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THE SHARPNESS OF KUZNETSOV'S $O(\sqrt{\Delta x})$ L^1-ERROR ESTIMATE FOR MONOTONE DIFFERENCE SCHEMES

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ABSTRACT. We derive a lower error bound for monotone difference schemes to the solution of the linear advection equation with BV initial data. A rigorous analysis shows that for any monotone difference scheme the lower L^1-error bound is $O(\sqrt{\Delta x})$, where Δx is the spatial stepsize.

1. INTRODUCTION

Conservative monotone difference schemes, which include the Lax-Friedrichs scheme, Godunov's scheme, and the Engquist-Osher scheme [3], play an important role in both theoretical analysis and practical computation for hyperbolic conservation laws. From the viewpoint of numerical computation, accuracy and error bounds are of particular interest. Harten, Hyman, and Lax [4] pointed out that the monotone difference schemes are of at most first-order accuracy and Kuznetsov [6] showed that their (upper) L^1-error bound is $O(\sqrt{\Delta x})$ as Δx goes to zero, where Δx is the spatial stepsize.

In this paper we demonstrate that *all* monotone schemes applied to linear first-order conservation laws in one dimension have a best possible $\sqrt{\Delta x}$ rate of convergence when applied to discontinuous data.

A $(p+q+1)$-point conservative finite difference scheme

$$\begin{aligned} v_j^{n+1} &= H(v_{j-p}^n, v_{j-p+1}^n, \dots, v_{j+q}^n) \\ &= v_j^n - \lambda[\bar{f}(v_{j-p+1}^n, \dots, v_{j+q}^n) - \bar{f}(v_{j-p}^n, \dots, v_{j+q-1}^n)] \end{aligned} \tag{1.1}$$

is said to be monotone if H is a monotone nondecreasing function of each of its arguments, and is said to be consistent with a scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad x \in \mathbb{R}, \quad t > 0, \tag{1.2a}$$

$$u|_{t=0} = u_0(x), \tag{1.2b}$$

if the numerical flux $\bar{f}$ satisfies

$$\bar{f}(w, \dots, w) = f(w), \tag{1.3}$$

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where $\lambda = \Delta t/\Delta x = \text{const}$, p and q are given nonnegative integers, and

$$v_j^0 = T_{\Delta x}(u_0)(x_j) = \frac{1}{\Delta x}\int_{x_j-\Delta x/2}^{x_j+\Delta x/2} u_0(x)\,dx\,, \quad x_j = j\Delta x. \tag{1.4}$$

Stability, convergence, and error estimates for monotone difference schemes can be found in [2], [6], and [8].

It is easy to see that if (1.2) is the linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad (a = \text{const})\,, \tag{1.5a}$$

$$u|_{t=0} = u_0(x)\,, \tag{1.5b}$$

then a linear $(p+q+1)$-point monotone difference scheme is of the form

$$v_j^{n+1} = \sum_{s=-p}^{q} a_s v_{j+s}^n\,, \tag{1.6}$$

where

$$a_s \geq 0 \quad \text{for} \quad s = -p\,, \dots\,, q. \tag{1.7}$$

The consistency condition (1.3) implies that

$$\sum_{s=-p}^{q} a_s = 1 \tag{1.8}$$

and

$$\sum_{s=-p}^{q} s a_s = -\lambda a. \tag{1.9}$$

Denote

$$\mathscr{S}_1 = \{s \mid a_s > 0\} \quad \text{and} \quad \mathscr{S}_1 \backslash s_0 = \{s \mid s \in \mathscr{S}_1 \text{ and } s \neq s_0\}\,,$$

where s_0 is an index which satisfies $a_{s_0} = \max_{s\in\mathscr{S}_1} a_s$. For the analysis of (1.6), we introduce

$$v_{\Delta x}(x\,,\,t) = v_j^n \quad \text{for} \quad (x\,,\,t) \in [x_{j-1/2}\,,\,x_{j+1/2}) \times [t_n\,,\,t_{n+1})\,, \tag{1.10}$$

where $x_{j+1/2} = (j+1/2)\Delta x$, $t_n = n\Delta t$, $n \in \mathbb{N}$ and $j \in \mathbb{Z}$.

