Journal of Computational Mathematics, Vol.22, No.2, 2004, 287–298.

A FAST NUMERICAL METHOD FOR INTEGRAL EQUATIONS OF THE FIRST KIND WITH LOGARITHMIC KERNEL USING MESH GRADING [*1)]

Qi-yuan Chen
(*Department of Applied Mathematics, Beijing University of Technology, Beijing 100081, China*)

Tao Tang
(*Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong*)
(*E-mail: ttang@hkbu.edu.hk*)

Zhen-huan Teng
(*LMAM & School of Mathematical Sciences, Peking University, Beijing 100871, China*)
(*E-mail: tengzh@math.pku.edu.cn*)

Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

The aim of this paper is to develop a fast numerical method for two-dimensional boundary integral equations of the first kind with logarithm kernels when the boundary of the domain is smooth and closed. In this case, the use of the conventional boundary element methods gives linear systems with dense matrix. In this paper, we demonstrate that the dense matrix can be replaced by a sparse one if appropriate graded meshes are used in the quadrature rules. It will be demonstrated that this technique can increase the numerical efficiency significantly.

Mathematics subject classification: 65R20, 45L10
Key words: Integral equations, Mesh grading, Fast numerical method.

1. Introduction

Consider the first kind integral equation with logarithmic kernel:

$$-\int_{\Gamma} \log|x-y| u(y) d\nu_y = f(x), \quad x = (x_1, x_2) \in \Gamma \tag{1.1}$$

where $\Gamma \subset \mathbf{R}^2$ is a smooth and closed curve in the plane, u is a unknown function, $f(x)$ is a given function, $|x-y|$ is the Euclidean distance and $d\nu_y$ is the element of arc length. It should be pointed out that the well-posedness of (1.1) may be subject to some additional conditions, see, e.g., [8]. Hsiao and Wendland [8] were the first to give a rigorous error analysis for the Galerkin method applied to this boundary integral equation where Γ is a smooth and closed curve. Later Costabel and Stephan [7] extended this analysis to treat the more difficult case where Γ is a polygon. More generally, Sloan and Spence [11] have investigated (1.1) for Γ either a closed contour or an open arc. In this case, singularities in the unknown function $u(x)$ are produced at corners and ends (see, [7, 11]).

[*] Received January 31, 2004.
[1)] The research of the third author was supported by the Special Funds for Major State Basic Research Projects of China, and the research of the second author was supported by the Hong Kong Research Grants Council and the International Research Team on Complex System of Chinese Academy of Sciences.

In this work we assume that the boundary Γ is a simple closed smooth curve with a twicely continuously differentiable parametrization. More precisely, let Γ be parameterized by the arclength,

$$\nu : [0, L] \to \Gamma,$$

where L is the length of Γ, $|d\nu/ds| = 1$ and $\nu(\sigma)$ is a periodic function with period of L. Then the integral equation (1.1) is transformed into the following form

$$-\int_{-L/2}^{L/2} \log|\nu(s) - \nu(\sigma)| u(\nu(\sigma)) d\sigma = f(\nu(s)), \quad s \in [-L/2, L/2]. \tag{1.2}$$

The conventional way in solving the equation (1.2) is to use n collocation points to obtain n collocation equations. Then for each fixed s the integral in (1.2) will be approximated by an appropriate quadrature rule using the information on the n collocation points. This approach leads to a linear system whose matrix is a full matrix. In this work, we will approximate the integral term by using a subset of the n collocation points. More precisely, we consider the case when the unknown function u is reasonably smooth and the curve Γ is smooth and closed. In this case, some suitable graded-meshes can be used as the quadrature points to handle the logarithmic kernel, which yields a linear system whose matrix is sparse. The graded-mesh concept was proposed by Rice [10]. It was then used to improve the formal order of convergence when solutions have weak singularity, see, e.g., [5, 14] for boundary integral equations and [2, 12] for weakly singular Volterra integral equations. In [2], the analysis of graded mesh convergence was specifically based on the fact that the solution was non-smooth (i.e. having an unbounded derivative at $t = 0$). A much more general analysis of graded mesh collocation for weakly singular Volterra integral equations (including logarithmic kernels) can be found in [3]. It is noticed that some other types of numerical methods have been proposed to handle the singularities in integrals equations [4, 9].

In this work the solution of the integral equation is assumed to be regular. In the earlier works for handling solution singularity such as [2, 13] a graded mesh is used as the collocation points. However, with a smooth solution we just need to use a uniform mesh for the collocation points. A graded mesh which is a *subset* of the uniform mesh will be employed to efficiently evaluate the integrals. We will show that the proposed approach can not only preserve the formal rate of convergence but also save a significant amount of CPU time. We point out that similar efforts have been made by Chen et al. [6] who developed a fast collocation method for integral equations with weakly singular kernels by constructing multiscale interpolating functions and collocation functionals that have vanishing moments.

A related integral equation to (1.1) is

$$-\int_\Gamma \log|x - y| u(y) d\nu_y + \omega = f(x), \ \ x \in \Gamma, \quad \int_\Gamma u(y)\, d\nu_y = b, \tag{1.3}$$

where the function u and the scalar ω are unknowns, and b is a given real number. It is known that the solution of interior or exterior Dirichlet problem for Laplace equation in two-dimension can be reduced to the integral equations (1.3).

