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ERROR BOUNDS FOR FRACTIONAL STEP METHODS FOR CONSERVATION LAWS WITH SOURCE TERMS*

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Abstract. Fractional step methods have been used to approximate solutions of scalar conservation laws with source terms. In this paper, the stability and accuracy of the basic fractional step algorithms are analyzed when these algorithms are used to compute discontinuous solutions of nonhomogeneous scalar conservation laws. The authors show that time-splitting methods for conservation laws with source terms always converge to the unique weak solution satisfying the entropy condition. In particular, it is proved that the L^1 errors in the splitting methods are bounded by $O(\sqrt{\Delta t})$, where Δt is the splitting time step. The L^1 convergence rate of a class of fully discrete splitting methods is also investigated.

Key words. splitting method, hyperbolic conservation laws, error estimate, monotone scheme, Euler method

AMS subject classifications. 65M10, 65M05, 35L65

1. Introduction. We consider the initial value problem for the nonhomogeneous scalar conservation law

$$u_t + (f(u))_x = g(u), \qquad (x,t) \in \mathbf{R} \times [0,T], \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbf{R}, \tag{1.2}$$

with $u_0 \in \mathrm{BV}(\mathbf{R}) \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, $f \in C^1(\mathbf{R})$, and g satisfies a Lipschitz condition, with a Lipschitz constant L, and $g(0) = 0$. For ease of exposition we shall consider the one-dimensional equation (1.1) in the analysis; its extension to the multi-dimensional case is straightforward (see §6). Also, we assume that $g = g(u)$, but the results of this paper can be easily extended to the more general case $g = g(x,u)$ which is assumed in [1], [13].

Most problems of technological interest are nonhomogeneous, or multi-dimensional, or both. The source term in the nonhomogeneous problem is due to physical or geometrical effects. (see, e.g., [2], [17]). The nonhomogeneous conservation laws have been investigated theoretically and numerically by several authors (see, e.g., [10], [13], [25], [3], [8], [12], [17]). In the scalar case Kruzkov [10] proved the existence and uniqueness of the solution.

In the present paper we shall consider time-splitting methods for the numerical solutions of (1.1). In the simplest case the first step is to use the method known as operator splitting to remove the nonhomogeneous term $g(u)$ from (1.1). That is, we solve the homogeneous scalar conservation law

$$u_t + (f(u))_x = 0. \tag{1.3}$$

The second step is to solve the ordinary differential equation

$$u_t = g(u). \tag{1.4}$$

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Let $S(t)$ denote the exact solution operator of (1.1) which satisfies the entropy conditions. Therefore, the solution of (1.1)–(1.2) can be expressed in the form $u(x,t) = S(t)u_0$. Similarly, let $S_1(t)$ and $S_2(t)$ denote the solution operators of (1.3) and (1.4), respectively. The first-order fractional step method is based on the approximation

$$S(t_n)u_0 \approx \left(S_2(\Delta t)S_1(\Delta t)\right)^n u_0, \qquad t_n = n\Delta t \in [0,T], \tag{1.5}$$

or on the one with the roles of S_2 and S_1 reversed, where Δt is the splitting time step. To maintain second-order accuracy, the Strang splitting [18] can be used, in which the solution $S(t_n)u_0$ is approximated by

$$S(t_n)u_0 \approx \left(S_2(\Delta t/2)S_1(\Delta t)S_2(\Delta t/2)\right)^n u_0, \tag{1.6}$$

or by the one with the roles of S_2 and S_1 reversed. It should be pointed out that first-order accuracy and second-order accuracy are based on the truncation errors for smooth solutions. For discontinuous solutions of conservation laws, it is not difficult to show that both approximations (1.5) and (1.6) are at most first-order accurate [4]. To analyze the principal properties of fractional step methods for discontinuous solutions, we shall concentrate on the splitting method (1.5). The main results in this paper for (1.5) can be easily extended to the scheme (1.6).

For multi-dimensional homogeneous conservation laws, a first-order fractional step method was introduced by Godunov [6], which was modified by Strang [18]. The stability, accuracy and convergence of their methods are analyzed by Crandall and Majda [4], who proved that both the Godunov splitting algorithm and Strang splitting algorithm converge to the unique weak solution satisfying the entropy condition. In the recent work of Teng [22], the convergence rates of both methods are investigated. However, the splitting methods for nonhomogeneous conservation laws have not been analyzed so far. In this work, we shall show that the splitting algorithm (1.5) converges to the entropy solution of (1.1)–(1.2) and a convergence rate is obtained. In practical calculations, the solution operators S_1 and S_2 should be replaced by certain discrete splitting operators, G_1 and G_2, respectively. That is, we need to consider one-dimensional difference approximations, $G_1(\Delta t) \approx S_1(\Delta t), G_2(\Delta t) \approx S_2(\Delta t)$, to define fully discrete splitting methods. In this work, we consider the case when G_1 is a monotone scheme and G_2 is the forward Euler method. A difference scheme,

$$G_1(\Delta t)u_j^n = u_j^n - \lambda\Big(\bar{f}(u_{j-p+1}^n, \ldots, u_{j+p}) - \bar{f}(u_{j-p}^n, \ldots, u_{j+p-1}^n)\Big) \tag{1.7}$$

$$u_j^0 = \frac{1}{\Delta x}\int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} u_0(x)dx, \tag{1.8}$$

where $\lambda = \Delta t/\Delta x$ is a constant and Δx is the spatial step length, is locally monotone on the interval, $[a,b]$, if the right-hand side of (1.7) is a nondecreasing function of all arguments as they vary over $[a,b]$. The basic properties of monotone schemes are provided in [5], [9], [11]. The forward Euler scheme

$$G_2(\Delta t)w(x) = w(x) + g(w(x))\Delta t \tag{1.9}$$

is the simplest and most well-known method for solving ordinary differential equations. The main results of the present work are given by the following two theorems.

THEOREM 1.1. *Let* $u_0 \in \mathrm{BV}(\mathbf{R}) \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, $f \in C^1(\mathbf{R})$, *and assume that* g *satisfies a Lipschitz condition and* $g(0) = 0$. *Let* $S(t)u_0$ *denote the unique weak* *solution of* (1.1)–(1.2) *satisfying the entropy condition*; *then the* L^1 *convergence rate of the semi-discrete fractional step algorithm* (1.5) *is* 1/2. *More precisely, for any* $t_n = n\Delta t \in [0,T]$, *the following estimate holds*:

$$\left\| S(t_n)u_0 - \Big(S_2(\Delta t)S_1(\Delta t)\Big)^n u_0 \right\|_{L^1(\mathbf{R})} \le C\sqrt{\Delta t}, \tag{1.10}$$

where C *is a constant independent of* Δt. *Similar results are valid with the roles of* S_2 *and* S_1 *reversed.*

THEOREM 1.2. *Let* $u_0 \in \mathrm{BV}(\mathbf{R}) \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, $f \in C^1(\mathbf{R})$, *and assume that* g *satisfies a Lipschitz condition and* $g(0) = 0$. *Assume the finite difference scheme* (1.7)–(1.8) *to be monotone and consistent with* (1.3), *the numerical flux* $\bar{f}$ *to be Lipschitz continuous, and* G_2 *to be the forward Euler operator. If* $\delta := \max\{\Delta x, \Delta t\}$ *is sufficiently small, then for any* $t_n = n\Delta t \in [0,T]$,

$$\left\| S(t_n)u_0 - \left(G_2(\Delta t)G_1(\Delta t)\right)^n u_0 \right\|_{L^1(\mathbf{R})} \le C\sqrt{\delta}, \tag{1.11}$$

where C *is a constant independent of* δ *and where the function* $(G_2(\Delta t)G_1(\Delta t))^n u_0$ *is a piecewise constant function, i.e., for* $(j-1/2)\Delta x \le x < (j+1/2)\Delta x$,

$$\left(G_2(\Delta t)G_1(\Delta t)\right)^n u_0(x) = \left(G_2(\Delta t)G_1(\Delta t)\right)^n u_0(x_j).$$

Similar results hold with the roles of G_2 *and* G_1 *reversed.*

Remark. In the one-dimensional case, the L^∞ condition for u_0 can be dropped since $\mathrm{BV}(\mathbf{R}) \subset L^\infty(\mathbf{R})$, but in the N-dimensional case ($N > 1$) the $L^\infty(\mathbf{R}^N)$ condition has to be imposed since $\mathrm{BV}(\mathbf{R}^N) \not\subset L^\infty(\mathbf{R}^N)$ (see [7]). In order to extend Theorems 1.1 and 1.2 to multi-dimensional nonhomogeneous conservation laws we still list the L^∞ condition for u_0 in the one-dimensional case.