In this paper we will prove the following theorem.

Theorem. *Any monotone difference scheme* (1.6)*, which is consistent with* (1.5)*, has the following L^1-error bounds: for any $M > 0$ and $t > 0$*

$$\begin{aligned} c(p\,,\,q)\,M\sum_{s\neq s_0}\sqrt{a_s}\sqrt{\frac{t}{\lambda}}\sqrt{\Delta x} &\leq \sup_{|u_0|_{\mathrm{BV}}\leq M} \|v_{\Delta x}(\cdot\,,\,t) - u(\cdot\,,\,t)\|_{L^1(\mathbb{R})} \\ &\leq M\left[2\sqrt{\sum_s s^2 a_s - \lambda^2 a^2}\sqrt{\frac{t}{\lambda}}\sqrt{\Delta x} + \Delta x\right]\,, \end{aligned} \tag{1.11}$$

provided that Δx is small enough. Here, $c(p\,,\,q) > 0$ is a constant depending only on p and q, $u(x\,,\,t)$ is the solution of (1.5) *and*

$$|u_0|_{\mathrm{BV}} = \sup_{|h|\neq 0}\frac{1}{|h|}\|u_0(\cdot + h) - u_0(\cdot)\|_{L^1(\mathbb{R})}. \tag{1.12}$$

Remark 1. The lower bound of (1.11) indicates that, except in a trivial case ($a_s = 0$ for $s \neq s_0$, a pure translation), any monotone difference scheme applied to a linear advection equation has an L^1 convergence order of at most one-half in the class BV of solutions.

Remark 2. Several authors have studied error estimates for difference schemes to first-order hyperbolic equations by using Fourier methods (see [1,5] and references therein). However, to the best of our knowledge, none of these includes a lower error bound for monotone difference schemes in the presence of discontinuous initial data.

2. Some lemmas

A key step in proving the lower error bound of (1.11) is to get a precise lower bound of a sum of terms with multi-indices running over a set J_n (see the right-hand side of (3.7)). Lemma 2 and Lemma 4 below provide a precise lower cardinality of J_n and precise lower bounds of the summand terms, respectively. Consequently, they yield the desired lower bound. Lemma 1 gives a multinomial equality, while Lemma 3 is a generalized de Moivre theorem [7], which gives an asymptotic formula for the multinomial probabilities.

Lemma 1. *If* $\mathbf{a} = (a_{-p}, \dots, a_q) \in \mathbb{R}^{p+q+1}$ *satisfies* (1.7) *and* (1.8), *then*

$$\sum_{|\alpha|=n} C_n(\alpha)\, \mathbf{a}^\alpha = 1, \tag{2.1}$$

where $\alpha = (\alpha_{-p}, \dots, \alpha_q) \in \mathbb{N}^{p+q+1}$, $|\alpha| = \alpha_{-p} + \dots + \alpha_q$, $\mathbf{a}^\alpha = a_{-p}^{\alpha_{-p}} \cdots a_q^{\alpha_q}$ *and* $C_n(\alpha)$ *is the multinomial coefficient defined by*

$$C_n(\alpha) = \frac{n!}{\alpha_{-p}! \cdots \alpha_q!}. \tag{2.2}$$

Proof. The above equality can be easily derived, and the proof is omitted. □

Denote

$$I_n = \{\alpha \,|\, |\alpha| = n, \quad \alpha \in \mathbb{N}^{p+q+1}\}, \tag{2.3}$$

(2.4a)

$$\mathbb{J}_n = \Big\{\xi \,\Big|\, \xi \in \mathbb{R}^{p+q+1},\ \xi_s = a_s n + y_s\sqrt{n}\ (s \in \mathscr{S}_1) \text{ and } \xi_s = 0\ (s \notin \mathscr{S}_1);$$
$$y_s \in [\bar{y}_s, \bar{y}_s + \Delta y_s]\ (s \in \mathscr{S}_1 \backslash s_0) \text{ and } \sum_{s\in\mathscr{S}_1} y_s = 0\Big\}$$

and

$$J_n = \{\alpha \,|\, \alpha \in \mathbb{J}_n \cap \mathbb{N}^{p+q+1}\}, \tag{2.4b}$$

where $\mathbf{a} = (a_{-p}, \dots, a_q)$ satisfies (1.7) and (1.8).