2. Numerical Approximations for Singular Integrals

First we describe several numerical quadrature methods for the weakly singular integration

$$I = \int_{-L/2}^{L/2} G(\sigma) g(\sigma) d\sigma = Kg, \tag{2.1}$$

where $g(\sigma)$ is a smooth function, $G(\sigma)$ is a singular kernel satisfying

$$|G^{(i)}(\sigma)| \le \begin{cases} C|\sigma|^{-i}, & i = 1, 2, \cdots \\ C|\log|\sigma||, & i = 0. \end{cases} \tag{2.2}$$

Both $G(\sigma)$ and $g(\sigma)$ are periodic functions with period L. Let

$$A := \{\alpha_{-n/2}, \alpha_{-n/2+1}, \cdots, \alpha_{n/2}\}, \qquad \alpha_i = \frac{2i}{n}\frac{L}{2} \tag{2.3}$$

be a *uniform mesh* of the interval $[-L/2, L/2]$, where n is an even integer. Let

$$B := \{\beta_{-m/2}, \beta_{-m/2+1}, \cdots, \beta_{m/2}\}, \quad \beta_l = \mathrm{sgn}(l)\left(\frac{2|l|}{m}\right)^q \frac{L}{2} \tag{2.4}$$

be a *graded mesh* with a grading exponent q for the interval $[-L/2, L/2]$, where m is an even integer.

A standard approximation for the singular integral (2.1) is to use the mid-point type rule:

$$I_n^{(1)} = \sum_{i=-n/2}^{n/2-1} \int_{\alpha_i}^{\alpha_{i+1}} \frac{1}{2} G(\sigma)\Big[g(\alpha_i) + g(\alpha_{i+1})\Big]\, d\sigma, \tag{2.5}$$

which has the following equivalent form:

$$I_n^{(1)} = \sum_{i=-n/2}^{n/2} \tilde{\lambda}_i g(\alpha_i), \tag{2.6}$$

where the weights $\{\tilde{\lambda}_i\}_{-n/2}^{n/2}$ are given by

$$\tilde{\lambda}_{-n/2} = \frac{1}{2}\int_{\alpha_{-n/2}}^{\alpha_{-n/2+1}} G(\sigma)\, d\sigma, \qquad \tilde{\lambda}_{n/2} = \frac{1}{2}\int_{\alpha_{n/2-1}}^{\alpha_{n/2}} G(\sigma)\, d\sigma$$

$$\tilde{\lambda}_i = \frac{1}{2}\int_{\alpha_{i-1}}^{\alpha_{i+1}} G(\sigma)\, d\sigma\,, \qquad -n/2+1 \le i \le n/2-1.$$

Since G and g are L-periodic, (2.6) can be written in the following equivalent form

$$I_n^{(1)} = \sum_{i=-n/2+1}^{n/2} \lambda_i^{(1)} g(\alpha_i), \tag{2.7}$$

where the weights $\{\lambda_i^{(1)}\}_{-n/2+1}^{n/2}$ are given by

$$\lambda_i^{(1)} = \frac{1}{2}\int_{\alpha_{i-1}}^{\alpha_{i+1}} G(\sigma)\, d\sigma\,, \qquad -n/2+1 \le i \le n/2. \tag{2.8}$$

In fact, the formula $I_n^{(1)}$ is the zero-order boundary element approximation to the weakly singular integration (2.1). The theory for the boundary element method gives the following error estimate (see, e.g., [1]).

Lemma 1. *Assume that $g \in C^2[-L/2, L/2]$ and G satisfies (2.2). If the numerical approximation $I_n^{(1)}$ is given by (2.5), then*

$$\left|I - I_n^{(1)}\right| \le C\frac{\log n}{n^2},$$

where C is a constant independent of n.

The formula $I_n^{(1)}$ requires the exact evaluation (or with higher accuracy) for the integrals of G, see (2.8). To avoid doing this, the following full discretization is proposed:

$$I_n^{(2)} = {\sum_{i=-n/2}^{n/2-1}}' \frac{1}{2}\Big[G(\alpha_i)g(\alpha_i) + G(\alpha_{i+1})g(\alpha_{i+1})\Big](\alpha_{i+1} - \alpha_i), \tag{2.9}$$

where and in what follows $\sum'$ means that $G(0)$ in the summation will be set to 0, which is due to the possible unboundedness of $G(0)$ as indicated in (2.2). The summation, by noting that G and g are L-periodic, can be written in the following equivalent form

$$I_n^{(2)} = \sum_{i=-n/2+1}^{n/2}{}' \lambda_i^{(2)} g(\alpha_i), \tag{2.10}$$

where the weights $\{\lambda_i^{(2)}\}_{-n/2+1}^{n/2}$ are given by

$$\lambda_i^{(2)} = \frac{1}{2} G(\alpha_i)(\alpha_{i+1} - \alpha_{i-1}) \qquad -n/2+1 \le i \le n/2. \tag{2.11}$$

It is noted that the approximation $I_n^{(2)}$ is the trapezoidal method for the singular integral (2.1). A standard analysis gives the following estimate (see, e.g., [1]):

Lemma 2. *Assume that $g \in C^2[-L/2, L/2]$ and G satisfies (2.2). If the numerical approximation $I_n^{(2)}$ is given by (2.9), then*

$$\left| I - I_n^{(2)} \right| \le C \frac{\log n}{n}, \tag{2.12}$$

where C is a constant independent of n.

