_{t_1}^{t^*} e^{-\lambda(t^*-t)}\int_{t_1}^t K(t,s)\mathrm{d}s\mathrm{d}t \\
\geq& -L|u^*(t^*) - u_*(t^*)|e^{-|\lambda|(T_1-t_1)}\int_{t_1}^{T_1}\int_{t_1}^t K(t,s)\mathrm{d}s\mathrm{d}t > -\frac{1}{2}|u^*(t^*) - u_*(t^*)|.
\end{aligned}$$

This is a contradiction and the proof is complete. □

Corollary 2.1. *Assume that $K(t,s) \geq 0$ for all $0 < s < t < \infty$ and that $G(u)$ is an increasing function with $G(0) = 0$.*

(i) Any solution $u(t)$ to (2.1) satisfies $u(t) \geq u_0 e^{-\lambda t}$ in the maximum interval of its existence.

(ii) If $u^*(t)$ is a positive global supersolution to (2.1) in the interval $[t_0, \infty)$, then any positive solution to (2.1) with $u(t) \leq u^*(t)$ for $t \in [0, t_0]$ exists globally.

Proof. The proof is trivial from Theorem 2.1. □

2.2. Razumikhin-type theorem for VIDEs

We now focus on presenting some conditions by a Razumikhin technique to investigate the tendency of global solutions to (1.1), namely,

(i) suppose that there exists a global solution $u(t)$ to (1.1), and

(ii) establish the Razumikhin-type theorem to show $u(t) \to \infty$.

Lemma 2.1. (Razumikhin-type theorem) *Suppose that there exists a global solution $u(t)$ to* (1.1)*, and assume that there exist a continuous function $V(u)$, positive constants c_1, c_2, $\rho > 1$, $r > 0$, and a continuous function $\epsilon(t) \geqslant t$ such that :*

(i) $c_1|u| \leq V(u) \leq c_2|u|$ *for all* $u \in \mathbb{R}$*;*

(ii) the function $V(u)$ along with the solution $u(t)$ to (1.1) *satisfies*

$$\inf_{t\geq 0} V(u(t)) > 0;$$

(iii) the derivative of the function $V(u)$ along with the solution $u(t)$ to (1.1) *satisfies that*

$$V'(t_1) = \left.\frac{\mathrm{d}V(u(t))}{\mathrm{d}t}\right|_{t=t_1} = \left.\frac{\partial}{\partial u}V(u(t))\right|_{t=t_1} u'(t_1) \geq r,$$

whenever $V(u(t_1)) \leq \rho \inf_{s\geq t} V(u(s))$ *for* $t_1 \geq \epsilon(t)$.

Then we have $\lim_{t\to\infty} u(t) = \infty$.

Proof. Let $W(t) = V(u(t))$ and $\eta(t) = \epsilon(t) + \frac{(2\rho-1)\underline{W}(t)}{r}$. The Condition (ii) ensures that $\underline{W}(0) > 0$. By Remark 2.1, the tendency of $W(t)$ to ∞ is resulted from the leaving-time $T_n := t_{\rho^n \underline{W}(0)}(W)$ of $W(t)$ satisfying

$$T_n \leq \eta(T_{n-1}) \quad \text{for } n \geq 0, \tag{2.2}$$

where $T_{-1} = 0$. It is obvious that (2.2) holds for $n = 0$. Suppose that (2.2) holds up to n and let $T^n := \inf\{t \geq \epsilon(t) : W(t) > \rho\underline{W}(T_n)\}$. Then $T^n \leq \eta(T_n)$.

Otherwise condition (iii) yields that $\mathrm{d}W(t)/\mathrm{d}t \geq r$ for all $t \in [\epsilon(T_n), \eta(T_n)]$. Hence the contradiction comes from

$$\begin{aligned} W(\eta(T_n)) &\geq W(\epsilon(T_n)) + r(\eta(T_n) - \epsilon(T_n)) \\ &\geq \underline{W}(\epsilon(T_n)) + r\frac{(2\rho-1)\underline{W}(T_n)}{r} \\ &\geq \underline{W}(T_n) + (2\rho-1)\underline{W}(T_n) = 2\rho\underline{W}(T_n). \end{aligned}$$

Again, condition (iii) implies that

$$T_{n+1} \leq T^n.$$

Otherwise $t_1 = \inf\{t \geq T^n : W(t) < \rho\underline{W}(T_n)\} < \infty$ and $\frac{\mathrm{d}}{\mathrm{d}t}W(t_1) \leq 0$, which contradicts to $\frac{\mathrm{d}}{\mathrm{d}t}W(t_1) \geq r > 0$.

Therefore $\lim_{t\to\infty} W(t) = \infty$ and the proof is completed by $|u(t)| \geq \frac{1}{c_2}W(t)$. □

By Corollary 2.1, the Liapunov function is chosen by $V(u)=|u|$, which satisfies Condition (i) in Lemma 2.1 with $c_1=c_2=1$. Moreover,

$$\inf_{t\geq 0} V(u(t))=\underline{u}(0),$$

and

$$\frac{\mathrm{d}V(u(t))}{\mathrm{d}t}=u'(t).$$

In the following two lemmas, to verify Condition (ii) in Lemma 2.1, we discuss the infimum of a solution $u(t)$, i.e., $\underline{u}(0)>0$. From the proofs, it is seen that the positivity is guaranteed not only by a sufficiently large initial value but also by the historical information.