Lemma 2. *For sufficiently large n,*

$$|J_n| \geq |\mathscr{S}_1|/2 \Big(\prod_{s\in\mathscr{S}_1\backslash s_0} |\Delta y_s|\Big) n^{(|\mathscr{S}_1|-1)/2}, \tag{2.5}$$

where $|J_n|$ *and* $|\mathscr{S}_1|$ *are the cardinalities of* J_n *and* $\mathscr{S}_1$, *respectively.*

Proof. Since the set J_n consists of all lattice points of $\mathbb{J}_n$, we have

$$\lim_{n\to\infty}\frac{|J_n|}{\operatorname{meas}(\mathbb{J}_n)}=1,$$

and thus for sufficiently large n,

$$|J_n|\geq\frac{1}{2}\operatorname{meas}(\mathbb{J}_n).$$

But from calculus we find that

$$\operatorname{meas}(\mathbb{J}_n)=|\mathscr{S}_1|\left(\prod_{s\in\mathscr{S}_1\backslash s_0}|\Delta y_s|\right)n^{(|\mathscr{S}_1|-1)/2}. \quad \square$$

Lemma 3. *If* $\alpha\in J_n$, *then*

$$C_n(\alpha)\,\mathbf{a}^\alpha\sim\frac{1}{(2\pi n)^{(|\mathscr{S}_1|-1)/2}\prod_{s\in\mathscr{S}_1}\sqrt{a_s}}\exp\left(-\sum_{s\in\mathscr{S}_1}\frac{{y_s}^2}{2a_s}\right) \tag{2.6}$$

uniformly for $y_s\in[\bar{y}_s,\ \bar{y}_s+\Delta y_s]$ $(s\in\mathscr{S}_1\backslash s_0)$, *i.e., as* $n\to\infty$,

$$\sup_{\substack{y_s\in[\bar{y}_s,\ \bar{y}_s+\Delta y_s]\\(s\in\mathscr{S}_1\backslash s_0)}}\left|C_n(\alpha)\,\mathbf{a}^\alpha\Big/\left\{\frac{1}{(2\pi n)^{(|\mathscr{S}_1|-1)/2}\prod_{s\in\mathscr{S}_1}\sqrt{a_s}}\exp\left(-\sum_{s\in\mathscr{S}_1}\frac{{y_s}^2}{2a_s}\right)\right\}-1\right|\to 0.$$

Proof. The proof depends on Stirling's formula

$$m!=\sqrt{2\pi m}\,e^{-m}m^m(1+R(m)),$$

where $R(m)\to 0$ as $m\to\infty$. Since $\alpha_s=a_sn+y_s\sqrt{n}\to\infty$ as $n\to\infty$ for $s\in\mathscr{S}_1$, we have, by using the definition (2.2) and Stirling's formula,

$$\begin{aligned}C_n(\alpha)\,\mathbf{a}^\alpha&=\frac{\sqrt{2\pi n}\,e^{-n}n^n\mathbf{a}^\alpha(1+R(n))}{(2\pi)^{|\mathscr{S}_1|/2}\left(\prod_{s\in\mathscr{S}_1}\sqrt{\alpha_s}\right)e^{-n}\alpha^\alpha\prod_{s\in\mathscr{S}_1}(1+R(\alpha_s))}\\&=\frac{1+R(n)}{(2\pi n)^{(|\mathscr{S}_1|-1)/2}\prod_{s\in\mathscr{S}_1}\sqrt{a_s+y_s/\sqrt{n}}}\\&\quad\times\frac{1}{\prod_{s\in\mathscr{S}_1}(1+y_s/(a_s\sqrt{n}))^{a_sn+y_s\sqrt{n}}\prod_{s\in\mathscr{S}_1}(1+R(\alpha_s))},\end{aligned}$$

where the last equality follows from the fact that

$$\alpha^\alpha=\mathbf{a}^\alpha\prod_{s\in\mathscr{S}_1}n^{\alpha_s}\prod_{s\in\mathscr{S}_1}(1+y_s/(a_s\sqrt{n}))^{\alpha_s}=\mathbf{a}^\alpha n^n\prod_{s\in\mathscr{S}_1}(1+y_s/(a_s\sqrt{n}))^{a_sn+y_s\sqrt{n}}.$$