2. Preliminaries. A bounded measurable function, u, is a weak solution of (1.1) and (1.2) if for all $\phi \in C^1(\mathbf{R} \times [0,T])$ with compact support in $\mathbf{R} \times [0,T)$ (i.e., $\phi \in C_0^1(\mathbf{R} \times [0,T))$),

$$\int_{\mathbf{R}} \int_0^T \Big(u\phi_t + f(u)\phi_x\Big)dxdt + \int_{\mathbf{R}} u_0(x)\phi(x,0)dx = -\int_{\mathbf{R}} \int_0^T g\phi dxdt.$$

The bounded-variation seminorm of a function $u \in \mathrm{BV}(\mathbf{R})$ is defined by

$$|u(\cdot,t)|_{\mathrm{BV}(\mathbf{R})} = \sup_{0<|h|<\infty} \int_{\mathbf{R}} \frac{|u(x+h,t) - u(x,t)|}{|h|} dx.$$

Weak solutions are not uniquely determined by their initial data and additional principles, entropy conditions, are needed to select the appropriate physical solution. A weak solution, u, of (1.1)–(1.2) is an entropy solution if for all $\phi \in C_0^1(\mathbf{R} \times (0,T))$ with $\phi(x,t) \ge 0$ and any $k \in \mathbf{R}$,

$$\int_{\mathbf{R}} \int_0^T \Big(|u-k|\phi_t + \mathrm{sign}(u-k)(f(u)-f(k))\phi_x + \mathrm{sign}(u-k)g\phi\Big)dxdt \ge 0. \tag{2.1}$$

PROPOSITION 2.1. (See [10].) *If* $u_0 \in L^\infty(\mathbf{R})$, $f \in C^1(\mathbf{R})$, *and* g *satisfies a Lipschitz condition with a Lipschitz constant* L, *then the nonhomogeneous conservation law*

(1.1) *possesses a unique weak solution* $u(x,t) = S(t)u_0 \in L^\infty(\mathbf{R}) \cap C([0,T], L_{loc}(\mathbf{R}))$ *with* $u(x,0) = u_0$ *satisfying the entropy condition* (2.1) *and the following inequality*:

$$\|S(t)u_0\|_{L^\infty(\mathbf{R})} \le e^{Lt}\Big(\|u_0\|_{L^\infty(\mathbf{R})} + |g(0)|t\Big).$$

Moreover, for any $v_0 \in L^\infty(\mathbf{R})$, *the following inequality holds*:

$$\|S(t)u_0 - S(t)v_0\|_{L^1(\mathbf{R})} \le e^{Lt}\|u_0 - v_0\|_{L^1(\mathbf{R})}.$$

Using the above results and the properties for the S_1 [10] and G_1 [5], [11] we can establish the following stability results, given in Propositions 2.2–2.4, which will play an important role in the proofs of Theorems 1.1 and 1.2. The detailed proof of them can be found in [20].

PROPOSITION 2.2. *If* u_0, f, *and* g *satisfy the conditions stated in Theorem* 1.1, *then there exists a unique weak solution*, $u(x,t) \equiv S(t)u_0$, *belonging to* $\mathrm{BV}(\mathbf{R}\times[0,T]) \cap L^1(\mathbf{R}\times[0,T]) \cap C([0,T], L^1(\mathbf{R}))$ *and satisfying the following entropy condition: for any nonnegative function* $\phi \in C_0^1(\mathbf{R}\times\overline{\mathbf{R}^+})$, *any* $k \in \mathbf{R}$, *and any* $\tau_1, \tau_2 \in [0,T]$ *such that* $\tau_1 \le \tau_2$,

$$-\int_{\mathbf{R}}\int_{\tau_1}^{\tau_2}\Big(|u-k|\phi_\tau + \mathrm{sign}(u-k)(f(u)-f(k))\phi_x + \mathrm{sign}(u-k)g(u)\phi\Big)dx\,d\tau$$
$$+\int_{\mathbf{R}}\Big(|u-k|\phi\Big)\Big|_{\tau=\tau_1}^{\tau=\tau_2} dx \le 0, \tag{2.2}$$

where $u = u(x,\tau), \phi = \phi(x,\tau)$, *and* $h(x,\tau)\,\big|_{\tau=\tau_1}^{\tau=\tau_2} = h(x,\tau_2) - h(x,\tau_1)$. *Moreover, the function* $S(t)u_0$ *satisfies the following properties*:

$$\|S(t)u_0\|_{L^\infty(\mathbf{R})} \le e^{Lt}\|u_0\|_{L^\infty(\mathbf{R})}, \tag{2.3}$$
$$|S(t)u_0|_{\mathrm{BV}(\mathbf{R})} \le e^{Lt}|u_0|_{\mathrm{BV}(\mathbf{R})}, \tag{2.4}$$
$$\|S(t)u_0\|_{L^1(\mathbf{R})} \le e^{Lt}\|u_0\|_{L^1(\mathbf{R})}, \tag{2.5}$$
$$\|S(\tau_2)u_0 - S(\tau_1)u_0\|_{L^1(\mathbf{R})} \le e^{LT}\Big(C_1|u_0|_{\mathrm{BV}(\mathbf{R})} + L\|u_0\|_{L^1(\mathbf{R})}\Big)|\tau_2 - \tau_1|, \tag{2.6}$$

where $C_1 = \max_{|u|\le e^{LT}\|u_0\|_{L^\infty(\mathbf{R})}} |f'(u)|$.

Following the notations in [4], we define $u_\Delta(x,t)$ as follows:

$$u_\Delta = \begin{cases} S_1(2(t-t_n))(S_2(\Delta t)S_1(\Delta t))^n u_0, & t \in [t_n, t_{n+1/2}), \\ S_2(2(t-t_{n+1/2}))S_1(\Delta t)(S_2(\Delta t)S_1(\Delta t))^n u_0, & t \in [t_{n+1/2}, t_{n+1}), \end{cases} \tag{2.7}$$

where $t_n = n\Delta t$ and $t_{n+1/2} = (n+1/2)\Delta t$. Further, let $x_{j+1/2} = (j+1/2)\Delta x$, $x_j = j\Delta x, X_j = [x_{j-1/2}, x_{j+1/2}), u_j^{n+1/2} = G_1(\Delta t)u_j^n$, and $u_j^{n+1} = G_2(\Delta t)u_j^{n+1/2}$. We define function $u_\delta(x,t)$ as follows:

$$u_\delta = \begin{cases} \frac{2}{\Delta t}(t-t_n)u_j^{n+1/2} + \frac{2}{\Delta t}(t_{n+1/2}-t)u_j^n, \\ \qquad x \in X_j,\ t \in [t_n, t_{n+1/2}), \\ \frac{2}{\Delta t}(t_{n+1}-t)u_j^{n+1} + \frac{2}{\Delta t}(t-t_{n+1/2})u_j^{n+1/2}, \\ \qquad x \in X_j,\ t \in [t_{n+1/2}, t_{n+1}). \end{cases} \tag{2.8}$$

Therefore, $u_\delta(x,\cdot)$ is a piecewise constant function in $\mathbf{R}$ and $u_\delta(\cdot,t) \in C([0,T])$. Also, $u_\delta(x_j,t_n) = u_j^n$ and $u_\delta(x_j,t_{n+1/2}) = u_j^{n+1/2}$.

PROPOSITION 2.3. *Let the functions u_0, f, and g be subject to the conditions stated in Theorem* 1.1. *Then the splitting solution u_Δ defined in* (2.7) *satisfies the following properties:*

$$\|u_\Delta(\cdot,t)\|_{L^\infty(\mathbf{R})} \le e^{Lt}\|u_0\|_{L^\infty(\mathbf{R})}, \tag{2.9}$$

$$|u_\Delta(\cdot,t)|_{\mathrm{BV}(\mathbf{R})} \le e^{Lt}|u_0|_{\mathrm{BV}(\mathbf{R})}, \tag{2.10}$$

$$\|u_\Delta(\cdot,t)\|_{L^1(\mathbf{R})} \le e^{Lt}\|u_0\|_{L^1(\mathbf{R})}, \tag{2.11}$$

$$\|u_\Delta(\cdot,\tau_2) - u_\Delta(\cdot,\tau_1)\|_{L^1(\mathbf{R})} \le 2e^{LT}\Big(C_1|u_0|_{\mathrm{BV}(\mathbf{R})} + L\|u_0\|_{L^1(\mathbf{R})}\Big)|\tau_2-\tau_1|, \tag{2.12}$$

where $0 \le \tau_1, \tau_2 \le T$, and C_1 is defined in Proposition 2.2.