Lemma 2.2. *Assume that $\lambda>0$ and the integration $I(t)$ of a kernel $k(z)$ is positive for all $t>0$, and suppose that a solution $u(t)$ to* (1.1) *exists globally, then we have $\underline{u}(0)\geqslant M$ whenever*

$$M:=e^{-\lambda\tau}u_0>\left(\frac{\lambda}{I(\tau)}\right)^{1/(p-1)} \quad \text{for some } \tau>0. \tag{2.3}$$

Proof. In view of $u(0)=u_0>M$, we suppose that $u(t^*)=M$ and $u(t)>M$ for $t\in[0,t^*)$. Then from Corollary 2.1 (i) and (2.3), we have $t^*\geq\tau$ and $u'(t^*)\leq 0$. While, on the other hand,

$$\begin{aligned} u'(t^*)=&-\lambda u(t^*)+\int_0^{t^*}k(t^*-s)u^p(s)\mathrm{d}s\\ >&-\lambda M+\int_{t^*-\tau}^{t^*}k(t^*-s)\mathrm{d}sM^p\geq 0. \end{aligned}$$

This is a contradiction implies that $\underline{u}(0)\geq M$ and the proof is complete. □

To indicate the influence of history information, we introduce a function

$$I_*(\tau):=\inf_{t\geqslant\tau}\int_0^{\tau}k(t-s)\mathrm{d}s \quad \text{for } \tau>0. \tag{2.4}$$

It is obvious that $I_*(\tau)>0$ for Type II kernels with $\beta\geq 1$, i.e., the historical information always influences the future behavior of solutions. Thus $\underline{u}(0)>0$ holds for all $u_0>0$.

Lemma 2.3. *Assume that $\lambda>0$ and $k(z)$ is of Type II with $\beta\geqslant 1$, then we have $\underline{u}(0)>0$ for any global solution to* (1.1) *with $u_0>0$.*

Proof. By Corollary 2.1 (i), we have $u(t)>0$ for all $t>0$, which implies that $\bar{u}(1)>0$ and $\underline{u}(0)\geq 0$. Hence $\underline{u}(0)=0$ yields that there exists a $t^*>1$ such that

$$u(t^*)=\bar{u}(t^*)<\frac{I_*(1)\bar{u}(1)^p}{\lambda} \quad \text{and} \quad u'(t^*)\leq 0.$$

But this contradicts to

$$
\begin{aligned}
u'(t^*) &\ge -\lambda u(t^*) + \int_0^{t^*} k(t^*-s)u(t)^p \mathrm{d}s \\
&\ge -\lambda u(t^*) + \bar{u}(1)^p \int_0^1 k(t^*-s)\mathrm{d}s \\
&\ge -\lambda u(t^*) + I_*(1)\bar{u}(1)^p > 0.
\end{aligned}
$$

The proof is complete. □

In the remainder, we will prove the tendency of a global solution to (1.1) is ∞ by Condition (iii) in Lemma 2.1.

Theorem 2.2. *Assume that* $\lambda > 0$ *and suppose that a solution to* (1.1) *with* $u_0 > 0$ *exists globally, then we have*

(i) for Type I, $\lim_{t\to\infty} u(t) = \infty$ *when the initial value* u_0 *satisfies* (2.3)*;*

(ii) for Type II with $\beta \ge 1$*, it always holds that* $\lim_{t\to\infty} u(t) = \infty$*; and*

(iii) for Type II with $0 < \beta < 1$*,* $\lim_{t\to\infty} u(t) = \infty$ *when the initial value* u_0 *satisfies* (2.3).

Proof. In the proof, we choose $V(u) = |u|$, $c_1 = c_2 = 1$. From Lemmas 2.2 and 2.3, we only need consider Condition (iii) in Lemma 2.1.

(i) By Lemma 2.2, $\underline{u}(0) \ge M = e^{-\lambda\tau}u_0$, which together with (2.3) implies that there exist $\rho > 1$ and $r > 0$ such that for all $u \ge \underline{u}(0)$ and

$$I(\tau)u^p - \rho\lambda u \ge r.$$

Therefore, Condition (iii) in Lemma 2.1 holds since

$$
\begin{aligned}
u'(t_1) &= -\lambda u(t_1) + \int_0^{t_1} k(t_1-s)u^p(s)\mathrm{d}s \\
&\ge -\lambda\rho\underline{u}(t) + \int_t^{t_1} k(t_1-s)u^p(s)\mathrm{d}s \\
&\ge -\lambda\rho\underline{u}(t) + I(\tau)\underline{u}(t)^p \geqslant r,
\end{aligned}
$$

whenever $u(t_1) \le \rho\underline{u}(t)$ for $t > 0$ and $t_1 \ge t + \tau$.

(ii)For any given $\rho > 1$ and $r > 0$, let $\epsilon(t) \geqslant t$ be such that

$$I(\epsilon(t)) \ge \frac{\lambda\rho\underline{u}(t) + r}{\underline{u}(0)^p},$$

then Condition (iii) in Lemma 2.1 holds since

$$
\begin{aligned}
u'(t_1) &\ge -\lambda\rho\underline{u}(t_1) + \int_0^{t_1} k(t_1-s)u^p(s)\mathrm{d}s \\
&\ge -\lambda\rho\underline{u}(t) + I(t_1)\underline{u}(0)^p \\
&\ge -\lambda\rho\underline{u}(t) + I(\epsilon(t))\underline{u}(0)^p \ge r,
\end{aligned}
$$

whenever $u(t_1) \le \rho \underline{u}(t)$ for some $t_1 \ge \epsilon(t)$.

The proof of (iii) is similar to (i) and the proof is complete. □

Remark 2.2. We remark that for a constant kernel $k(z) = 1$, all solutions to (1.1) are convex and hence they are strictly increasing after some while. Therefore, the Razumikhin-type theorem is not in need for the investigation of the tendency of global solutions to (1.1). While, for variable kernels such as $k(z) = 2 + \sin(\omega z)$ for large ω, a similar result is obtained by the Razumikhin technique.