Here we have used (1.8) and $\sum_{s\in\mathscr{S}_1}y_s=0$. In order to prove (2.6), we need to verify the following formulas:

$$\sqrt{a_s+\frac{y_s}{\sqrt{n}}}\sim\sqrt{a_s} \tag{2.7}$$

and

$$F(n)=\prod_{s\in\mathscr{S}_1}\left(1+\frac{y_s}{a_s\sqrt{n}}\right)^{a_sn+y_s\sqrt{n}}\sim\exp\left(\sum_{s\in\mathscr{S}_1}\frac{{y_s}^2}{2a_s}\right). \tag{2.8}$$

The first, (2.7), is obvious. Now we prove (2.8). Since

$$\ln F(n) = \sum_{s\in\mathscr{S}_1} (a_s n + y_s\sqrt{n})\ln\Big(1 + \frac{y_s}{a_s\sqrt{n}}\Big),$$

we find, by applying Taylor's formula to $\ln(1 + y_s/(a_s\sqrt{n}))$, that

$$\begin{aligned}\ln F(n) &= \sum_{s\in\mathscr{S}_1} (a_s n + y_s\sqrt{n})\left[\frac{y_s}{a_s\sqrt{n}} - \frac{1}{2}\Big(\frac{y_s}{a_s\sqrt{n}}\Big)^2 + \frac{1}{3}\Big(\frac{\tilde{y}_s}{a_s\sqrt{n}}\Big)^3\right] \\ &= \sum_{s\in\mathscr{S}_1}\left[y_s\sqrt{n} + \frac{{y_s}^2}{2a_s} + \frac{1}{a_s\sqrt{n}}\Big(-\frac{{y_s}^3}{2a_s} + \frac{{\tilde{y}_s}^3}{3{a_s}^2} + \frac{y_s{\tilde{y}_s}^3}{3\sqrt{n}{a_s}^3}\Big)\right],\end{aligned}$$

where $\tilde{y}_s \in (0, y_s)$ and $y_s \in [\bar{y}_s, \bar{y}_s + \Delta y_s]$. Now using $\sum_{s\in\mathscr{S}_1} y_s = 0$, we obtain that for sufficiently large n

$$\ln F(n) = \sum_{s\in\mathscr{S}_1}\Big(\frac{{y_s}^2}{2a_s} + O\Big(\frac{1}{\sqrt{n}}\Big)\Big).$$

This verifies (2.8) and hence (2.6) is proved. □

Lemma 4. *If parameters* $\bar{y}_s$ *and* Δy_s, *defined in* $\mathbb{J}_n$, *are given by*

$$\bar{y}_s = \begin{cases}\dfrac{\sqrt{a_s}}{s - s_0}, & s - s_0 > 0,\\[2mm] \dfrac{2\sqrt{a_s}}{s - s_0}, & s - s_0 < 0,\end{cases} \qquad \Delta y_s = \frac{\sqrt{a_s}}{|s - s_0|} \quad (s \in \mathscr{S}_1\backslash s_0) \tag{2.9}$$

and $\mathbf{a}$ *satisfies* (1.7)–(1.9), *then for sufficiently large* n,

$$\min_{\alpha\in J_n} C_n(\alpha)\,\mathbf{a}^\alpha \ge \frac{0.5}{(2\pi n)^{(|\mathscr{S}_1|-1)/2}\prod_{s\in\mathscr{S}_1}\sqrt{a_s}}\exp\{-2(|\mathscr{S}_1| - 1)|\mathscr{S}_1|\}, \tag{2.10}$$

$$\min_{\alpha\in J_n}\Big|\sum_{s=-p}^{q} s\alpha_s + \lambda a n\Big| \ge \sqrt{n}\sum_{s\ne s_0}\sqrt{a_s}. \tag{2.11}$$

Proof. By using Lemma 3, we have for sufficiently large n,

$$C_n(\alpha)\,\mathbf{a}^\alpha \ge \frac{0.5}{(2\pi n)^{(|\mathscr{S}_1|-1)/2}\prod_{s\in\mathscr{S}_1}\sqrt{a_s}}\exp\Big(-\sum_{s\in\mathscr{S}_1}\frac{{y_s}^2}{2a_s}\Big). \tag{2.12}$$