PROPOSITION 2.4. *Let the functions u_0, f, g and the operators G_1, G_2 be subject to the conditions stated in Theorem* 1.2. *Then the fully discrete solution u_δ defined in* (2.8) *satisfies the following properties:*

$$\|u_\delta(\cdot,t)\|_{L^\infty(\mathbf{R})} \le e^{LT}\|u_0\|_{L^\infty(\mathbf{R})}, \tag{2.13}$$

$$|u_\delta(\cdot,t)|_{\mathrm{BV}(\mathbf{R})} \le e^{LT}|u_0|_{\mathrm{BV}(\mathbf{R})}, \tag{2.14}$$

$$\|u_\delta(\cdot,t)\|_{L^1(\mathbf{R})} \le e^{LT}\|u_0\|_{L^1(\mathbf{R})}, \tag{2.15}$$

$$\|u_\delta(\cdot,\tau_2) - u_\delta(\cdot,\tau_1)\|_{L^1(\mathbf{R})} \le 2e^{LT}\Big(C_2|u_0|_{\mathrm{BV}(\mathbf{R})} + L\|u_0\|_{L^1(\mathbf{R})}\Big)|\tau_2-\tau_1|, \tag{2.16}$$

for all $t, \tau_1, \tau_2 \in [0,T]$, where $C_2 = (2p+1)\bar{L}$, with $2p+1$ the number of difference grid points and $\bar{L}$ the Lipschitz constant of $\bar{f}$.

3. A general error bound. Throughout this paper, C denotes a positive constant independent of $\epsilon, \Delta t$, and Δx, but possibly with different values at different places. Moreover, in the subsequent sections, u' and q' denote $u(x',\tau')$ and $q(x',\tau')$, respectively, and u, q, and ω should be understood as $u(x,\tau), q(x,\tau)$, and $\omega(x,x',\tau,\tau') = \tilde{\omega}(x-x',\tau-\tau')$, respectively. We first introduce a nonnegative function $\theta \in C^\infty$ satisfying $\theta(\xi) = \theta(-\xi), \theta(\xi) = 0$ for $|\xi| \ge 1$, $\mathrm{sign}(\xi)\theta'(\xi) \le 0$, and $\int_{\mathbf{R}} \theta(\xi)d\xi = 1$. For $\epsilon > 0$, let

$$\tilde{\omega}(x,t) = \theta^\epsilon(t)\theta^\epsilon(x), \qquad \theta^\epsilon(t) = \frac{1}{\epsilon}\theta(t/\epsilon), \quad \theta^\epsilon(x) = \frac{1}{\epsilon}\theta(x/\epsilon).$$

It can be shown that $\theta^\epsilon \in C_0^\infty(\mathbf{R})$, $\tilde{\omega} \in C_0^\infty(\mathbf{R}^2)$, and

$$\theta^\epsilon(\xi) = 0 \quad \text{if} \quad |\xi| \ge \epsilon; \qquad \|\theta^\epsilon\|_{L^1(\mathbf{R})} = 1; \qquad \|\theta_t^\epsilon\|_{L^1(\mathbf{R})} = \frac{2}{\epsilon}\theta(0); \tag{3.1}$$

$$\tilde{\omega}(x,t) = 0 \quad \text{if} \quad |x| \ge \epsilon \quad \text{or} \quad |t| \ge \epsilon; \qquad \|\tilde{\omega}_t\|_{L^1(\mathbf{R}^2)} = \|\tilde{\omega}_x\|_{L^1(\mathbf{R}^2)} = \frac{2}{\epsilon}\theta(0). \tag{3.2}$$

Assume functions $p, q \in L^\infty(\mathbf{R}\times[0,T])$ satisfying

$$\|p(\cdot+h,\tau_2) - p(\cdot,\tau_1)\|_{L^1(\mathbf{R})} \le A\max\{|h|, |\tau_2-\tau_1|\}, \tag{3.3}$$

$$\|q(\cdot+h,\tau_2) - q(\cdot,\tau_1)\|_{L^1(\mathbf{R})} \le B\max\{|h|, |\tau_2-\tau_1|\}, \tag{3.4}$$

for any $h \in \mathbf{R}$ and $\tau_2, \tau_1 \in [0,T]$, where A and B are positive constants independent of h, τ_1 and τ_2. For a given $t \in (0,T]$, let $S = \mathbf{R}\times[0,t]$, $ds = dxd\tau$ and $ds' = dx'd\tau'$. For $p, q \in L^\infty(\mathbf{R}\times[0,T])$ satisfying (3.3) and (3.4), we define the functional Λ as follows:

$$\Lambda(p,q,t) := -\int_S\int_S \Big(|p-q'|\omega_\tau + \operatorname{sign}(p-q')(f(p)-f(q'))\omega_x\Big)dsds' + \int_S\int_{\mathbf{R}} \{|p-q'|\omega\}\Big|_{\tau=0}^{\tau=t} dxds', \tag{3.5}$$

where $S=\mathbf{R}\times[0,t], p=p(x,\tau), q'=q(x',\tau')$, and $\omega=\omega(x,x',\tau,\tau')=\tilde{\omega}(x-x',\tau-\tau')$. Our global error bound result, Lemma 3.1, is based on Kuznetsov's lemma [11], which states that if $p,q\in L^\infty(\mathbf{R}\times[0,T])$ satisfy (3.3) and (3.4), respectively, then for any $t\in[\epsilon,T]$,

$$\begin{aligned}&\|p(\cdot,t)-q(\cdot,t)\|_{L^1(\mathbf{R})}\\ &\le \|p(\cdot,0)-q(\cdot,0)\|_{L^1(\mathbf{R})}+\Lambda(p,q,t)+\Lambda(q,p,t)+(A+B)\epsilon.\end{aligned} \tag{3.6}$$

Propositions 2.2–2.4 indicate that u, u_Δ, and u_δ belong to $L^\infty(\mathbf{R}\times[0,T])\cap L^1(\mathbf{R}\times[0,T])$ and satisfy (3.3) and (3.4).

LEMMA 3.1. *Assume function $q\in L^\infty(\mathbf{R}\times[0,T])\cap L^1(\mathbf{R}\times[0,T])$ satisfying the condition* (3.4). *Let u_0, f, and g be subject to the conditions stated in Theorem* 1.1. *Moreover, if u, the entropy solution of* (1.1)–(1.2), *and q satisfy the condition*

$$\Lambda(q,u,t_n) \le -2\sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_S \operatorname{sign}(u-q')g(q')\omega dsdx'd\tau' + C\gamma, \tag{3.7}$$

with $\gamma=\gamma(\epsilon,\Delta t,\delta)>0$, where $S=\mathbf{R}\times[0,t_n]$, then the following result holds:

$$\begin{aligned}&\|u(\cdot,t_n)-q(\cdot,t_n)\|_{L^1(\mathbf{R})}\\ &\le \|u(\cdot,0)-q(\cdot,0)\|_{L^1(\mathbf{R})}+C\left(\epsilon+\gamma+\Delta t+\frac{\Delta t}{\epsilon}\right).\end{aligned} \tag{3.8}$$

Remark. An alternative definition of the functional Λ is

$$\tilde{\Lambda}(p,q,t) = \Lambda(p,q,t) - \int_S\int_S \operatorname{sign}(p-q')g(p)\omega dsds', \tag{3.9}$$

which incorporates the inhomogeneous term into $\tilde{\Lambda}$ (see [23], [24]). By the entropy condition (2.2) we have $\tilde{\Lambda}(u,q,t_n)\le 0$ for any $q\in L^\infty(\mathbf{R}\times[0,T])$. The Kuznetsov's lemma can also be generalized for $\tilde{\Lambda}$ with no problem. However, if we use the fact $\tilde{\Lambda}(u,q,t_n)\le 0$ and try to bound $\tilde{\Lambda}(q,u,t_n)$, then the global error estimate cannot be obtained. For example, in the case $q=u_\Delta$, the splitting method yields (cf. (4.6))

$$\begin{aligned}\tilde{\Lambda}(u_\Delta,u,t_n) &\le 2\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_{m+1/2}}^{t_{m+1}} \operatorname{sign}(u_\Delta-u')g(u_\Delta)\omega d\tau dxds'\\ &\qquad -\int_S\int_S \operatorname{sign}(u_\Delta-u')g(u_\Delta)\omega dsds' + \frac{C\Delta t}{\epsilon}\\ &= \sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_{m+1/2}}^{t_{m+1}} \Big\{\operatorname{sign}(u_\Delta(x,r)-u')g(u_\Delta(x,r))\\ &\qquad \times\omega(x,x',r,\tau')\Big\}\Big|_{r=\tau-\Delta t/2}^{r=\tau} d\tau dxds' + \frac{C\Delta t}{\epsilon}.\end{aligned}$$

Since the function sign$(\cdot)$ is discontinuous, it seems impossible to obtain an $o(\Delta t)$ error bound from the last integral term. To avoid the discontinuity of the sign function, we replace the integral term in (3.9) by the integral term in (3.7). We also estimate both $\Lambda(u,q,t_n)$ and $\Lambda(q,u,t_n)$ so that the resulting inequality does not involve subtraction of two sign functions (cf. (3.11)).