Remark 2.3. We investigate the tendency of a global solution as $t \to \infty$ by a Razumikhin technique. Indeed, the tendency is strongly dependent on the global integrability of the kernels and the historical information of solutions. Indeed, from the discussion in the next subsection, there always exists a bounded global solution to (1.1) for Type I kernels and there exist no global positive solutions to (1.1) for Type II kernels with $\beta \ge 1$. But the existence of global solutions to (1.1) for Type II kernels with $0 < \beta < 1$ is still open, since the historical influence decays as $t \to \infty$, i.e., $I_*(\tau) \equiv 0$ for all $\tau > 0$.

2.3. Blowup and global existence for VIDEs

Following our framework in [6], we have presented some conditions such that global solutions to (1.1) tend to ∞ and $\underline{u}(0) > 0$. Then according to Definition 2.2, the leaving-times

$$t_n = t_{\rho^n \underline{u}(0)}(u), \quad n = 0, 1, \cdots$$

are well-defined for any given $\rho > 1$ and the blowup results come from the summability of the spending-times $\sum_{n=0}^{\infty}(t_{n+1} - t_n)$. Again the fluctuation of solutions to (1.1) yields that the maximum value of solutions in $[t_n, t_{n+1}]$ is extremely large. Hence we separate the periodic into $[t_n, s_n]$ and $[s_n, t_{n+1}]$ by the arriving-times

$$s_n = t_{\rho^n \underline{u}(0)}^{\rho^{n+1}\underline{u}(0)}(u), \quad n = 0, 1, \cdots.$$

Remark 2.4. For an increasing solution, the arriving-times coincide with the leaving-times and the blowup analysis goes back to our previous work [6]. Even for a fluctuation solution, the analysis in the interval $[t_n, s_n]$ is also similar as our previous work. Hence the main contribution in this subsection is the analysis in the interval $[s_n, t_{n+1}]$. Anyways, it is clear that

(i) $t_n < s_n \le t_{n+1}$;

(ii) the solution $u(t)$ in $[t_n, s_n]$ is bounded by $\rho^n \underline{u}(0)$ and $\rho^{n+1}\underline{u}(0)$;

(iii) the solution $u(t)$ in $[s_n, t_{n+1}]$ is bounded from below by $\rho^n \underline{u}(0)$;

(iv) $u(s_n) = u(t_{n+1}) = \rho^{n+1}\underline{u}(0)$.

Different from the tendency of global solutions, the blowup behavior is only related to the local information of the kernel. Precisely, in the following lemmas, we show that any unbounded solution to (1.1) blows up in finite time by the two steps:

(i) the spending-times satisfy $\lim_{n\to\infty}(t_{n+1}-t_n)=0$;

(ii) the spending-times are summable.

Lemma 2.4. *Assume that the integral $I(t)$ of the kernel is an increasing function and that a global solution $u(t)$ tends to ∞ and $\underline{u}(0)>0$, then for any given $\rho>1$, the leaving-times $t_n=t_{\rho^n\underline{u}(0)}(u)$ $(n=0,1,\cdots)$ satisfy*

$$\lim_{n\to\infty}(t_{n+1}-t_n)=0.$$

Proof. By Remark 2.1, the leaving-time t_n and arriving-time s_n of the solution are well-defined and the limitation is established by (i) and (ii).

(i) $\lim_{n\to\infty}(s_n-t_n)=0$ is simple, since the solution is bounded by $\rho^n\underline{u}(0)$ and $\rho^{n+1}\underline{u}(0)$.

Suppose that there exists a time subsequence (also denoted by t_n,s_n) such that $s_n-t_n\geq\underline{h}>0$. Then for sufficiently large n and $t_n+\dfrac{1}{2}\underline{h}\leq t\leq s_n$, we have

$$\begin{aligned}u'(t)\geqslant&-\lambda u(t)+\int_{t-\frac{1}{2}\underline{h}}^{t}k(t-s)u^p(s)\mathrm{d}s\\ \geqslant&-\lambda\rho\rho^n\underline{u}(0)+\rho^{pn}\underline{u}(0)^p\int_{t-\frac{1}{2}\underline{h}}^{t}k(t-s)\mathrm{d}s\\ \geqslant&\rho^{pn}\left(-\lambda\rho^{1+(1-p)n}\underline{u}(0)+I\left(\frac{1}{2}\underline{h}\right)\underline{u}(0)^p\right)\geqslant\frac{1}{2}I\left(\frac{1}{2}\underline{h}\right)\underline{u}(0)^p\rho^{pn}.\end{aligned}$$

On the other hand by the mean-value theorem, there exists a $\xi\in\left[t_n+\dfrac{1}{2}\underline{h},s_n\right]$ such that

$$s_n-\left(t_n+\frac{1}{2}\underline{h}\right)=\frac{u(s_n)-u\left(t_n+\frac{1}{2}\underline{h}\right)}{u'(\xi)},$$

which yields a contradiction, given by

$$\frac{1}{2}\underline{h}\leq\frac{(\rho-1)\rho^n}{\frac{1}{2}I\left(\frac{1}{2}\underline{h}\right)\rho^{pn}\underline{u}(0)^{p-1}}\to0\ \text{ as } n\to\infty.$$

(ii) To derive $\lim_{n\to\infty}(t_{n+1}-s_n)=0$, we need the following representation of solutions to (1.1), i.e., for $t\geq s_n$

$$u(t)=e^{-\lambda(t-s_n)}u(s_n)+\int_{s_n}^{t}\int_0^s e^{-\lambda(t-s)}k(s-r)u^p(r)\mathrm{d}r\mathrm{d}s.$$

Moreover, let

$$J(t) := \int_0^t I(z)\mathrm{d}z.$$

Then $J(t)$ is positive for all $t > 0$, since $I(t) > 0$ for all $t > 0$.