Since $y_s \in [\bar{y}_s, \bar{y}_s + \Delta y_s]$ for $s \ne s_0$, we have, on account of (2.8),

$$\sum_{s\in\mathscr{S}_1}\frac{{y_s}^2}{2a_s} \le \frac{{y_{s_0}}^2}{2a_{s_0}} + \sum_{s\in\mathscr{S}_1\backslash s_0}\frac{2}{(s - s_0)^2} \le \frac{{y_{s_0}}^2}{2a_{s_0}} + 2(|\mathscr{S}_1| - 1). \tag{2.13}$$

On the other hand, from $\sum_{s\in\mathscr{S}_1} y_s = 0$ and $a_{s_0} = \max_s a_s$, we see that

$${y_{s_0}}^2 = \Big(\sum_{s\in\mathscr{S}_1\backslash s_0} y_s\Big)^2 \le \Big(\sum_{s\in\mathscr{S}_1\backslash s_0}\frac{2\sqrt{a_s}}{|s - s_0|}\Big)^2 \le 4a_{s_0}(|\mathscr{S}_1| - 1)^2,$$

or

$$\frac{{y_{s_0}}^2}{2a_{s_0}} \le 2(|\mathscr{S}_1| - 1)^2.$$

Substituting this into (2.13) gives

$$\sum_{s\in\mathscr{S}_1} \frac{{y_s}^2}{2a_s} \le 2(|\mathscr{S}_1| - 1)|\mathscr{S}_1|,$$

and combining this with (2.12) yields (2.10).

We now turn to (2.11). By using (2.4b), (1.9), and (2.9) , we have that

$$\begin{aligned}\left|\sum_s s\alpha_s + \lambda an\right| &= \sqrt{n}\left|\sum_{s\in\mathscr{S}_1} sy_s\right| = \sqrt{n}\left|\sum_{s\in\mathscr{S}_1\backslash s_0} (s-s_0)y_s\right| \\ &\ge \sqrt{n}\sum_{s\neq s_0}\sqrt{a_s}, \quad \forall\alpha\in J_n\subset\mathbb{J}_n.\end{aligned}$$

This concludes the proof of (2.11). □

3. Proof of the main theorem

Proof of Theorem. By using (1.6) repeatedly for $n := 0, \ldots, n-1$, we can express v_j^n in terms of the initial data v_j^0 as

$$v_j^n = \sum_{\alpha\in I_n} C_n(\alpha)\,\mathbf{a}^\alpha v^0_{j+\sum_s s\alpha_s}, \tag{3.1}$$

where $\sum_s$ is the sum over s from $-p$ to q. It is also known that the solution of (1.5) is of the form

$$u(x, t) = u_0(x-at). \tag{3.2}$$

Thus, we have

$$v_j^n - u(x, t_n) = \sum_{\alpha\in I_n} C_n(\alpha)\,\mathbf{a}^\alpha\left(v^0_{j+\sum_s s\alpha_s} - u_0(x-at_n)\right), \tag{3.3}$$

where we have used the equality (2.1). The upper error bound of (1.11) is a special case of the error estimate for scalar conservation laws [6], but the coefficient given here is more accurate. We will not present the proof here.

In order to prove the lower error bound of (1.11) , we only need to verify that the first inequality holds for the Riemann initial data

$$u_0(x) = \begin{cases} M/2 & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -M/2 & \text{for } x < 0. \end{cases} \tag{3.4}$$

From (1.4) and (3.4), we see that

$$v^0_{j+\sum_s s\alpha_s} = u_0\left(x_j + \Delta x\sum_s s\alpha_s\right),$$

and hence

$$
\begin{aligned}
&v^0_{j+\sum_s s\alpha_s} - u_0(x_j - at_n) \\
&\quad = \begin{cases}
-M, & -\sum_s s\alpha_s > j > \lambda an, \\
-M/2, & -\sum_s s\alpha_s = j > \lambda an \text{ or } -\sum_s s\alpha_s > j = \lambda an, \\
0, & j > (\lambda an) \vee \left(-\sum_s s\alpha_s\right) \text{ or } j < (\lambda an) \wedge \left(-\sum_s s\alpha_s\right), \\
M/2, & -\sum_s s\alpha_s = j < \lambda an \text{ or } -\sum_s s\alpha_s < j = \lambda an, \\
M, & -\sum_s s\alpha_s < j < \lambda an,
\end{cases}
\end{aligned}
$$

where $c \vee d = \max\{c, d\}$ and $c \wedge d = \min\{c, d\}$. Substituting this into (3.3) yields for $j \neq \lambda an$