In order to obtain Lemma 3.1, we need to use the following inequality; we defer its proof to the Appendix and simply give the result here:

$$\begin{aligned}(3.10)\qquad &\Lambda(u,q,t_n)\\ &\leq 2\sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_S \operatorname{sign}(u-q')g(u)\omega ds dx' d\tau' + C\left(\Delta t+\frac{\Delta t}{\epsilon}\right).\end{aligned}$$

Note that g satisfies a Lipschitz condition. From (3.6), (3.7), and (3.10), we have, for all $t_n \in [\epsilon, T]$,

$$\begin{aligned}(3.11)\qquad &\|u(\cdot,t_n)-q(\cdot,t_n)\|_{L^1(\mathbf{R})}\\ &\leq 2\sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_S \operatorname{sign}(u-q')(g(u)-g(q'))\omega ds dx' d\tau'\\ &\qquad +\|u(\cdot,0)-q(\cdot,0)\|_{L^1(\mathbf{R})}+C\left(\gamma+\epsilon+\Delta t+\frac{\Delta t}{\epsilon}\right)\\ &\leq 2\int_S\int_S |g(u)-g(q')|\omega ds ds' + \|u(\cdot,0)-q(\cdot,0)\|_{L^1(\mathbf{R})}\\ &\qquad +C\left(\gamma+\epsilon+\Delta t+\frac{\Delta t}{\epsilon}\right)\\ &\leq C\int_S\int_S |u-q'|\omega ds ds' + \|u(\cdot,0)-q(\cdot,0)\|_{L^1(\mathbf{R})}\\ &\qquad +C\left(\gamma+\epsilon+\Delta t+\frac{\Delta t}{\epsilon}\right).\end{aligned}$$

Using the properties of ω, we can easily show that

$$(3.12)\qquad \int_S\int_{\mathbf{R}}\Big|u(x,\tau)-q(x',\tau')\Big|\omega dx ds' \leq \|u(\cdot,\tau)-q(\cdot,\tau)\|_{L^1(\mathbf{R})}+C\epsilon.$$

From (3.11) and (3.12), we obtain that for all $t_n \in [\epsilon, T]$

$$\begin{aligned}&\|u(\cdot,t_n)-q(\cdot,t_n)\|_{L^1(\mathbf{R})} \leq C\int_0^{t_n}\|u(\cdot,\tau)-q(\cdot,\tau)\|_{L^1(\mathbf{R})}d\tau\\ (3.13)\qquad &\qquad +\|u(\cdot,0)-q(\cdot,0)\|_{L^1(\mathbf{R})}+C\left(\gamma+\epsilon+\Delta t+\frac{\Delta t}{\epsilon}\right).\end{aligned}$$

In the case $t_n \in [0,\epsilon]$, we obtain from (2.6) and (3.4) that $\|u(\cdot,t_n)-q(\cdot,t_n)\|_{L^1(\mathbf{R})} \leq \|u(\cdot,0)-q(\cdot,0)\|_{L^1(\mathbf{R})}+C\epsilon$. This, together with the Gronwall-type inequality (3.13), yields (3.8). Therefore, we have proved Lemma 3.1.

4. Proof of Theorem 1.1. We begin by estimating $\Lambda(u_\Delta,u,t_n)$, where u_Δ is defined by (2.7) and u is the unique entropy solution of (1.1)–(1.2).

LEMMA 4.1.

$$-\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_m}^{t_{m+1/2}}\Big(|u_\Delta-u'|\omega_\tau+2\mathrm{sign}(u_\Delta-u')(f(u_\Delta)-f(u'))\omega_x\Big)d\tau dx ds'$$

$$\leq -\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\Big(|u_\Delta-u'|\omega\Big)\Big|_{t_m}^{t_{m+1/2}}dxds'; \tag{4.1}$$

$$-\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_{m+1/2}}^{t_{m+1}}\Big(|u_\Delta-u'|\omega_\tau+2\mathrm{sign}(u_\Delta-u')g(u_\Delta)\omega\Big)d\tau dxds'$$

$$=-\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\Big(|u_\Delta-u'|\omega\Big)\Big|_{t_{m+1/2}}^{t_{m+1}}dxds'. \tag{4.2}$$

Proof. Given $\phi(x,\tau)$ belonging to $C_0^1(\mathbf{R}\times\overline{\mathbf{R}^+})$ and $k\in\mathbf{R}$. Let $\psi(x,\tau)=\phi(x,\tau/2)$ and

$$u_1^{(m)}(r)=S_1(r)(S_2(\Delta t)S_1(\Delta t))^m u_0, \qquad u_2^{(m)}(r)=S_2(r)S_1(\Delta t)(S_2(\Delta t)S_1(\Delta t))^m u_0,$$

where $0\leq r\leq\Delta t$. Using the change of variable $r=2(\tau-t_m)$ and using the definition of u_Δ, we have

$$\begin{aligned}
&-\int_{\mathbf{R}}\int_{t_m}^{t_{m+1/2}}\Big(|u_\Delta-k|\phi_\tau+2\mathrm{sign}(u_\Delta-k)(f(u_\Delta)-f(k))\phi_x\Big)d\tau dx \\
&=-\int_{\mathbf{R}}\int_0^{\Delta t}\Big(|u_1^{(m)}-k|\psi_r^{(m)}+\mathrm{sign}(u_1^{(m)}-k)(f(u_1^{(m)})-f(k))\psi_x^{(m)}\Big)drdx \\
&\leq-\int_{\mathbf{R}}\Big(|u_1^{(m)}-k|\psi^{(m)}\Big)\Big|_{r=0}^{r=\Delta t}dx, \\
&=-\int_{\mathbf{R}}\Big(|u_\Delta(x,\tau)-k|\phi(x,\tau)\Big)\Big|_{\tau=t_{m+1/2}}^{\tau=t_{m+1}}dx.
\end{aligned} \tag{4.3}$$

where $\psi^{(m)}(x,r)=\psi(x,t_{2m}+r)$, and where the second step follows from the entropy condition for $u_1^{(m)}$, since $u_1^{(m)}$ is the (unique) entropy solution of (1.3). By setting $k=u(x',\tau'),\phi(x,\tau)=\omega(x,x',\tau,\tau')$ in (4.3) and integrating the resulting inequality with respect to x' and τ', we obtain (4.1). Next, using the change of variable $r=2(\tau-t_{m+1/2})$ and using the integration by parts, we obtain

$$\begin{aligned}
&-\int_{t_{m+1/2}}^{t_{m+1}}\Big(|u_\Delta-k|\phi_\tau+2\mathrm{sign}(u_\Delta-k)g(u_\Delta)\phi\Big)d\tau \\
&=-\int_0^{\Delta t}\Big(|u_2^{(m)}-k|\psi_r^{(m+1/2)}+\mathrm{sign}(u_2^{(m)}-k)g(u_2^{(m)})\psi^{(m+1/2)}\Big)dr \\
&=-\Big(|u_2^{(m)}-k|\psi^{(m+1/2)}\Big)\Big|_{r=0}^{r=\Delta t}+\int_0^{\Delta t}\mathrm{sign}(u_2^{(m)}-k)\Big((u_2^{(m)})_r-g(u_2^{(m)})\Big)\psi dr,
\end{aligned} \tag{4.4}$$

where $\psi^{(m+1/2)}(x,r)=\psi(x,t_{2m+1}+r)$. Since $u_2^{(m)}(r)$ satisfies (1.4) for $r\in[0,\Delta t]$, it follows that the last term of (4.4) is 0. Then (4.2) is obtained. □

Remark. Lemma 4.1 characterizes the main property of the time-splitting method. In the time interval $[t_m,t_{m+1/2})$ we used the entropy condition for the solution operator S_1, while in $[t_{m+1/2},t_{m+1})$ we noticed that S_2 is an ordinary differential equation (ODE) solution operator.