Similarly, we can define a numerical approximation to (2.1) by using the graded mesh (2.4):

$$I_m^{(3)} = \sum_{j=-m/2}^{m/2-1}{}' \frac{1}{2} \Big[G(\beta_j) g(\beta_j) + G(\beta_{j+1}) g(\beta_{j+1}) \Big] (\beta_{j+1} - \beta_j), \tag{2.13}$$

which can be written in the following equivalent form

$$I_m^{(3)} = \sum_{j=-m/2+1}^{m/2}{}' \mu_j g(\beta_j), \tag{2.14}$$

where the weights $\{\mu_j\}_{-m/2+1}^{m/2}$ are given by

$$\mu_j = \frac{1}{2} G(\beta_j)(\beta_{j+1} - \beta_{j-1}) \qquad -m/2+1 \le j \le m/2.$$

Lemma 3. *Assume that $g \in C^2[-L/2, L/2]$ and G satisfies (2.2). If the graded mesh in $I_m^{(3)}$ is given by (2.4) with $q = 2$, then*

$$\left| I - I_m^{(3)} \right| \le C \frac{\log m}{m^2}, \tag{2.15}$$

where C is a constant independent of m.

Proof. It can be verified that the truncation error in each graded mesh interval $[\beta_j, \beta_{j+1}]$ has the form:

$$\begin{aligned} E_j &:= \frac{1}{2} \Big(G(\beta_j) g(\beta_j) + G(\beta_{j+1}) g(\beta_{j+1}) \Big) (\beta_{j+1} - \beta_j) - \int_{\beta_j}^{\beta_{j+1}} G(\sigma) g(\sigma) d\sigma \\ &= \frac{1}{12} (\beta_{j+1} - \beta_j)^3 \left(G(\tilde{\beta}_j) g(\tilde{\beta}_j) \right)'', \qquad \tilde{\beta}_j \in (\beta_j, \beta_{j+1}) \end{aligned} \tag{2.16}$$

provided that $Gg \in C^{(2)}[\beta_j, \beta_{j+1}]$. Applying the assumptions (2.2) and (2.4) to the right hand side of (2.16) gives

$$|E_j| \le C \left| \frac{j+\gamma}{m} \right|^3 \left(\frac{1}{m} \right)^3 \left(\frac{m}{j+\theta} \right)^2 \le C \frac{|j|}{m^4}, \tag{2.17}$$

where $0 < \text{sgn}(j)\gamma < 1$ and $0 < \text{sgn}(j)\theta < 1$. Therefore,

$$\begin{aligned} |I - I_m^{(3)}| &\leq \left| \int_{[-L/2,L/2]\setminus[\beta_{-1},\beta_1]} G(\sigma)g(\sigma)d\sigma - I_m^{(3)} \right| + \left| \int_{\beta_{-1}}^{\beta_1} G(\sigma)g(\sigma)\, d\sigma \right| \\ &\leq \frac{C}{m^4} \sum_{j=-m/2}^{m/2} |j| + C \int_{-(2/m)^2 L/2}^{(2/m)^2 L/2} |\log \sigma|\, d\sigma \\ &\leq \frac{C}{m^2} + C\frac{\log m}{m^2}, \end{aligned}$$

which gives the desired estimate (2.15).

An immediate observation from Lemma 3 is that although $I_n^{(2)}$ and $I_m^{(3)}$ are both based on the trapezoidal methods the latter one improves the efficiency significantly. This can be seen from the estimate (2.12) and the following estimate:

$$\left| I - I_{\sqrt{n}}^{(3)} \right| \leq C\frac{\log n}{n}. \tag{2.18}$$

This result implies that to achieve the same accuracy only $\mathcal{O}(\sqrt{n})$ integration points are needed for the trapezoidal-type methods with an appropriate graded mesh. It is seen from (2.18) that with $\mathcal{O}(\sqrt{n})$ grid points only first-order accuracy can be obtained. To improve this, we can use a Simpson-type method with a graded mesh:

$$I_m^{(4)} = \sum_{j=-m/2}^{m/2-1}{}' \frac{1}{6}\Big[G(\beta_j)g(\beta_j) + 4G(\beta_{j+1/2})g(\beta_{j+1/2}) + G(\beta_{j+1})g(\beta_{j+1})\Big](\beta_{j+1} - \beta_j), \tag{2.19}$$

where $\beta_{j+1/2} = (\beta_j + \beta_{j+1})/2$. The above approximation can be re-written as

$$I_m^{(4)} = \sum_{l=-m}^{m}{}' \mu_j g(\beta_{l/2}), \tag{2.20}$$

where the weights $\{\mu_l\}_{-m}^m$ are given by

$$\begin{aligned} \mu_{-m} &= \tfrac{1}{6}G(\beta_{-m/2})\big(\beta_{-m/2+1} - \beta_{-m/2}\big); && \mu_m = \frac{1}{6}G(\beta_{m/2})(\beta_{m/2} - \beta_{m/2-1}); \\ \mu_l &= \tfrac{1}{6}G(\beta_{l/2})(\beta_{l/2+1} - \beta_{l/2-1}), && l = \text{even number}; \\ \mu_l &= \tfrac{2}{3}G(\beta_{l/2})(\beta_{(l+1)/2} - \beta_{(l-1)/2}), && l = \text{odd number}. \end{aligned}$$

Lemma 4. *Assume that $g \in C^4[-L/2, L/2]$ and G satisfies* (2.2)*. If the numerical approximation $I_m^{(4)}$ is given by* (2.19) *and the corresponding graded mesh is given by* (2.4) *with $q = 4$, then we have*

$$\left| I - I_m^{(4)} \right| \leq C\frac{\log m}{m^4}, \tag{2.21}$$

where C is a constant independent of m.