We suppose that there exists a time subsequence (also denoted by t_n, s_n) such that $t_{n+1} - s_n \geqslant \bar{h} > 0$. Then there exists an integer $N > 0$ such that for all $n \geq N$,

$$\rho^{1-(p-1)n} \leq e^{-\frac{1}{2}\lambda\bar{h}} J\left(\frac{1}{2}\bar{h}\right) \underline{u}(0)^{p-1}.$$

Thus for $t \geq s_n + \frac{3}{4}\bar{h}$,

$$\begin{aligned}
u(t) &= e^{-\lambda(t-s_n)}\rho^{n+1}\underline{u}(0) + \int_{s_n}^t \int_0^s e^{-\lambda(t-s)}k(s-r)u^p(r)\mathrm{d}r\mathrm{d}s \\
&\geq \rho^{pn}\underline{u}(0)^p \int_{s_n}^t \int_{s_n}^s e^{-\lambda(t-s)}k(s-r)\mathrm{d}r\mathrm{d}s \\
&\geq \rho^{pn}\underline{u}(0)^p \int_{t-\frac{1}{2}\bar{h}}^t \int_{t-\frac{1}{2}\bar{h}}^s e^{-\lambda(t-s)}k(s-r)\mathrm{d}r\mathrm{d}s \\
&\geq \rho^{pn}\underline{u}(0)^p e^{-\frac{1}{2}\lambda\bar{h}} \int_{t-\frac{1}{2}\bar{h}}^t \int_{t-\frac{1}{2}\bar{h}}^s k(s-r)\mathrm{d}r\mathrm{d}s \\
&\geq \rho^{pn}\underline{u}(0)^p e^{-\frac{1}{2}\lambda\bar{h}} J\left(\frac{1}{2}\bar{h}\right) \geq \rho^{n+1}\underline{u}(0).
\end{aligned}$$

This is contradiction by the definition of the leaving-time t_{n+1}. The proof is complete. □

Lemma 2.5. *Assume that the positive kernel satisfies $k(z) \geq k_* z^{\beta-1}$ in a neighborhood of $z = 0$ for some $k_* > 0$ and $\beta > 0$, $\underline{u}(0) > 0$ and that the leaving-times $t_n = t_{\rho^n \underline{u}(0)}(u)$, $n = 0, 1, \cdots$, are well-defined for some $\rho > 1$. Then the spending-times are summable, i.e., $\sum_{n=0}^{\infty}(t_{n+1} - t_n) < \infty$.*

Proof. It follows from the proof of Lemma 2.4 that the leaving-times $t_n = t_{\rho^n \underline{u}(0)}(u)$ $(n = 0, 1, \cdots)$ satisfy $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$, which implies that for sufficiently large $n \geqslant n_0$

$$e^{-\lambda(s-r)} \geq \frac{1}{2} \quad \text{and} \quad k(s-r) \geq k_*(s-r)^{\beta-1} \quad \text{for all } t_n \leq r < s \leq t_{n+1}. \tag{2.5}$$

Therefore, the summability is established by $\sum_{n=0}^{\infty}(s_n - t_n) < \infty$ and $\sum_{n=0}^{\infty}(t_{n+1} - s_n) < \infty$, respectively.

(i) $\sum_{n=0}^{\infty}(s_n - t_n) < \infty$ is similar to our previous analysis.

Continuing from the proof of Lemma 2.4, one obtains from (2.5) that

$$\begin{aligned} u(s_n) &= e^{-\lambda(s_n-t_n)}\rho^n\underline{u}(0) + \int_{t_n}^{s_n}\int_0^s e^{-\lambda(s_n-s)}k(s-r)u^p(r)\mathrm{d}r\mathrm{d}s \\ &\geq \rho^{np}\underline{u}(0)^p\int_{t_n}^{s_n}\int_{t_n}^s e^{-\lambda(s_n-s)}k(s-r)\mathrm{d}r\mathrm{d}s \\ &\geq \frac{\underline{u}(0)^p}{2\beta(\beta+1)}\rho^{np}k_*(s_n-t_n)^{\beta+1}, \end{aligned}$$

which together with $u(s_n)=\rho^{n+1}\underline{u}(0)$ implies that for $n\geqslant n_0$,

$$\begin{aligned} s_n-t_n &\leq \left(\frac{2\beta(\beta+1)}{k_*\,\underline{u}(0)^{p-1}}\right)^{1/(\beta+1)}\left(\frac{\rho^{n+1}}{\rho^{np}}\right)^{1/(\beta+1)} \\ &= \left(\frac{2\beta(\beta+1)}{k_*\underline{u}(0)^{p-1}}\right)^{1/(\beta+1)}\frac{\rho^{1/(\beta+1)}}{\rho^{1/(\beta+1)}-1}\frac{\rho^{n/(\beta+1)}-\rho^{(n-1)/(\beta+1)}}{\rho^{np/(\beta+1)}}. \end{aligned}$$

Hence

$$\sum_{n=n_0}^{\infty}(s_n-t_n)\leq\left(\frac{2\beta(\beta+1)}{k_*\underline{u}(0)^{p-1}}\right)^{1/(\beta+1)}\frac{\rho^{1/(\beta+1)}}{\rho^{1/(\beta+1)}-1}\int_{\rho^{(n_0-1)/(\beta+1)}}^{\infty}\frac{1}{x^p}\mathrm{d}x<\infty.$$

(ii) The representation is also needed for $\sum_{n=0}^{\infty}(t_{n+1}-s_n)<\infty$.