As in the proof for (A.2), the following inequality can be established:

$$
\begin{aligned}
(4.5)\quad &-\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_{m+1/2}}^{t_{m+1}} \operatorname{sign}(u_\Delta-u')(f(u_\Delta)-f(u'))\omega_x d\tau dx ds' \\
&+\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_m}^{t_{m+1/2}} \operatorname{sign}(u_\Delta-u')(f(u_\Delta)-f(u'))\omega_x d\tau dx ds' \le C\Delta t/\epsilon.
\end{aligned}
$$

Adding the inequalities (4.1), (4.2), and (4.5) yields

$$
(4.6)\quad \Lambda(u_\Delta,u,t_n)\le 2\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_{m+1/2}}^{t_{m+1}} \operatorname{sign}(u_\Delta-u')g(u_\Delta)\omega d\tau dx ds' + \frac{C\Delta t}{\epsilon}.
$$

By changing notations, $\tau\leftrightarrow\tau', x\leftrightarrow x'$, and noting that $\omega(x,x',\tau,\tau')=\omega(x',x,\tau',\tau)$, we get

$$
\Lambda(u_\Delta,u,t_n)\le -2\sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_S \operatorname{sign}(u-u'_\Delta)g(u'_\Delta)\omega ds dx' d\tau' + \frac{C\Delta t}{\epsilon}.
$$

Noting that $u_\Delta(\cdot,0)=u(\cdot,0)$, Lemma 3.1 gives

$$
(4.7)\quad \|u(\cdot,t_n)-u_\Delta(\cdot,t_n)\|_{L^1(\mathbf{R})}\le C\left(\epsilon+\Delta t+\frac{\Delta t}{\epsilon}\right).
$$

Hence, setting $\epsilon=\sqrt{\Delta t}$ in (4.7) yields Theorem 1.1.

5. Proof of Theorem 1.2.

We now estimate $\Lambda(u_\delta,u,t_n)$, where u_δ is defined by (2.8) and u is the entropy solution of (1.1)–(1.2).

LEMMA 5.1.

$$
\begin{aligned}
&-\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_m}^{t_{m+1/2}} \Big(|u_\delta-u'|\omega_\tau + 2\operatorname{sign}(u_\delta-u')(f(u_\delta)-f(u'))\omega_x\Big) d\tau dx ds' \\
(5.1)\quad &+\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\Big(|u_\delta-u'|\omega\Big)\Big|_{\tau=t_m}^{\tau=t_{m+1/2}} dx ds' \le \frac{C\delta}{\epsilon};
\end{aligned}
$$

Proof. We shall use the following notations:

$$
\begin{aligned}
&T_t^+v^m = v^{m+1/2}-v^m, \qquad T_x^+v_j=v_{j+1}-v_j;\\
&F(v(x,t),w)=\operatorname{sign}(v(x,t)-w)(f(v(x,t))-f(w)).
\end{aligned}
$$

It can be verified that the left-hand side of (5.1) is equal to $I_{13}+I_{14}+I_{15}+I_{16}$, where

$$
\begin{aligned}
I_{13} &:= -\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_m}^{t_{m+1/2}} |u_\delta(x,t_{m+1/2})-u'|\omega_\tau d\tau dx ds' \\
&\quad+\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\Big(|u_\delta-u'|\omega\Big)\Big|_{\tau=t_m}^{\tau=t_{m+1/2}} dx ds',\\
I_{14} &:= -\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_m}^{t_{m+1/2}} 2F(u_\delta(x,t_m),u')\omega_x d\tau dx ds',
\end{aligned}
$$

$$I_{15} := \sum_{m=0}^{n-1} \int_S \int_{\mathbf{R}} \int_{t_m}^{t_{m+1/2}} \Big(|u_\delta(x, t_{m+1/2}) - u'| - |u_\delta(x,\tau) - u'| \Big) \omega_\tau d\tau dx ds',$$

$$I_{16} := \sum_{m=0}^{n-1} \int_S \int_{\mathbf{R}} \int_{t_m}^{t_{m+1/2}} 2\Big(F(u_\delta(x, t_m), u') - F(u_\delta(x,\tau), u') \Big) \omega_x d\tau dx ds'.$$

Using (2.16) and the following facts (cf. the proof for (A.2)),

$$\Big| |u_\delta(x, t_{m+1/2}) - u'| - |u_\delta(x,\tau) - u'| \Big| \le |u_\delta(x, t_{m+1/2}) - u_\delta(x,\tau)|,$$

$$\Big| F(u_\delta(x, t_m), u') - F(u_\delta(x,\tau), u') \Big| \le C|u_\delta(x,t_m) - u_\delta(x,\tau)|,$$

we can show that $I_{15} \le C\delta/\epsilon$ and $I_{16} \le C\delta/\epsilon$. Also, we can show that

$$I_{13} = \sum_{m=0}^{n-1} \int_S \int_{\mathbf{R}} \Big(|u_\delta(x, t_{m+1/2}) - u'| - |u_\delta(x, t_m) - u'| \Big) \omega(x, x', t_m, t') dx ds'.$$

Using the Mean Value theorem, we have

$$\begin{aligned} I_{13} &= \sum_{m=0}^{n-1} \int_S \sum_j T_t^+ |u_j^m - u'| \int_{x_{j-1/2}}^{x_{j+1/2}} \omega(x, x', t_m, \tau') dx ds' \\ &= \sum_{m=0}^{n-1} \int_S \sum_j T_t^+ |u_j^m - u'| \Delta x \omega(x_j^*, x', t_m, \tau') ds', \end{aligned} \tag{5.2}$$

where $x_j^* \in (x_{j-1/2}, x_{j+1/2})$. Similarly, for I_{14} we have

$$\begin{aligned} I_{14} = &- \sum_{m=0}^{n-1} \int_S \int_{t_m}^{t_{m+1/2}} \sum_j 2F(u_\delta(x_j, t_m), u')(\omega(x_{j+1/2}, x', \tau, \tau') \\ &- \omega(x_{j-1/2}, x', \tau, \tau')) d\tau ds' \\ = &\sum_{m=0}^{n-1} \int_S \sum_j 2T_x^+ F(u_j^m, u') \int_{t_m}^{t_{m+1/2}} \omega(x_{j+1/2}, x', \tau, \tau') d\tau ds' \\ = &\sum_{m=0}^{n-1} \int_S \sum_j T_x^+ F(u_j^m, u') \Delta t \omega(x_{j+1/2}, x', t_m^*, \tau') ds', \end{aligned} \tag{5.3}$$

where $t_m^* \in (t_m, t_{m+1/2})$. Using the following results,

$$\sum_{m=0}^{n-1} \|u_\delta(\cdot, t_{m+1/2}) - u_\delta(\cdot, t_m)\|_{L^1(\mathbf{R})} \le C,$$

$$\sum_{m=0}^{n-1} \|u_\delta(\cdot + \Delta x, t_m) - u_\delta(\cdot, t_m)\|_{L^1(\mathbf{R})} \le C,$$

$$\int_S |\omega(x_1, x', \tau_1, \tau') - \omega(x_2, x', \tau_2, \tau')| ds' \le \frac{C\delta}{\epsilon},$$

$$\text{if} \quad \max\{|x_1 - x_2|, |\tau_1 - \tau_2|\} \le \delta,$$

we obtain from (5.2) and (5.3) that

$$
\begin{aligned}
(5.4)\qquad & I_{13}+I_{14}\\
&\le \sum_{m=0}^{n-1}\sum_j \int_S \Big(\Delta x T_t^+|u_j^m-u'| + \Delta t T_x^+ F(u_j^m,u')\Big)\omega(x_j,x',t_m,\tau')ds' + \frac{C\delta}{\epsilon}.
\end{aligned}
$$

Since $u_j^{m+1/2} = G_1(\Delta t)u_j^m$ with G_1 a monotone scheme, there exists a Lipschitz continuous numerical entropy flux [5] satisfying

$$
(5.5)\ \bar F(v,\dots,v)=F(v,u'),\quad \Delta x T_t^+|u_j^m-u'| + \Delta t T_x^+\bar F(u_{j-p}^m,\dots,u_{j+p-1}^m)\le 0.
$$

The first part of (5.5), together with (5.4), yields

$$
\begin{aligned}
& I_{13}+I_{14}\\
&\le \sum_{m=0}^{n-1}\sum_j \int_S \Big(\Delta x T_t^+|u_j^m-u'| + \Delta t T_x^+ \bar F(u_{j-p}^m,\dots,u_{j+p-1}^m)\Big)\omega(x_j,x',t_m,\tau')ds'\\
&+\sum_{m=0}^{n-1}\sum_j \int_S \Delta t T_x^+\Big(\bar F(u_j^m,\dots,u_j^m) - \bar F(u_{j-p}^m,\dots,u_{j+p-1}^m)\Big)\omega(x_j,x',t_m,\tau')ds'\\
&+\frac{C\delta}{\epsilon}.
\end{aligned}
$$