The proof for Lemma 4 is similar to that for Lemma 3. An immediate application of the above estimate is the following:

$$\left| I - I_{\sqrt{n}}^{(4)} \right| \leq C\frac{\log n}{n^2}. \tag{2.22}$$

We close this section by pointing out that the results in Lemmas 3 and 4 also hold for the graded meshes not exactly of the form (2.4) but with some slight extensions:

$$|\beta_j| \geq C_1 \left|\tfrac{j}{m}\right|^q \qquad \text{for } j = -m/2, \cdots, m/2 \tag{2.23}$$

$$|\beta_{j+1} - \beta_j| \leq C_2 \left| \left(\tfrac{j+1}{m}\right)^q - \left(\tfrac{j}{m}\right)^q \right| \qquad \text{for } j = -m/2, \cdots, m/2 - 1. \tag{2.24}$$

A mesh satisfying the above properties is called *generalized graded mesh* with grading exponent q. A generalized graded mesh will be used when a higher-order quadrature rule is needed.

3. Graded Mesh Method for Singular Integral Equations

We rewrite the integral equations (1.1) by using a variable substitution $\varrho = \sigma - s$:

$$-\int_{-L/2}^{L/2} \log|\nu(s) - \nu(\sigma+s)| u(\nu(\sigma+s))d\sigma = f(\nu(s)), \quad s \in [-L/2, L/2], \tag{3.1}$$

where we have used the fact that $\nu(\sigma)$ is a L-periodic function. Similarly, the integral equation (1.3) can be written in the following form:

$$\begin{cases} -\int_{-L/2}^{L/2} \log|\nu(s) - \nu(\sigma+s)| u(\nu(\sigma+s))d\sigma + \omega &= f(\nu(s)), \; s \in [-L/2, L/2], \\ \int_{-L/2}^{L/2} u(\nu(\sigma))\, d\sigma &= b. \end{cases} \tag{3.2}$$

Notice that the singular integrals in (3.1) and (3.2) are of the type (2.1) with

$$G(\sigma) = \log|\nu(s) - \nu(\sigma+s)| \text{ and } g(\sigma) = u(\nu(\sigma+s)),$$

where s is regarded as a parameter. It can be verified that the kernel G above satisfies the condition (2.2). Hence the approximation methods in the last section can be applied.

In this section, the basic idea will be demonstrated by considering the integral equation (3.1); the extension to the equation (3.2) is straightforward. The key idea of our proposed method is that we use a uniform mesh A of the form (2.3) as the set of collocation points for the integral equation (3.1) but when numerically approximating the integral term in (3.1) we use a subset of A as the quadrature points. More precisely, the integral equation (3.1) is collocated on the uniform grid A:

$$-\int_{-L/2}^{L/2} \log|\nu(\sigma) - \nu(\sigma+\alpha_i)| u(\nu(\sigma+\alpha_i))d\sigma = f(\nu(\alpha_i)), \quad \alpha_i \in A, \tag{3.3}$$

and the graded-mesh approximations (2.13) and (2.19) will be employed to approximate the integral term in (3.3).

3.1. First-order graded-mesh method

We approximate the singular integral in (3.3) using the numerical quadrature $I_m^{(3)}$, i.e. (2.13), with $n = m^2$ and $q = 2$. Here n is the total number of collocation points, m is the total number of quadrature points, and q is the grading exponent of the graded mesh defined by (2.4). This gives a first-order approximation to (3.3):

$$\sum_{j=-\sqrt{n}/2}^{\sqrt{n}/2} \mu_j^{(i)} U(\beta_j + \alpha_i) = F(\alpha_i), \qquad -n/2 \le i \le n/2, \tag{3.4}$$

where

$$U(\sigma) = u(\nu(\sigma)), \qquad F(\sigma) = f(\nu(\sigma));$$
$$\mu_{-\sqrt{n}/2}^{(i)} = \frac{1}{2}\log|\nu(\beta_{-\sqrt{n}/2} + \alpha_i) - \nu(\alpha_i)|\beta_{-\sqrt{n}/2+1} - \beta_{-\sqrt{n}/2},$$
$$\mu_{\sqrt{n}/2}^{(i)} = \frac{1}{2}\log|\nu(\beta_{\sqrt{n}/2} + \alpha_i) - \nu(\alpha_i)|\beta_{\sqrt{n}/2} - \beta_{\sqrt{n}/2-1}$$
$$\mu_j^{(i)} = \frac{1}{2}\log|\nu(\beta_j + \alpha_i) - \nu(\alpha_i)|\beta_{j+1} - \beta_{j-1} \qquad -\sqrt{n}/2 + 1 \le j \le \sqrt{n}/2 - 1.$$

We emphasize that the index j in the sum (3.4) is from $-\sqrt{n}/2$ to $\sqrt{n}/2$, which implies that the total number of the summation terms in (3.4) is $\sqrt{n}$. This is different with the boundary element approach which uses n summation terms. The difference enables us to make the resulting matrix from (3.4) sparse. We also notice that the graded mesh in terms of n (rather than m) is of the form

$$B' := \{\beta_{-\sqrt{n}/2}, \beta_{-\sqrt{n}/2+1}, \cdots, \beta_{\sqrt{n}/2}\}, \quad \beta_j = \text{sgn}(j)\frac{(2j)^2}{n}\frac{L}{2}. \tag{3.5}$$

It is clear that $B' \subset A$, where A is the set of the uniform collocation point.