From (2.5), we have the following estimation

$$\begin{aligned} u(t_{n+1}) &= e^{-\lambda(t_{n+1}-s_n)}\rho^{n+1}\underline{u}(0) + \int_{s_n}^{t_{n+1}}\int_0^s e^{-\lambda(t_{n+1}-s)}k(s-r)u^p(r)\mathrm{d}r\mathrm{d}s \\ &\geq e^{-\lambda(t_{n+1}-s_n)}\rho^{n+1}\underline{u}(0)+\rho^{np}\underline{u}(0)^p\int_{s_n}^{t_{n+1}}\int_{s_n}^s e^{-\lambda(t_{n+1}-s)}k(s-r)\mathrm{d}r\mathrm{d}s \\ &\geq e^{-\lambda(t_{n+1}-s_n)}\rho^{n+1}\underline{u}(0)+\rho^{np}\underline{u}(0)^pk_*\frac{1}{2\beta(\beta+1)}(t_{n+1}-s_n)^{\beta+1}, \end{aligned}$$

which together with $u(t_{n+1})=\rho^{n+1}\underline{u}(0)$ and $\lambda(t_{n+1}-s_n)>1-e^{-\lambda(t_{n+1}-s_n)}$ implies that

$$\lambda(t_{n+1}-s_n)\rho^{n+1}\geq\frac{1}{2\beta(\beta+1)}\rho^{np}\underline{u}(0)^{p-1}k_*(t_{n+1}-s_n)^{\beta+1}.$$

Therefore

$$\sum_{n=n_0}^{\infty}(t_{n+1}-s_n)\leq\left(\frac{2\beta(\beta+1)\lambda}{\underline{u}(0)^{p-1}k_*}\right)^{1/\beta}\frac{\rho^{1/\beta}}{\rho^{1/\beta}-1}\int_{\rho^{(n_0-1)/\beta}}^{\infty}\frac{1}{x^p}\mathrm{d}x<\infty.$$

As a result of (i) and (ii), the proof is complete. □

It is ready to formulate our main results on the blowup and global existence of solutions to (1.1).

Theorem 2.3. *Let $\lambda > 0$ and $k(z)$ be of Type I. Then*

(i) *$u(t)$ to (1.1) blows up in finite time for sufficiently large u_0;*

(ii) *a global solution $u(t)$ to (1.1) exists for $0 < u_0 < \left(\frac{\lambda}{k^\infty\,\Gamma(\beta)}\right)^{1/(p-1)}$.*

Proof. (i) Suppose that a solution $u(t)$ with

$$u(0) > \left(\frac{\lambda e\beta}{k_\infty}\right)^{1/(p-1)} e^{\lambda}$$

exists globally. Then it follows from Theorem 2.2 (i) that $\lim_{t\to\infty} u(t) = \infty$, since

$$I(1) = \int_0^1 z^{\beta-1} k_1(z)\exp(-z)\mathrm{d}z \geqslant \frac{k_\infty}{e\beta}.$$

Thus for any $\rho > 1$, its leaving-times $t_n = t_{\rho^n \underline{u}(0)}(u)$ $(n = 0, 1, \cdots)$ satisfy $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$ by Lemma 2.4. By Lemma 2.5, this contradicts to the summability of the spending-times.

(ii). It is easy to see that

$$u^*(t) \equiv u^* = \left(\frac{\lambda}{k^\infty\,\Gamma(\beta)}\right)^{1/(p-1)}$$

is a supersolution to (1.1) on the interval $[0, \infty)$, since for all $t \geqslant 0$

$$\begin{aligned}&\frac{\mathrm{d}u^*}{\mathrm{d}t}(t) + \lambda u^*(t) - \int_0^t k(t-s)u^*(s)^p \mathrm{d}s\\ =&\lambda u^* - \int_0^t k(t-s)(u^*)^p \mathrm{d}s \geq u^*\lambda\left(1 - \frac{I(t)}{k^\infty\,\Gamma(\beta)}\right) \geq 0.\end{aligned}$$

Hence Corollary 2.1 (ii) implies that solutions to (1.1) exist globally whenever $0 < u_0 < u^*$ and the proof is complete. □

Theorem 2.4. *Let $\lambda > 0$ and $k(z)$ be of Type II. Then*

(i) *for $\beta \geq 1$, any positive solution $u(t)$ to (1.1) blows up in finite time for all $u_0 > 0$;*

(ii) *for $0 < \beta < 1$, positive solutions to (1.1) blow up in finite time when u_0 is large enough.*

Proof. The blowup results directly come from Theorem 2.2 (ii) and (iii), Lemmas 2.4 and 2.5. Hence the proof is complete. □

Remark 2.5. (i) The blowup result in Theorem 2.3 is a generalization of ordinary differential equations (the corresponding kernel $k(z) = \delta(z)$ is globally integrable). While the blowup result in Theorem 2.4 (i) is a generalization of constant kernels.

(ii) It is seen from Lemmas 2.4 and 2.5 that the blowup is only determined by the local behavior of the kernels.

(iii) Solutions blow up in finite time if they are strictly increasing after some while.

(iv) For Type II kernels, global solutions either tend to zero or fluctuate in the whole interval $[0,\infty)$.

3. Applications to SLVDEs on bounded domains

In this section, we deal with the blowup and global existence of solutions to

$$u_t = \Delta u + \int_0^t k(t-s)u^p(s,x)\mathrm{d}s, \qquad t>0,\ x\in\Omega, \tag{3.1a}$$

$$u(0,x) = u_0(x) \gneqq 0, \qquad x\in\Omega, \tag{3.1b}$$

$$u(t,x) \equiv 0, \qquad x\in\partial\Omega, \tag{3.1c}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^1-smooth boundary and the kernel is of Type I or Type II. By Kaplan's method, the blowup and global existence are related to the corresponding results of (1.1), where the linear coefficient $\lambda > 0$ is the first eigenvalue of the Laplacian operator. Hence as applications of Theorems 2.3 and 2.4, the blowup and global existence of solutions to SLVDEs on bounded domains are drawn in this section.