(3.5)

$$
\begin{aligned}
v^n_j - u(x_j, t_n) &= \sum_{\alpha \in I_n} C_n(\alpha)\, \mathbf{a}^\alpha (v^0_{j+\sum_s s\alpha_s} - u_0(x_j - at_n)) \\
&= \begin{cases}
-M \sum_{\{\alpha \in I_n\} \cap \{j < -\sum_s s\alpha_s\}} C_n(\alpha)\, \mathbf{a}^\alpha \\
\qquad -M/2 \sum_{\{\alpha \in I_n\} \cap \{j = -\sum_s s\alpha_s\}} C_n(\alpha)\, \mathbf{a}^\alpha & \text{for } j > \lambda an, \\
M \sum_{\{\alpha \in I_n\} \cap \{j > -\sum_s s\alpha_s\}} C_n(\alpha)\, \mathbf{a}^\alpha \\
\qquad +M/2 \sum_{\{\alpha \in I_n\} \cap \{j = -\sum_s s\alpha_s\}} C_n(\alpha)\, \mathbf{a}^\alpha & \text{for } j < \lambda an.
\end{cases}
\end{aligned}
$$

For simplicity, we assume $t = t_n$ for some $n\,(= t/\Delta t)$. Since $u(x, t_n)$ is a two-piecewise constant function and $|v^n_j| \le M/2$, we have

$$
\begin{aligned}
\|v_{\Delta x}(\cdot, t) - u(\cdot, t)\|_{L^1(\mathbf{R})} &= \|v_{\Delta x}(\cdot, t_n) - u(\cdot, t_n)\|_{L^1(\mathbf{R})} \\
&\ge \Delta x \sum_j |v^n_j - u(x_j, t_n)| - M\Delta x \\
&\ge \Delta x \sum_{j \neq \lambda an} |v^n_j - u(x_j, t_n)| - M\Delta x.
\end{aligned} \tag{3.6a}
$$

We divide the sum in the last term of (3.6a) into two parts,

$$
\sum_{j \neq \lambda an} |v^n_j - u(x_j, t_n)| = \mathrm{I}_+ + \mathrm{I}_-, \tag{3.6b}
$$

where

$$
\mathrm{I}_+ = \sum_{j > \lambda an} |v^n_j - u(x_j, t_n)|
$$

and

$$
\mathrm{I}_- = \sum_{j < \lambda an} |v^n_j - u(x_j, t_n)|.
$$

Substituting (3.5) into I_+ gives

$$\begin{aligned}
\mathrm{I}_+ &= M \sum_{j>\lambda an} \sum_{\{\alpha\in I_n\}\cap\{j<-\sum_s s\alpha_s\}} C_n(\alpha)\,\mathbf{a}^\alpha \\
&\quad + M/2 \sum_{j>\lambda an} \sum_{\{\alpha\in I_n\}\cap\{j=-\sum_s s\alpha_s\}} C_n(\alpha)\,\mathbf{a}^\alpha \\
&\geq M/2 \sum_{j>\lambda an} \sum_{\{\alpha\in I_n\}\cap\{j\leq-\sum_s s\alpha_s\}} C_n(\alpha)\,\mathbf{a}^\alpha \\
&= M/2 \sum_{\{\alpha\in I_n\}\cap\{\lambda an<-\sum_s s\alpha_s\}} \left(-\sum_s s\alpha_s - [\lambda an]\right) C_n(\alpha)\,\mathbf{a}^\alpha \\
&\geq M/2 \sum_{\{\alpha\in I_n\}\cap\{\lambda an<-\sum_s s\alpha_s\}} \left(-\sum_s s\alpha_s - \lambda an\right) C_n(\alpha)\,\mathbf{a}^\alpha,
\end{aligned}$$

where $[\eta]$ means the largest integer less than or equal to η. Similarly, we have

$$\mathrm{I}_- \geq M/2 \sum_{\{\alpha\in I_n\}\cap\{\lambda an>-\sum_s s\alpha_s\}} \left(\sum_s s\alpha_s + \lambda an\right) C_n(\alpha)\,\mathbf{a}^\alpha.$$