We can also make a simple index transformation to change the negative index in (3.4) positive so that a proper matrix equation can be formed. This can be done by letting $l = j + n/2 + j(i)$, which gives

$$\sum_{l=j(i)}^{\sqrt{n}+j(i)} \mu_{J(l)}^{(i)} U(\beta_{J(l)} + \alpha_i) = F(\alpha_i), \qquad \alpha_i \in A, \tag{3.6}$$

where $j(i) = \left[\sqrt{n/4 - i/2} + 1/2\right]$ and $J(l) = \text{mod}(l + \sqrt{n}/2, \sqrt{n}) - \sqrt{n}/2$. Here $[x]$ denotes the largest possible integer not exceeding x. It is noted that the variable $\beta_{J(l)} + \alpha_i$ satisfies

$$-L/2 = \beta_{J(j(i))} + \alpha_i < \beta_{J(j(i)+1)} + \alpha_i < \cdots < \beta_{J(\sqrt{n}+j(i))} + \alpha_i = L/2. \tag{3.7}$$

Hence the equation (3.6) can be written in the following matrix form:

$$A_n U_n = F_n, \tag{3.8}$$

where $U_n = (U(\alpha_{-n/2+1}), \cdots, U(\alpha_{n/2}))^T$ and $F_n = (F(\alpha_{-n/2+1}), \cdots, F(\alpha_{n/2}))^T$. The matrix $A_n = (a_{p\,q}^{(n)})$ is an $n \times n$ sparse matrix defined by

$$a_{p\,q}^{(n)} = \begin{cases} \mu_{J(l)}^{(p)}, & \text{if } q = \beta_{J(l)} + \alpha_p,\ (l = j(p), \cdots, \sqrt{n} + j(p)), \\ 0, & \text{otherwise}, \end{cases} \tag{3.9}$$

where $p = -n/2 + 1, \cdots, n/2$. This implies that in each row of A_n there are only $\sqrt{n}$ non-zero elements.

Assertion 3.1. *Assume $n = m^2$, where n is the total number of collocation points and m the total number of quadrature points. If the singular integral in* (3.3) *is approximated by using the numerical quadrature $I_m^{(3)}$ with the graded mesh* (3.5)*, then the resulting matrix of the linear system* (3.8) *is sparse. More precisely, there are $\sqrt{n}$ non-zero elements out of the n entries in each row.*

3.2. Second-order graded-mesh method

It is seen from (2.18) that the method proposed in the last subsection is of first-order accuracy only. Now we describe a second-order method also using the graded-mesh approach. We now let $n = m^2$ (m is the total number of quadrature points) and $q = 4$ in the definition of (2.4). The larger value of the grading exponent introduces a technical trouble: some points β_j in (2.4) may not belong to the set of the collocation point A. Below we will fix it by using the idea of the generalized graded mesh (2.23)-(2.24).

Assume the total number of quadrature points $m = \sqrt{n}$ and the grading exponent $q = 4$ in (2.4). Let

$$\beta_j = \text{sgn}(j)\left[\left(\frac{2j}{\sqrt{n}}\right)^4 \frac{n}{2} + \frac{1}{2}\right]\frac{2}{n}\frac{L}{2}, \text{ for } j \in J_n := \left\{\pm n_0, \cdots, \pm\sqrt{n}/2\right\} \tag{3.10}$$

with $n_0 = \left[(n/4)^{1/3} + 1/2\right]$. Then it can be verified that

$$B'' := \{\beta_{-\sqrt{n}/2}, \cdots, \beta_{-n_0}, \beta_{n_0}, \cdots, \beta_{\sqrt{n}/2}\} \tag{3.11}$$

satisfies the requirements (2.23)-(2.24), and more importantly, that each point in B'' is an even point in A, i.e., $B'' \subset A$ and there is an i such that $\beta_j = \alpha_{2i}$. Therefore, all of the mid-points $\beta_{j+1/2} := (\beta_j + \beta_{j+1})/2$ are also in the set of the uniform mesh point. Now we split the singular integral in (3.3) into three parts

$$\begin{aligned} & -\int_{-L/2}^{\beta_{-n_0}} - \int_{\beta_{-n_0}}^{\beta_{n_0}} - \int_{\beta_{n_0}}^{L/2} \log|\nu(\sigma) - \nu(\sigma+\alpha_i)| u(\nu(\sigma+\alpha_i)) d\sigma \\ =:& \quad \mathsf{T}_1 + \mathsf{T}_2 + \mathsf{T}_3, \end{aligned} \tag{3.12}$$

and approximate T_2 by using the second-order uniform mesh method $I_{2n_0}^{(1)}$, and T_1 and T_3 by using the second-order graded mesh method $I_m^{(4)}$ with the generalized graded mesh B'' in (3.10)-(3.11). Substituting these approximations to the integral equation (3.3) gives

$$\sum_{k=n\beta_{-n_0}/L}^{n\beta_{n_0}/L} \lambda_k^{(i)} U(\alpha_{k+i}) + \sum_{k \in K_n} \mu_k^{(i)} U(\beta_{k/2} + \alpha_i) = F(\alpha_i), \qquad \alpha_i \in A, \tag{3.13}$$

where

$$K_n = \left\{-\sqrt{n}, -\sqrt{n}+1, \cdots, -2n_0, 2n_0, \cdots, \sqrt{n}-1, \sqrt{n}\right\}, \tag{3.14}$$