3.1. Kaplan's method

Let Ω be a bounded domain with C^1-boundary. Then there exists a unique continuous positive solution ϕ to the eigenvalue problem

$$\Delta\phi = -\lambda_1\phi, \qquad x\in\Omega\subseteq\mathbb{R}^N, \tag{3.2a}$$

$$\phi(x)\equiv 0, \qquad x\in\partial\Omega, \tag{3.2b}$$

$$\int_\Omega \phi(x)\mathrm{d}x = 1, \tag{3.2c}$$

where $\lambda_1 > 0$ is Kaplan's first eigenvalue only dependent on the domain. Let $\Phi(t) := \int_\Omega u(t,x)\phi(x)\mathrm{d}x$. Then the solution $u(t,x)$ to (3.1) blows up in finite time if and only if so does $\Phi(t)$. It follows from (3.1) that

$$\Phi'(t) = -\lambda_1\Phi(t) + \int_0^t k(t-s)\int_\Omega u^p(s,x)\phi(x)\mathrm{d}x\mathrm{d}s. \tag{3.3}$$

Applying Jensen's inequality to (3.3) produces

$$\Phi'(t) \geq -\lambda_1 \Phi(t) + \int_0^t k(t-s)\Phi^p(s)\mathrm{d}s, \qquad t > 0, \tag{3.4a}$$

$$\Phi(0) = \int_\Omega \phi(x)u_0(x)\mathrm{d}x > 0. \tag{3.4b}$$

Therefore, $\Phi(t)$ is a supersolution to the VIDEs

$$v'(t) = -\lambda_1 v(t) + \int_0^t k(t-s)v^p(s)\mathrm{d}s, \qquad t > 0, \tag{3.5a}$$

$$v(0) = \Phi(0). \tag{3.5b}$$

Thus the finite blowup of $\Phi(t)$ is resulted from the conclusions of Theorems 2.3 and 2.4.

For the global existence, we seek a supersolution in such a form $u(t,x) = v(t)\phi(x)$ to (3.1) with an initial value $u(0,x) = v_0\phi(x) \gneqq 0$. Using the relationship

$$(\partial_t - \Delta)u = (v'(t) + \lambda_1 v(t))\phi(x),$$
$$(v(s)\phi(x))^p = \|\phi\|_\infty^p \left(\frac{\phi(x)}{\|\phi\|_\infty}\right)^p v^p(s) \leq \|\phi\|_\infty^{p-1} v^p(s)\phi(x),$$

$u(t,x)$ is a supersolution to (3.1) when $v(t)$ is a positive solution to

$$v'(t) = -\lambda_1 v(t) + \|\phi\|_\infty^{p-1} \int_0^t k(t-s)v^p(s)\mathrm{d}s, \qquad t > 0, \tag{3.6a}$$

$$v(0) = v_0 > 0. \tag{3.6b}$$

Thus, for Type I kernels, Theorem 2.3 (ii) provides a global solution $v(t)$ to (3.6), which produces the existence of a global solution $u(t,x)$ to (3.1) when $u_0(x) \leq v_0\phi(x)$, by the comparison theorem for SLVDEs yielded by Lemma 3.2 in [13] (see also [8, 19, 42] in detail).

3.2. Blowup and critical exponents

Theorem 3.1. *Assume that Ω is a bounded domain with C^1-boundary.*

(i) Let $k(z)$ be of Type I. Then for all $\beta > 0$,

> *solutions to (3.1) blow up in finite time when $u_0(x) \gneqq 0$ is large enough; a global solution exists when $u_0(x) \gneqq 0$ is small.*

The property implies that the critical exponent is $p^* = 1$ for all $\beta > 0$.

(ii) Let $k(z)$ be of Type II. Then

(a) *for* $\beta \geq 1$*, solutions to* (3.1) *blow up in finite time for all* $u_0(x) \gneqq 0$*;*

(b) *for* $0 < \beta < 1$*, solutions to* (3.1) *blow up in finite time when* $u_0(x) \gneqq 0$ *is large enough.*

The property (a) implies that the critical exponent is $p^* = \infty$ *for all* $\beta \geq 1$.

Proof. From Kaplan's method in Section 3.1, we let ϕ be the unique solution to (3.2). Then $\Phi(t) = \int_\Omega u(t,x)\phi(x)dx$ is a supersolution to (3.5). Therefore, the blowup results are proved by Theorems 2.3 and 2.4.

From Theorem 2.3 (ii), the solution $v(t)$ to (3.6) exists globally when $v_0 > 0$ is sufficiently small, which implies that any solution to (3.1) exists globally whenever $v_0 \leq v_0\phi$. The proof is complete. □

Remark 3.1. The result in Theorem 3.1 (i) is a generalization of the local parabolic problem (1.3) on bounded domains and the results for Type II in (ii) show that the critical exponent is strongly dependent on the global integrability of the kernel. While the existence of a global solution for Type II kernels with $0 < \beta < 1$ is still open.

4. Applications to SLVDEs on $\mathbb{R}^N$

As another application, we analyze the blowup and global existence of solutions to SLVDEs on $\mathbb{R}^N$ of the form

$$u_t = \Delta u + \int_0^t k(t-s)u^p(s,x)ds, \qquad t>0,\ x\in\mathbb{R}^N, \tag{4.1a}$$

$$u(0,x) = u_0(x) \gneqq 0, \qquad x\in\mathbb{R}^N, \tag{4.1b}$$

$$u(t,\infty) \equiv 0. \tag{4.1c}$$

By Fujita's approach in [10], the blowup and global existence are also related to the results of (1.1). However, the linear coefficient has a coupled relationship with the initial value.