Adding I_+ and I_- yields

$$\begin{aligned}
\mathrm{I}_+ + \mathrm{I}_- &\geq M/2 \sum_{\{\alpha\in I_n\}\cap\{\lambda an\neq-\sum_s s\alpha_s\}} \left|\sum_s s\alpha_s + \lambda an\right| C_n(\alpha)\,\mathbf{a}^\alpha \\
&= M/2 \sum_{\alpha\in I_n} \left|\sum_s s\alpha_s + \lambda an\right| C_n(\alpha)\,\mathbf{a}^\alpha.
\end{aligned}$$

By the definitions (2.3) and (2.4b) we know that $J_n \subset I_n$ and, furthermore, assume that the parameters in $\mathbb{J}_n$ are given by (2.9), so that (2.10) and (2.11) hold. Then, on account of (2.5), we obtain

(3.7)

$$\begin{aligned}
\mathrm{I}_+ + \mathrm{I}_- &\geq M/2 \sum_{\alpha\in J_n} \left\{\left|\sum_s s\alpha_s + \lambda an\right| C_n(\alpha)\,\mathbf{a}^\alpha\right\} \\
&\geq \frac{M}{2}\,|J_n| \min_{\alpha\in J_n}\left|\sum_s s\alpha_s + \lambda an\right| \min_{\alpha\in J_n} C_n(\alpha)\,\mathbf{a}^\alpha \\
&\geq \frac{M}{2}\,\frac{\sqrt{|\mathscr{S}_1|}}{2}\left(\prod_{s\in\mathscr{S}_1\backslash s_0} \frac{\sqrt{a_s}}{|s-s_0|}\right) n^{(|\mathscr{S}_1|-1)/2}\sqrt{n}\sum_{s\neq s_0}\sqrt{a_s} \\
&\quad \times \frac{0,5}{(2\pi n)^{(|\mathscr{S}_1|-1)/2}\prod_{s\in\mathscr{S}_1}\sqrt{a_s}} \exp\{-2(|\mathscr{S}_1|-1)|\mathscr{S}_1|\} \\
&= \frac{\sqrt{|\mathscr{S}_1|}}{8}\left(\prod_{s\in\mathscr{S}_1\backslash s_0}\frac{1}{|s-s_0|}\right)\frac{\exp\{-2(|\mathscr{S}_1|-1)|\mathscr{S}_1|\}}{(2\pi)^{(|\mathscr{S}_1|-1)/2}\sqrt{a_{s_0}}} M\left(\sum_{s\neq s_0}\sqrt{a_s}\right)\sqrt{n}.
\end{aligned}$$

It follows from $|s - s_0| \le (p+q)$, $1 \le |\mathscr{S}_1| \le (p+q+1)$, and $a_{s_0} \le 1$, that

(3.8)
$$\begin{aligned}\mathrm{I}_+ + \mathrm{I}_- &\ge \frac{1}{8(p+q)^{(p+q)}(2\pi)^{(p+q)/2}} \exp\{-2(p+q)(p+q+1)\} M \left(\sum_{s \ne s_0} \sqrt{a_s}\right)\sqrt{n} \\ &= 2\,c(p, q)\, M \left(\sum_{s\ne s_0} \sqrt{a_s}\right)\sqrt{n}.\end{aligned}$$

We can see that $c(p, q) > 0$ is a constant which depends only on p and q. Since the Riemann initial data (3.4) satisfies $|u_0(\cdot)|_{\mathrm{BV}(\mathbf{R})} = M$, combining (3.6) and (3.8) yields the desired lower error bound, provided

$$\Delta x \le \left[c(p, q) \sum_{s \ne s_0} \sqrt{a_s}\right]^2 t/\lambda.$$

This completes the proof of the main theorem. □

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