and the weighs $\lambda_k^{(i)}$ and $\mu_k^{(i)}$ are defined by

$$\begin{aligned} \lambda_k^{(i)} &= \frac{1}{2}\int_{\alpha_k}^{\alpha_{k+1}} \log|\nu(\sigma+\alpha_i) - \nu(\alpha_i)| d\sigma, && k = n\beta_{-n_0}/L; \\ \lambda_k^{(i)} &= \frac{1}{2}\int_{\alpha_{k-1}}^{\alpha_k} \log|\nu(\sigma+\alpha_i) - \nu(\alpha_i)| d\sigma, && k = n\beta_{n_0}/L; \\ \lambda_k^{(i)} &= \frac{1}{2}\int_{\alpha_{k-1}}^{\alpha_{k+1}} \log|\nu(\sigma+\alpha_i) - \nu(\alpha_i)| d\sigma, && n\beta_{-n_0}/L < k < n\beta_{n_0}/L; \\ \mu_{2l}^{(i)} &= \frac{1}{6}\log|\nu(\beta_l+\alpha_i) - \nu(\alpha_i)|(\beta_{l+1} - \beta_{l-1}), && 2l \in K_n; \\ \mu_{2l+1}^{(i)} &= \frac{2}{3}\log|\nu(\beta_{l+1/2}+\alpha_i) - \nu(\alpha_i)|(\beta_{l+1} - \beta_l), && 2l+1 \in K_n. \end{aligned}$$

Similarly, in order to avoid the use of negative index in (3.13) we use the index transformations $l = k+i$ and $l = k + n/2 + j(i)$ in the two summations in (3.13) respectively. By doing this we obtain

$$\sum_{l=n\beta_{-n_0}/L+i}^{n\beta_{n_0}/L+i} \lambda_{l-i}^{(i)} U(\alpha_l) + \sum_{l \in K_n + \sqrt{n} + 2j(i)} \mu_{J(l)}^{(i)} U(\beta_{J(l)/2} + \alpha_i) = F(\alpha_i), \quad \alpha_i \in A, \tag{3.15}$$

where

$$\begin{aligned} j(i) &= \left[\sqrt{n}/2(1/2 - i/n)^{1/4} + 1/2\right], \\ J(l) &= 2\Big(\mathrm{mod}(l/2 + \sqrt{n}/2, \sqrt{n}) - \sqrt{n}/2\Big). \end{aligned}$$

Therefore, the variable $\beta_{J(l/2)} + \alpha_i$ (after module by L) of $U(\sigma)$ will be in the interval $[-n/2, n/2]$ and satisfies

$$\begin{aligned} & -L/2 = \beta_{J(j(i))} + \alpha_i < \beta_{J(j(i)+1)} + \alpha_i < \cdots < \beta_{J(-n_0)} + \alpha_i \\ & < \beta_{J(-n_0)} + \alpha_1 + \alpha_i < \beta_{J(-n_0)} + \alpha_2 + \alpha_i < \cdots < \beta_{J(n_0)} + \alpha_i \\ & < \beta_{J(n_0+1)} + \alpha_i < \cdots < \beta_{J(j(i)+\sqrt{n})} + \alpha_i = L/2. \end{aligned}$$

The above procedure again gives a linear system of the form

$$A_n U_n = F_n, \tag{3.16}$$

where A_n is a sparse matrix. The number of summations from T_1 and T_3 are about $\sqrt{n}$ each: the use second order method $I_m^{(4)}$ requires the information of the mid-point, which doubles the non-zero entries of that for the first-order method. Moreover, the number of summations from T_2 is $2n\beta_{n_0}/L$ which is less than $4\sqrt[3]{n}$.

Assertion 3.2. *Assume $n = m^2$, where n is the total number of collocation points and m the total number of quadrature points. If the singular integral in* (3.3) *is approximated by using the numerical quadrature* (3.12)*, with the graded mesh* (3.10)-(3.11)*, then the resulting matrix of the linear system* (3.16) *is sparse. More precisely, there are less than $2\sqrt{n} + 4\sqrt[3]{n}$ non-zero elements out of the n entries in each row.*

4. Numerical Experiments

As pointed out earlier, the numerical scheme designed for (1.1) in the last section can be extended to the equations (1.3) directly. In this section, we will consider a numerical example for (1.3) to demonstrate the efficiency of the proposed graded-mesh methods.

Example 4.1. *Consider the boundary integral equations* (1.3)*, where Γ is a unit circle, $b = 0$ and $f(x) = x_1$. It can be shown that the exact solution is $u(x_1, x_2) = 2x_1$ on Γ and $\omega = 0$.*

If we let $\nu(s) = (\cos s, \sin s)$, $-\pi \le s \le \pi$, then the integral equations (1.3) become

$$-\int_{-\pi}^{\pi} U(s) \log \sqrt{(\cos s - \cos t)^2 + (\sin s - \sin t)^2}\, ds + \omega = \cos t, \quad -\pi \le t \le \pi, \quad (4.1)$$

$$\int_{-\pi}^{\pi} U(s)\, ds = 0, \quad (4.2)$$

where $U(s) = u(\cos s, \sin s)$. The exact solution of (4.1)-(4.2) is $U(t) = 2\cos t$ and $\omega = 0$.

To demonstrate the efficiency of the proposed graded mesh methods, we also include the first-order and the second-order uniform mesh methods in our computations, which are obtained by approximating the integral terms in (4.1)-(4.2) with the first-order numerical quadrature $I_n^{(2)}$ and the second-order numerical quadrature $I_n^{(1)}$, respectively. In both cases, the resulting linear system $A_nU_n = F_n$ has a dense matrix.