4.1. Fujita's approach

Let

$$G(\sigma,x) := (4\pi\sigma)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4\sigma}\right), \qquad \sigma>0,\ x\in\mathbb{R}^N,$$

and

$$\Phi_\sigma(t) := \int_{\mathbb{R}^N} G(\sigma,x)u(t,x)dx.$$

In view of

$$\Delta G(\sigma,x) = \frac{|x|^2 - 2N\sigma}{4\sigma^2} G(\sigma,x)$$

and

$$\int_{\mathbb{R}^N} [G(\sigma,x)\Delta u(t,x) - u(t,x)\Delta G(\sigma,x)]\,\mathrm{d}x = 0,$$

from (4.1) and Jensen's inequality, one obtains that

$$\begin{aligned}\Phi_\sigma'(t) &= \int_{\mathbb{R}^N} \Delta u(t,x)G(\sigma,x)\mathrm{d}x + \int_0^t k(t-s)\int_{\mathbb{R}^N} u^p(s,x)G(\sigma,x)\mathrm{d}x\mathrm{d}s \\ &\geq -\frac{N}{2\nu}\Phi_\sigma(t) + \int_0^t k(t-s)\Phi_\sigma^p(s)\mathrm{d}s.\end{aligned}$$

By [14], the initial value $\Phi_\sigma(0) = \int\limits_{\mathbb{R}^N} G(\sigma,x)u_0(x)\mathrm{d}x$ satisfies

$$\Phi_\sigma(0) \geq C\sigma^{-N/2},$$

where the constant $C > 0$ is dependent on the initial function u_0. Therefore, $\Phi_\sigma(t)$ is a supersolution to

$$v'(t) = -\frac{N}{2\sigma}v(t) + \int_0^t k(t-s)v^p(s)\mathrm{d}s, \qquad t > 0, \tag{4.2a}$$

$$v(0) = \Phi_\sigma(0) \tag{4.2b}$$

and the blowup results still come from Theorems 2.3 and 2.4.

For the global existence, we consider a supersolution to (4.1) in a form of $u(t,x) = \delta(t+1)^\eta G(t+1,x)$ with an initial value $u(0,x) = \delta G(1,x)$. A direct calculation yields that

$$(\partial_t - \Delta)u = \delta\eta(t+1)^{\eta-1}G(t+1,x) = \delta\eta(4\pi)^{-\frac{N}{2}}(t+1)^{\eta-1-\frac{N}{2}}\exp\left(-\frac{|x|^2}{4(t+1)}\right),$$

and for all $0 \leq s \leq t$ and $x \in \mathbb{R}^N$

$$\begin{aligned}u^p(s,x) =& \delta^p(s+1)^{p\eta}(4\pi(s+1))^{-\frac{pN}{2}}\exp\left(-\frac{p|x|^2}{4(s+1)}\right) \\ \leq& \delta^p(4\pi)^{-\frac{pN}{2}}(s+1)^{p\eta-\frac{pN}{2}}\exp\left(-\frac{|x|^2}{4(t+1)}\right).\end{aligned}$$

Hence the global existence will be yielded by the inequality

$$\eta \geqslant \delta^{p-1}(t+1)^{1+\frac{N}{2}-\eta}\int_0^t k(t-s)(s+1)^{p\eta-\frac{pN}{2}}\mathrm{d}s \tag{4.3}$$

for sufficiently small $\delta > 0$, $\eta > 0$ and all $t \geq 0$.

4.2. Blowup and critical exponents

Theorem 4.1. *Assume that* $\Omega=\mathbb{R}^N$ *is an unbounded domain.*

(i) *Let* $k(z)$ *be of Type I. Then the critical exponent of* (4.1) *satisfies*

$$p^* \in \left[1+\frac{2}{N}, 1+\frac{2+4\beta}{N}\right].$$

Namely,

(a) *if* $1<p<1+\frac{2}{N}$, *solutions to* (4.1) *blow up in finite time for all* $u_0(x)\gneqq 0$;

(b) *if* $p>1+\frac{2+4\beta}{N}$, *solutions to* (4.1) *blow up in finite time for sufficiently larger* $u_0(x)$ *and a global solution exists when* u_0 *is small.*

(ii) *Let* $k(z)$ *be of Type II. Then*

(a) *for* $\beta\geq 1$, *solutions to* (4.1) *blow up in finite time for all* $u_0(x)\gneqq 0$;

(b) *for* $0<\beta<1$, *solutions to* (4.1) *blow up in finite time when* $u_0(x)\gneqq 0$ *is large enough.*

The property (a) implies that the critical exponent is $p^*=\infty$ *for all* $\beta\geq 1$.

Proof. We only consider the results in (i).

(a) Suppose that there exists a global solution to (4.1) with a given initial value $u_0\gneqq 0$. Then for any fixed positive $\sigma>0$, $\Phi_\sigma(t)$ is a global supersolution to (4.2), which implies that a solution $v(t)$ with $v_0=\Phi_\sigma(0)$ to (4.2) exists globally.

On the other hand, it follows from $1<p<1+2/N$ that there exists a sufficiently large $\sigma>0$ such that

$$C\sigma^{-\frac{N}{2}+\frac{1}{p-1}} > e^{-\frac{N}{2}}\left(\frac{N}{k_\infty\,\Gamma(\beta)}\right)^{1/(p-1)}$$

and

$$I(\sigma)\geq \frac{1}{2}k_\infty\,\Gamma(\beta).$$

Hence (2.3) holds for $\tau=\sigma$ and Theorem 2.2 (i) ensures that the global solution $v(t)$ to (4.2) with $v_0=\Phi_\sigma(0)$ tends to ∞ as $t\to\infty$. By Lemmas 2.4 and 2.5, one obtains a contradiction that $v(t)$ blows up in finite time.