The inequality in (5.5) indicates that the first term of the right-hand side of the above inequality is nonpositive. Also, since $\bar F$ satisfies a Lipschitz condition, the second term of the right-hand side of the above inequality can be bounded by

$$
\sum_{m=0}^{n-1}\sum_j \int_S \Delta t|\bar F(u_j^m,\dots,u_j^m) - \bar F(u_{j-p}^m,\dots,u_{j+p-1}^m)||T_x^+\omega(x_j,x',t_m,\tau')|ds' \le \frac{C\delta}{\epsilon}.
$$

Therefore, we have proved that $I_{13}+I_{14}\le C\delta/\epsilon$. This completes the proof of Lemma 5.1. □

LEMMA 5.2.

$$
\begin{aligned}
& -\sum_{m=0}^{n-1}\int_S\int_{\mathbf R}\int_{t_{m+1/2}}^{t_{m+1}} \Big(|u_\delta-u'|\omega_\tau + 2\mathrm{sign}(u_\delta-u')g(u_\delta)\omega\Big)d\tau dx ds'\\
(5.6)\qquad & +\sum_{m=0}^{n-1}\int_S\int_{\mathbf R}\Big(|u_\delta-u'|\omega\Big)\Big|_{\tau=t_{m+1/2}}^{\tau=t_{m+1}} dx ds' \le C\delta.
\end{aligned}
$$

Proof. Using the integration by parts and the definition of u_δ, we obtain

$$
\begin{aligned}
& -\int_{t_{m+1/2}}^{t_{m+1}}\Big(|u_\delta-u'|\omega_\tau + 2\mathrm{sign}(u_\delta-u')g(u_\delta)\omega\Big)d\tau\\
&= -\Big(|u_\delta-u'|\omega\Big)\Big|_{\tau=t_{m+1/2}}^{\tau=t_{m+1}}\\
&\quad+\int_{t_{m+1/2}}^{t_{m+1}} 2\mathrm{sign}(u_\delta-u')\Big(\frac{u_\delta(x,t_{m+1})-u_\delta(x,t_{m+1/2})}{\Delta t} - g(u_\delta)\Big)\omega d\tau\\
&= -\Big(|u_\delta-u'|\omega\Big)\Big|_{\tau=t_{m+1/2}}^{\tau=t_{m+1}}\\
&\quad+\int_{t_{m+1/2}}^{t_{m+1}} 2\mathrm{sign}(u_\delta-u')\Big(g(u_\delta(x,t_{m+1/2})) - g(u_\delta(x,\tau))\Big)\omega d\tau.
\end{aligned}
$$

Observe that

$$\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_{m+1/2}}^{t_{m+1}} 2\text{sign}(u_\delta - u')\Big(g(u_\delta(x,t_{m+1/2})) - g(u_\delta(x,\tau))\Big)\omega d\tau dx ds'$$
$$\le 2L\sum_{m=0}^{n-1}\int_{\mathbf{R}}\int_{t_{m+1/2}}^{t_{m+1}} |u_\delta(x,t_{m+1/2}) - u_\delta(x,\tau)| d\tau dx \le C\Delta t \le C\delta.$$

Using the above two results we obtain (5.6). □

Similar to the proof for (A.2) the following inequality can be obtained:

$$\begin{aligned} &-\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_{m+1/2}}^{t_{m+1}} \text{sign}(u_\delta - u')(f(u_\delta) - f(u'))\omega_x d\tau dx ds' \\ &+\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_m}^{t_{m+1/2}} \text{sign}(u_\delta - u')(f(u_\delta) - f(u'))\omega_x d\tau dx ds' \le C\delta/\epsilon. \end{aligned} \tag{5.7}$$

Adding the inequalities (5.1), (5.6), and (5.7) gives

$$\begin{aligned} \Lambda(u_\delta, u, t_n) &\le 2\sum_{m=0}^{n-1}\int_S\int_{\mathbf{R}}\int_{t_{m+1/2}}^{t_{m+1}} \text{sign}(u_\delta - u')g(u_\delta)\omega d\tau dx ds' + \frac{C\delta}{\epsilon} + C\delta \\ &= -2\sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_S \text{sign}(u - u'_\delta)g(u'_\delta)\omega ds dx' d\tau' + \frac{C\delta}{\epsilon} + C\delta. \end{aligned} \tag{5.8}$$

Note that $\|u_\delta(\cdot,0) - u(\cdot,0)\|_{L^1(\mathbf{R})} \le C\delta$. Then Theorem 1.2 can be obtained from Lemma 3.1 by setting $q = u_\delta$ and $\epsilon = \sqrt{\delta}$.

6. Extensions. In this section we describe some extensions of the error bound results given in §1. For simplicity, the proofs are omitted since they are for the most part analogous to the ones given before.

6.1. Conditions for the initial data and the source term. In this subsection, we discuss the conditions for u_0 and g. Unlike the homogeneous case [4], [22], $u_0 \in L^1(\mathbf{R})$ is required when the nonhomogeneous term is given. This condition is used to ensure the continuity in time, i.e., the estimates (2.6), (2.12), and (2.16). These properties are very important in obtaining Theorems 1.1 and 1.2, at least under the general framework used in the present work. Without the assumption $u_0 \in L^1(\mathbf{R})$, none of the estimates (2.6), (2.12), and (2.16) holds. To see this, we consider the case when $f \equiv 0, g(u) = u$, and $u_0 \equiv 1$. In this case, the exact solution is $u(x,t) = e^t$. Therefore, for any $\tau_2 \ne \tau_1$, we have $\|u(\cdot,\tau_2) - u(\cdot,\tau_1)\|_{L^1(\mathbf{R})} = +\infty$, $\|u_\Delta(\cdot,\tau_2) - u_\Delta(\cdot,\tau_1)\|_{L^1(\mathbf{R})} = +\infty$, and $\|u_\delta(\cdot,\tau_2) - u_\delta(\cdot,\tau_1)\|_{L^1(\mathbf{R})} = +\infty$. Also, without the assumption $u_0 \in L^1(\mathbf{R})$, the error estimates in Theorems 1.1 and 1.2 cannot be obtained. Consider the same example (i.e., $f \equiv 0, g(u) = u$, and $u_0 \equiv 1$), we have for any $[a,b] \subset \mathbf{R}$,

$$\|u(\cdot,\Delta t) - u_\delta(\cdot,\Delta t)\|_{L^1([a,b])} = (b-a)\left(e^{\Delta t} - 1 - \Delta t\right). \tag{6.1}$$

If $b - a \to \infty$, then the right-hand side of (6.1) goes to infinity, no matter how small Δt is. Therefore, the error $\|u(\cdot,t) - u_\delta(\cdot,t)\|_{L^1(\mathbf{R})}$ cannot be bounded without the assumption $u_0 \in L^1(\mathbf{R})$.

In most practical problems, $g(0) = 0$ is satisfied (see, e.g., [2], [17]). This condition is also assumed in [1]. Again, without this requirement, the estimates (2.6), (2.12), and (2.16) do not hold. This can be seen by considering the case when $f \equiv 0, g(u) = 1 + u$ and $u_0 \equiv 0$. In this case, the exact solution is $u(x,t) = e^t - 1$, and therefore $\|u(\cdot, \Delta t) - u_\delta(\cdot, \Delta t)\|_{L^1(\mathbf{R})}$ is unbounded.

For the nonhomogeneous problem (1.1)–(1.2), one of the basic problems is the so-called Riemann problem. In this case,

$$u_0(x) = \begin{cases} u_l & \text{if } x \leq \alpha \\ u_r & \text{if } x > \alpha, \end{cases} \tag{6.2}$$

where $\alpha, u_l, u_r \in \mathbf{R}$ are constants. Hence, for the Riemann problem, $u_0 \notin L^1(\mathbf{R})$, but it is easy to see that $u_0 \in L_{loc}(\mathbf{R})$. In order to cover the case $u_0 \notin L^1(\mathbf{R}), u_0 \in L_{loc}(\mathbf{R})$, we give the following results; their proofs are similar to the proofs of Theorems 1.1 and 1.2.

THEOREM 6.1. *Let $u_0 \in \mathrm{BV}(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $f \in C^1(\mathbf{R})$ and assume that g satisfies a Lipschitz condition and $g(0) = 0$. Let $S(t)u_0$ denote the unique weak solution of (1.1)–(1.2) satisfying the entropy condition. Then for all $[a,b] \subset \mathbf{R}$ with a, b finite,*

$$\begin{aligned} &\max_{t_n = n\Delta t \in [0,T]} \left\| S(t_n)u_0 - \left(S_2(\Delta t)S_1(\Delta t)\right)^n u_0 \right\|_{L^1([a,b])} \\ &\leq C\left(1 + \|u_0\|_{L^1([a-ct_n, b+ct_n])}\right)\sqrt{\Delta t}, \end{aligned} \tag{6.3}$$

where $c = \max_{|u| \leq e^{LT}\|u_0\|_{L^\infty(\mathbf{R})}} |f'(u)|$ and C is a constant independent of a, b, and Δt.