The total number of the collocation points used is from $n = 4^2$ to $n = 4^9$. The linear system $A_nU_n = F_n$ is solved using the conjugate gradient (CG) residual method. The number of iteration steps, required to obtain a given accuracy, is about 5 for $4^3 \le n \le 4^7$ and is independent of the quadrature rules used. The number of iteration steps increases to $10 \sim 20$ for $n = 4^8$ or $n = 4^9$. Numerical tests were carried on a Dell PC with Pentium IV 2200 MHz CPU.

Table 1 shows the L^2 -errors, the rate of convergence and the CPU times used with various numbers of collocation point, obtained by using the first-order approximation methods. It is seen that for a fixed n the total CPU time used by using the graded-mesh approach is significantly less than that with a uniform mesh. However, for a fixed n the L^2-error associated with the graded-mesh is larger than that with a uniform mesh. To have a better idea about the real gain about using the graded-mesh method, we plot in Fig. 1 the CPU time usage against the L^2-errors. It is seen from Fig. 1 that to obtain a given accuracy the CPU time required for a uniform mesh approach is about 10 times more than that for the graded-mesh approach. Also shows in Table 1 is the CPU time used for making up the matrix A_n. Due to its sparsity nature, forming the matrix from the graded-mesh method uses much less time than that with a uniform mesh.

Table 2 shows the L^2 -errors, the rate of convergence and the CPU times used with various numbers of collocation point, obtained by using the second-order approximation methods. Fig.

Table 1: The data for the first-order graded-mesh method. For comparison, the corresponding data for the first-order uniform-mesh method is listed in the brackets.

n	$\|u-u_h^k\|_{L^2(\Gamma)}$	Rate of convergence	Time for making A_n (sec)	Time in CG iteration (sec)
4^2	1.58e-1 (6.27e-2)		9.17e-7 (3.62e-6)	6.56e-5 (9.84e-4)
4^3	5.18e-2 (1.60e-2)	0.80 (0.98)	1.83e-6 (1.46e-5)	1.97e-4 (3.15e-3)
4^4	1.03e-2 (4.01e-3)	0.83 (1.00)	3.63e-6 (5.86e-5)	5.96e-4 (1.60e-2)
4^5	4.99e-3 (1.00e-3)	0.85 (1.00)	7.23e-6 (2.38e-4)	2.04e-3 (2.53e-1)
4^6	1.47e-3 (2.51e-4)	0.88 (1.00)	1.49e-5 (9.36e-4)	2.26e-2 (3.96e+0)
4^7	4.18e-4 (6.26e-5)	0.91 (1.00)	2.37e-5 (3.57e-3)	3.36e-1 (6.25e+1)
4^8	1.17e-4 (1.57e-5)	0.92 (1.00)	4.76e-5 (1.50e-2)	4.35e+0 (2.10e+3)
4^9	3.17e-5 (3.91e-6)	0.94 (1.00)	9.55e-5 (6.47e-2)	8.15e+1 (9.26e+4)

Table 2: The data for the second-order graded-mesh method. For comparison, the corresponding data for the second-order uniform-mesh method is listed in the brackets.

n	$\|u-u_h^k\|_{L^2(\Gamma)}$	Convergence rate	Total CPU time (sec)
4^2	2.36e-1 (1.97e-2)		1.58e-4 (1.00e-3)
4^3	2.07e-2 (1.38e-3)	1.76 (1.92)	5.22e-4 (3.22e-3)
4^4	1.29e-3 (8.82e-5)	1.80 (1.98)	1.23e-3 (1.63e-2)
4^5	9.29e-5 (5.55e-6)	1.90 (1.99)	3.59e-3 (2.54e-1)
4^6	6.47e-6 (3.47e-7)	1.92 (2.00)	4.07e-2 (3.96e+0)
4^7	4.42e-7 (2.17e08)	1.94 (2.00)	5.36e-1 (6.25e+1)
4^8	2.98e-8 (1.36e-9)	1.95 (2.00)	6.10e+0 (2.10e+3)
4^9	2.00e-9 (8.49e-11)	1.95 (2.00)	1.37e+2 (9.26e+4)

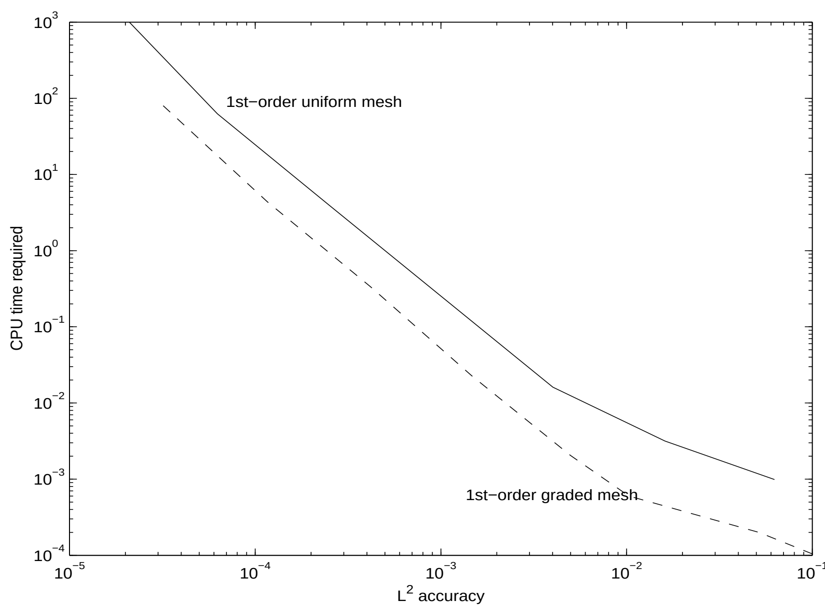

Figure 1: Accuracy versus CPU-time for the two first-order approaches: the broken line is for the graded-mesh method and the solid line is for the uniform mesh approach.