(b) The blowup result is trivial. In the following, let $p>1+\frac{2+4\beta}{N}$ and η be a sufficiently small such that

$$\eta<\frac{N}{2} \quad\text{and}\quad \beta+1-\frac{(p-1)N}{2}+(p-1)\eta<0 \tag{4.4}$$

and a continuous function $h(t)$ be defined by

$$h(t)=\frac{1}{t+1}\int_0^t\left(1-\frac{s+1}{t+1}\right)^{\beta-1}\exp(-(t-s))\left(\frac{s+1}{t+1}\right)^{p\eta-\frac{pN}{2}}\mathrm{d}s.$$

It follows from the facts that $h(0)=0$,

$$\begin{aligned}&\frac{1}{t+1}\int_{\frac{1}{2}t}^t\left(1-\frac{s+1}{t+1}\right)^{\beta-1}\exp(-(t-s))\left(\frac{s+1}{t+1}\right)^{p\eta-\frac{pN}{2}}\mathrm{d}s\\ \le&2^{-p\eta+\frac{pN}{2}}\int_{\frac{1}{2}t}^t\left(1-\frac{s+1}{t+1}\right)^{\beta-1}\mathrm{d}\frac{s+1}{t+1}\\ \le&2^{-p\eta+\frac{pN}{2}}\int_{\frac{t+2}{2t+2}}^1(1-r)^{\beta-1}\mathrm{d}r\\ =&2^{-p\eta+\frac{pN}{2}}\frac{1}{\beta}\left(\frac{t}{2t+2}\right)^{\beta}\leqslant 2^{-p\eta+\frac{pN}{2}-\beta}\frac{1}{\beta}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{t+1}\int_0^{\frac{1}{2}t}\left(1-\frac{s+1}{t+1}\right)^{\beta-1}\exp(-(t-s))\left(\frac{s+1}{t+1}\right)^{p\eta-\frac{pN}{2}}\mathrm{d}s\\ \le&\exp\left(-\frac{1}{2}t\right)\int_0^{\frac{1}{2}t}\left(1-\frac{s+1}{t+1}\right)^{\beta-1}\left(\frac{s+1}{t+1}\right)^{p\eta-\frac{pN}{2}}\mathrm{d}\frac{s+1}{t+1}\\ =&\exp\left(-\frac{1}{2}t\right)\int_{\frac{1}{t+1}}^{\frac{t+2}{2t+2}}(1-r)^{\beta-1}r^{p\eta-\frac{pN}{2}}\mathrm{d}r\\ \le&\exp\left(-\frac{1}{2}t\right)(t+1)^{-p\eta+\frac{pN}{2}}\int_0^{\frac{t+2}{2t+2}}(1-r)^{\beta-1}\mathrm{d}r\\ \le&\exp\left(-\frac{1}{2}t\right)(t+1)^{-p\eta+\frac{pN}{2}}\frac{1}{\beta}\left(\frac{t}{2t+2}\right)^{\beta},\end{aligned}$$

that $h(t)$ is bounded in the whole interval $[0,\infty)$, which together with (4.4) implies that

$$(t+1)^{1+\frac{N}{2}-\eta}\int_0^t k(t-s)(s+1)^{p\eta-\frac{pN}{2}}\mathrm{d}s\le k^{\infty}\,(t+1)^{\beta+1-\frac{(p-1)N}{2}+(p-1)\eta}h(t)$$

is bounded in the whole interval $[0,\infty)$. Therefore there exists a sufficiently small $\delta>0$ such that (4.3) and the proof is complete. □

Remark 4.1. For a global integrable kernel, the critical exponents of VIDEs and SLVDEs on bounded domains are same as the local problems. But the critical exponent of SLVDEs on

$\mathbb{R}^N$ is more complex, since the linear coefficient of corresponding VIDEs couples with the initial condition. For the local problem, the critical exponent is $p^* = 1+2/N$ (see [1, 10]), but for the non-local problems, in Theorem 4.1 (i) we only present the upper and below bounds of the critical exponent and the existence of global solutions to (4.1) for

$$1+\frac{2}{N} \leq p \leq 1+\frac{2+2\beta}{N}$$

is still open. For Type II kernels, the critical exponent is $p^* = \infty$ for $\beta \geqslant 1$, but the formulation of the critical exponent for $0<\beta<1$ is unknown.

5. Conclusions and future work

In this paper, the blowup and global existence of the solutions to VIDEs with a linear dissipative term are considered for local and global integrable kernels. It is shown that the blowup behavior depends not only on the global integrability of the kernels but also on the historical effect. For a global integrable kernel, i.e., Type I, the blowup results of VIDEs and SLVDEs on bounded domains are similar to the local problems. For Type II kernels with $\beta \geq 1$, the blowup results of VIDEs, SLVDEs on bounded domains and $\mathbb{R}^N$ are totally different from the local problems since the historical information always effects the future behavior. However, there are still two open problems to be solved, given by

(i) the existence of global solutions to (1.1) for Type II kernel with $0<\beta<1$;

(ii) the exact value of the critical exponent of SLVDEs (4.1) on $\mathbb{R}^N$ for Type I kernels.

In the future, we not only focus on the theoretical analysis for the two problems above, but also construct the efficient numerical scheme for numerical study of blowup in finite time, see [4, 5, 7, 31, 34, 40, 41]. In fact, it is very important to investigate the blowup behaviors of solutions in a numerical viewpoint.

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