THEOREM 6.2. *Let $u_0 \in \mathrm{BV}(\mathbf{R}) \cap L^\infty(\mathbf{R})$, $f \in C^1(\mathbf{R})$, and assume that g satisfies a Lipschitz condition and $g(0) = 0$. Assume the finite difference scheme (1.7)–(1.8) to be monotone and consistent with (1.3), the numerical flux $\bar{f}$ to be Lipschitz continuous, and G_2 to be the forward Euler operator. Then for any $[a,b] \in \mathbf{R}$ with a, b finite,*

$$\begin{aligned} &\max_{t_n = n\Delta t \in [0,T]} \left\| S(t_n)u_0 - \left(G_2(\Delta t)G_1(\Delta t)\right)^n u_0 \right\|_{L^1([a,b])} \\ &\leq C\left(1 + \|u_0\|_{L^1([a-ct_n, b+ct_n])}\right)\sqrt{\delta}, \end{aligned} \tag{6.4}$$

where $c = \max_{|u| \leq e^{LT}\|u_0\|_{L^\infty(\mathbf{R})}} |f'(u)|$, $\delta = \max\{\Delta x, \Delta t\}$, and C is a constant independent of a, b, and δ.

6.2. Multi-dimensional case. The results in the present work can be easily extended to the multi-dimensional nonhomogeneous conservation laws

$$\begin{array}{ll} v_t + \sum_{i=1}^N (f_i(v))_{x_i} = G(v, \underline{x}), & \text{for } t \in (0,T], \underline{x} \in \mathbf{R}^N, \\ v(\underline{x}, 0) = v_0(\underline{x}), & \text{for } \underline{x} \in \mathbf{R}^N, \end{array} \tag{6.5}$$

where $\underline{x} = (x_1, \ldots, x_N) \in \mathbf{R}^N$, $f_i \in C^1(\mathbf{R})$, $v_0 \in L^\infty(\mathbf{R}^N) \cap BV(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$, G satisfies a Lipschitz condition in v uniformly with respect to $\underline{x}$, and in $\underline{x}$ uniformly with respect to v; $G(0, \underline{x}) \equiv 0$. It can be shown that then there is a unique function $v(\underline{x}, t)$, called the entropy solution of (6.5), which satisfies $v \in L^\infty(\mathbf{R}^N) \cap BV(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$, $v(\cdot, 0) = v_0$, and the entropy condition: for each $\phi \in C_0^1(\mathbf{R}^N \times (0,T))$, $\phi \geq 0$, and $k \in \mathbf{R}$

$$\int_0^T \int_{\mathbf{R}^N} \left(|v-k|\phi_t + \sum_{i=1}^N \operatorname{sign}(v-k)\left(f_i(v) - f_i(k)\right)\phi_{x_i} + \operatorname{sign}(v-k)G(v, \underline{x})\phi\right) d\underline{x}dt \geq 0.$$

Denote the entropy solution of (6.5) by $S^{[\underline{f},G]}(t)v_0$ with $\underline{f} := (f_1, \dots, f_N) \in \mathbf{R}^N$. That is, $S^{[\underline{f},G]}(t)v_0 = v(\underline{x}, t)$, where v is the entropy solution of (6.5). Set $S^{(f_i,0)}(t) = S^{[(0,\dots,0,f_i,0,\dots,0),0]}(t)$. That is, $S^{(f_i,0)}$ is the solution operator of the following equation:

$$\frac{\partial v}{\partial t}(x_1, \dots, x_N, t) + \frac{\partial f_i(v)}{\partial x_i} = 0,$$

where $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \in \mathbf{R}^{N-1}$ is regarded as parameters. Also, we define $S^{(0,G)}(t)$ as the solution operator of the following equation:

$$\frac{\partial v}{\partial t}(\underline{x}, t) = G(v(\underline{x}, t), \underline{x}).$$

Further, we define $\{h(i)\}_{i=1}^N$ such that $h(i) \neq h(j)$ if $i \neq j$ and $h(i) \in \{f_1, \dots, f_N\}$. Using the techniques developed in [22] and in this work, we can show that for $n\Delta t \in [0, T]$,

$$\left\| S^{[\underline{f},G]}(n\Delta t)v_0 - \left(S^{(0,G)}(\Delta t) \prod_{i=1}^{N} S^{(h(i),0)}(\Delta t) \right)^n v_0 \right\|_{L^1(\mathbf{R}^N)} \leq C\sqrt{\Delta t}, \tag{6.6}$$

where C is a positive constant independent of Δt.

7. Conclusions. In this work, we have proved that the nonlinear semigroup $S(t)$, the exact solution operator of (1.1), can be approximated by $(S_2(\Delta t)S_1(\Delta t))^n$, as $|t - n\Delta t| < \Delta t \ll 1$, where S_1 and S_2 are the solution operators of (1.3) and (1.4), respectively. In the homogeneous case, i.e., $g \equiv 0$, setting $p = u$, the exact solution of (1.3), in (3.6) and using the entropy condition, we have $\Lambda(u, q, t) \leq 0$ for all $q \in L^\infty(\mathbf{R})$. Therefore, it is not necessary to estimate the term $\Lambda(u, q, t_n)$ in this case. However, for nonhomogeneous conservation laws, this term plays an important role in the error analysis and therefore a detail analysis for $\Lambda(u, q, t_n)$ is required.

In the homogeneous case, several authors provided convergence rates for various difference schemes or splitting methods (see, e.g., [11], [14], [15], [16], [19], [22], [24]). Although their problems are different, the convergence rates obtained are half in most of the cases. Therefore, the convergence rates obtained in this paper might be the best possible. In particular, the convergence rate in Theorem 1.2 should be the optimal one, since even in the homogeneous case the optimum rate for monotone schemes is 1/2 (see [21]). However, no numerical evidence has been found so far to confirm that the convergence order in (1.10) of Theorem 1.1, or in the general case (6.6), is the best possible (see also the numerical experiments in [22]). It remains to be seen if the optimum convergence rates in these two cases are higher than 1/2.

A. Appendix. Again, for ease of notation, we denote $u(x, \tau)$, $p(x, \tau)$, $u(x', \tau')$, and $q(x', \tau')$ by u, p, u', and q', respectively, in most of the integrands. Also, we assume in this appendix that $p, q \in L^\infty(\mathbf{R} \times [0, T]) \cap L^1(\mathbf{R} \times [0, T])$ satisfy (3.3) and (3.4), respectively. In order to obtain (3.10), we first prove the following inequalities:

$$\begin{aligned} &-\sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1/2}} \int_{\mathbf{R}} \int_S |p - q'| \omega_\tau ds dx' d\tau' \\ &+\sum_{m=0}^{n-1} \int_{t_{m+1/2}}^{t_{m+1}} \int_{\mathbf{R}} \int_S |p - q'| \omega_\tau ds dx' d\tau' \leq \frac{C\Delta t}{\epsilon}, \end{aligned} \tag{A.1}$$

$$-\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1/2}}\int_{\mathbf{R}}\int_S \operatorname{sign}(p-q')(f(p)-f(q'))\omega_x ds dx' d\tau'$$

$$(A.2)\qquad +\sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_S \operatorname{sign}(p-q')(f(p)-f(q'))\omega_x ds dx' d\tau' \le C\frac{\Delta t}{\epsilon},$$

$$\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1/2}}\int_{\mathbf{R}}\int_{\mathbf{R}}\Big\{|p-q'|\omega\Big\}\Big|_{\tau=0}^{\tau=t_n} dx dx' d\tau'$$

$$(A.3)\qquad -\sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_{\mathbf{R}}\Big\{|p-q'|\omega\Big\}\Big|_{\tau=0}^{\tau=t_n} dx dx' d\tau' \le C\left(\Delta t+\frac{\Delta t}{\epsilon}\right).$$

Proof of (A.1). By letting $\tau' \to \tau' - \Delta t/2$ and $\tau \to \tau - \Delta t/2$, we rewrite the first term of (A.1) as

$$-\sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_{\Delta t/2}^{t_n+\Delta t/2}\int_{\mathbf{R}} |p(x,\tau-\Delta t/2)-q(x',\tau'-\Delta t/2)|\omega_\tau dx d\tau dx' d\tau',$$

where we have used the fact that $\omega_\tau(x,x',\tau-\Delta t/2,\tau'-\Delta t/2) = \omega_\tau(x,x',\tau,\tau')$. Therefore, the left-hand side of (A.1) can be written as $I_1+I_2+I_3$, where

$$I_1 := \sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_0^{\Delta t/2}\int_{\mathbf{R}}\Big(|p-q'|\omega_\tau\Big) dx d\tau dx' d\tau';$$

$$I_2 := \sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_{\Delta t/2}^{t_n}\int_{\mathbf{R}}\Big(|p-q'| - |p(x,\tau-\Delta t/2)-q(x',\tau'-\Delta t/2)|\Big)\omega_\tau dx d\tau dx' d\tau';$$

$$I_3 := -\sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_{t_n}^{t_n+\Delta t/2}\int_{\mathbf{R}}\Big|p(x,\tau-\Delta t/2) - q(x',\tau'-\Delta t/2)\Big|\omega_\tau dx d\tau dx' d\tau'.$$