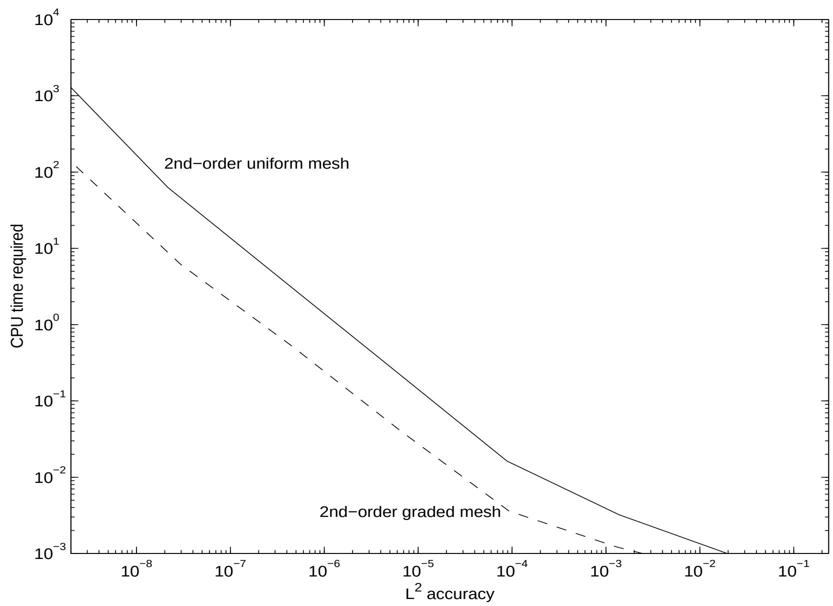

Figure 2: Accuracy versus CPU-time for the two second-order approaches: the broken line is for the graded-mesh method and the solid line is for the uniform mesh approach.

2 presents the CPU time usage against the L^2-errors for the second-order graded-mesh method and the second-order uniform mesh approach. Again it is observed that the CPU time required for a uniform mesh approach is about 10 times larger than that for the graded-mesh approach when n is reasonably large.

Acknowledgments. We are grateful to Professor Houde Han of Tsinghua University for many useful discussions. The third author would like to thank Prof. Weiwei Sun of the City University of Hong Kong for many helpful suggestions. Part of this work was carried out while this author was visiting the City University.

References

[1] K. Atkinson and W. Han, Theoretical Numerical Analysis: a Functional Analysis Framework, Springer, New York, 2001.

[2] H. Brunner, The numerical solution of weakly singular Volterra integral equations by collocation on graded meshes, *Math. Comp.*, **45** (1985), 417-437.

[3] H. Brunner, A. Pedas and G. Vainikko, The piecewise polynomial collocation method for nonlinear weakly singular Volterra integral equations, *Math. Comp.*, **68** (1999), 1079-1095.

[4] Y. Z. Cao, T. Herdman and Y. Xu, A hybrid collocation method for Volterra integral equations with weakly singular kernels, *SIAM J. Numer. Anal.*, **41** (2003), pp. 364-381.

[5] G. A. Chandler, Mesh grading for boundary integral equations, in Computational Techniques and Applications, J. Noye and C. Fletcher eds., Elsevier Science, North-Holland, Amsterdam, New York, 1984, 289-296.

[6] Z. Chen, C. A. Micchelli and Y. Xu, Fast collocation methods for second kind integral equations. *SIAM J. Numer. Anal.*, **40** (2002), 344–375.

[7] M. Costabel and E. Stephan, Boundary integral equations for mixed boundary value problems in polygonal domain and Galerkin approximation, in Mathematical Models and Methods in Mechanics, Banach Center Publication volume 15, PWN-Polish Scientific Publications, Warsaw, 1985, 175-251.

[8] G. C. Hsiao and W. L. Wendland, A finite element method for some integral equations of the first kind, *J. Math. Anal. Appl.*, **58** (1977), 449-481.

[9] Q. Hu, Stieltjes derivatives and beta-polynomial spline collocation for Volterra integro-differential equations with singularities, *SIAM J. Numer. Anal.*, **33** (1996), pp. 208-220.

[10] J. R. Rice, On the degree of convergence of nonlinear spline approximation, in Approximation with Special Emphasis on Spline Functions, I. J. Schoenberg, ed., Academic Press, New York, 1969.

[11] I. H. Sloan and A. Spence, The Galerkin method for integral equations of the first kind with logarithmic kernel: theory, *IMA J. Numer. Anal.,* **8** (1988), 105-122.

[12] T. Tang, Superconvergence of numerical solutions to weakly singular Volterra integro-differential equations, *Numer. Math.*, **61** (1992), 373-382.

[13] T. Tang, A note on collocation methods for Volterra integro-differential equations with weakly singular kernels, *IMA J. Numer. Anal.,* **13** (1993), 93-99.

[14] Y. Yan and I. H. Sloan, Mesh grading for integral equations of the first kind with logarithmic kernel, *SIAM J. Numer. Anal.*, **26** (1989), 574-587.