Since $p, q \in L^1(\mathbf{R}\times[0,T])$, we have

$$I_1 \le \int_S\int_0^{\Delta t/2}\int_{\mathbf{R}} |p||\omega_\tau| dx d\tau ds' + \int_S\int_0^{\Delta t/2}\int_{\mathbf{R}} |q'||\omega_\tau| dx d\tau ds' \le C\Delta t/\epsilon.$$

Similarly, we can show that I_3 has the same bound. Finally, noting that

$$\begin{aligned} I_2 &\le \int_{\Delta t/2}^{t_n}\int_{\mathbf{R}}\int_{\Delta t/2}^{t_n}\int_{\mathbf{R}}\Big(|p(x,\tau-\Delta t/2)-p(x,\tau)| \\ &\qquad + |q(x',\tau'-\Delta t/2)-q(x',\tau')|\Big)|\omega_\tau| ds ds' \\ &\le \frac{C}{\epsilon}\Big(\int_{\Delta t/2}^{t_n}\|p(\cdot,\tau-\Delta t/2)-p(\cdot,\tau)\|_{L^1(\mathbf{R})}d\tau \\ &\qquad + \int_{\Delta t/2}^{t_n}\|q(\cdot,\tau'-\Delta t/2)-q(\cdot,\tau')\|_{L^1(\mathbf{R})}d\tau'\Big) \end{aligned}$$

and using (3.3) and (3.4), we obtain $I_2 \le C\Delta t/\epsilon$. □

Proof of (A.2). Since $\text{sign}(p-q')(f(p)-f(q')) = f(p\vee q') - f(p\wedge q')$, where $a\vee b = \max\{a,b\}, a\wedge b = \min\{a,b\}$, the left-hand side of (A.2) can be written as $I_4 + I_5 + I_6$, where

$$I_4 := \sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_0^{\Delta t/2}\int_{\mathbf{R}}\Big(f(p\vee q') - f(p\wedge q')\Big)\omega_x dxd\tau dx'd\tau';$$

$$I_5 := \sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_{\Delta t/2}^{t_n}\int_{\mathbf{R}}\Big\{f\Big(p\vee q'\Big) - f\Big(p(x,\tau-\Delta t/2)\vee q(x',\tau'-\Delta t/2)\Big)$$
$$-f\Big(p\wedge q'\Big) + f\Big(p(x,\tau-\Delta t/2)\wedge q(x',\tau'-\Delta t/2)\Big)\Big\}\omega_x dxd\tau dx'd\tau';$$

$$I_6 := -\sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_{t_n}^{t_n+\Delta t/2}\int_{\mathbf{R}}\Big\{f\Big(p(x,\tau-\Delta t/2)\vee q(x',\tau'-\Delta t/2)\Big)$$
$$-f\Big(p(x,\tau-\Delta t/2)\wedge q(x',\tau'-\Delta t/2)\Big)\Big\}\omega_x dxd\tau dx'd\tau'.$$

Let

$$M = \max\{\|p\|_{L^\infty(\mathbf{R}\times[0,T])}, \|q\|_{L^\infty(\mathbf{R}\times[0,T])}\},\quad M_1 = \max_{|u|\le M}|df(u)/du|.$$

The term I_4 can be bounded by

$$\int_S\int_0^{\Delta t/2}\int_{\mathbf{R}} M_1|p-q'||\omega_x|dxd\tau ds' \le C\Delta t/\epsilon.$$

Similarly, $I_6 \le C\Delta t/\epsilon$. The following simple observations are very useful:

$$|a\vee b - c\vee d| \le |a\vee b - c\vee b| + |c\vee b - c\vee d| \le |a-c| + |b-d|,$$
$$|a\wedge b - c\wedge d| \le |a\wedge b - c\wedge b| + |c\wedge b - c\wedge d| \le |a-c| + |b-d|,$$

which lead to

$$I_5 \le 2M_1\int_{\Delta t/2}^{t_n}\int_{\mathbf{R}}\int_{\Delta t/2}^{t_n}\int_{\mathbf{R}}\Big(|p-p(x,\tau-\Delta t/2)|$$
$$+|q'-q(x',\tau'-\Delta t/2)|\Big)|\omega_x|dsds'.$$

By (3.3) and (3.4) we obtain that $I_5 \le C\Delta t/\epsilon$. This completes the proof of (A.2). □

Proof of (A.3). The left-hand side of (A.3) is equivalent to $I_7 + I_8 + I_9 + I_{10}$, where

$$I_7 := \sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_{\mathbf{R}}\Big(|p(x,t_n) - q(x',\tau'-\Delta t/2)|$$
$$-|p(x,t_n) - q(x',\tau')|\Big)\omega(x,x',t_n,\tau')dxdx'd\tau',$$

$$I_8 := \sum_{m=0}^{n-1}\int_{t_{m+1/2}}^{t_{m+1}}\int_{\mathbf{R}}\int_{\mathbf{R}}\Big(|p(x,0) - q(x',\tau')|$$
$$-|p(x,0) - q(x',\tau'-\Delta t/2)|\Big)\omega(x,x',0,\tau')dxdx'd\tau',$$

$$I_9 := \sum_{m=0}^{n-1} \int_{t_{m+1/2}}^{t_{m+1}} \int_{\mathbf{R}} \int_{\mathbf{R}} |p(x,t_n) - q(x',\tau' - \Delta t/2)| \Big(\omega(x,x',t_n,\tau' - \Delta t/2) - \omega(x,x',t_n,\tau')\Big) dx dx' d\tau',$$

$$I_{10} := \sum_{m=0}^{n-1} \int_{t_{m+1/2}}^{t_{m+1}} \int_{\mathbf{R}} \int_{\mathbf{R}} |p(x,0) - q(x',\tau' - \Delta t/2)| \Big(\omega(x,x',0,\tau') - \omega(x,x',0,\tau' - \Delta t/2)\Big) dx dx' d\tau'.$$

Similar to the proofs of (A.1) and (A.2) we can show that $I_7 \le C\Delta t, I_8 \le C\Delta t, I_9 \le C\Delta t/\epsilon, I_{10} \le C\Delta t/\epsilon$. $\square$

We now turn to the proof of (3.10). Setting $p = u$ in (A.1), (A.2), and (A.3) and adding the resulting inequalities yields

$$\begin{aligned} &\Lambda(u,q,t_n) \\ &\le -2 \sum_{m=0}^{n-1} \int_{t_{m+1/2}}^{t_{m+1}} \int_{\mathbf{R}} \int_{S} \Big(|u-q'|\omega_\tau + \operatorname{sign}(u-q')(f(u)-f(q'))\omega_x\Big) ds dx' d\tau' \\ &+2 \sum_{m=0}^{n-1} \int_{t_{m+1/2}}^{t_{m+1}} \int_{\mathbf{R}} \int_{\mathbf{R}} \Big\{|u-q'|\omega\Big\} \Big|_{\tau=0}^{\tau=t_n} dx dx' d\tau' + C\left(\Delta t + \frac{\Delta t}{\epsilon}\right). \end{aligned} \tag{A.4}$$

Since u is the entropy solution of (1.1)–(1.2), it satisfies the entropy condition (2.2). By letting $k = q(x',\tau'), \psi(x,\tau) = \omega(x,x',\tau,\tau'), \tau_1 = 0$, and $\tau_2 = t_n$ in (2.2), we obtain

$$\begin{aligned} &-\int_S \Big(|u-q'|\omega_\tau + \operatorname{sign}(u-q')(f(u)-f(q'))\omega_x\Big) ds \\ &+ \int_{\mathbf{R}} \Big\{|u-q'|\omega\Big\} \Big|_{\tau=0}^{\tau=t_n} dx \le \int_S \operatorname{sign}(u-q') g(u) \omega ds. \end{aligned}$$

Integrating the above inequality with respect to x' in $\mathbf{R}$ and τ' in $\cup_{m=0}^{n-1}[t_{m+1/2}, t_{m+1}]$, respectively, and using (A.4), we obtain (3.10). $\square$

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