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ON THE PIECEWISE SMOOTHNESS OF ENTROPY SOLUTIONS TO SCALAR CONSERVATION LAWS FOR A LARGER CLASS OF INITIAL DATA

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Abstract. We prove that if the initial data do not belong to a certain subset of C^k, then the solutions of scalar conservation laws are piecewise C^k smooth. In particular, our initial data allow centered compression waves, which was the case not covered by Dafermos (1974) and Schaeffer (1973). More precisely, we are concerned with the structure of the solutions in some neighborhood of the point at which only a C^{k+1} shock is generated. It is also shown that there are finitely many shocks for smooth initial data (in the Schwartz space) except for a certain subset of $\mathscr{S}(\mathbb{R})$ of the first category. It should be pointed out that this subset is smaller than those used in previous works. We point out that Thom's theory of catastrophes, which plays a key role in Schaeffer (1973), cannot be used to analyze the larger class of initial data considered in this paper.

Keywords: Piecewise smooth solutions; conservation laws; a set of first category.

Mathematics Subject Classification 2000: 35L65

1. Introduction

Consider the Cauchy problem for the hyperbolic conservation law:

$$\begin{aligned} u_t + f(u)_x = 0 \quad & \text{in } \mathbb{R} \times (0, \infty), \\ u = \phi \quad & \text{on } \mathbb{R} \times \{t = 0\}, \end{aligned} \tag{1.1}$$

where f is C^{k+1} smooth and uniformly convex, i.e. $\partial^2 f/\partial u^2 \geq \epsilon > 0$, the initial data are C^k smooth and bounded, with $3 \leq k \leq \infty$. In general, the problem (1.1) does not admit a global smooth solution even if the initial data are smooth, but for arbitrary bounded measurable initial data a unique global weak solution does exist. The structure of entropy solution has been studied by many authors, e.g. Chen–Zhang [1], Dafermos [2], Lax [5], Li–Wang [6,7], Oleinik [8], Schaeffer [9], and Tadmor–Tassa [10].

The main results of this work will be obtained by using the minimizing process of $F(x, t, u)$ introduced by Lax [5]: for each initial function ϕ, we define a function in $H \times \mathbb{R}$:

$$F(x, t, u) = tg(u) + \Phi(x - ta(u)), \tag{1.2}$$

where $H = \mathbb{R} \times (0, \infty)$, $a(u) = f'(u)$,

$$g(u) = ua(u) - f(u), \quad \Phi(y) = \int_0^y \phi(x)dx.$$

Lax has proved that for almost all (x, t) there exists a unique value of u which minimizes $F(x, t, \bullet)$, and $u(x, t)$, the function defined (almost everywhere) to equal the function minimizing $F(x, t, \bullet)$, is in fact the solution of (1.1). Using the convexity hypothesis and the boundedness of ϕ, we have $F(x, t, u) \to +\infty$ as $u \to \pm\infty$. Therefore, $F(x, t, \bullet)$ always has a minimum, and a minimizing value u must be a critical point of F, a solution of the equation

$$(\partial F/\partial u)(x, t, u) = 0. \tag{1.3}$$

Note that

$$(\partial F/\partial u)(x, t, u) = ta'(u)\{u - \phi(x - ta(u))\}, \tag{1.4}$$

and neither of the factors outside the brackets vanishes. It can be verified that the differential of $\partial F/\partial u$ never vanishes when $\partial F/\partial u = 0$, so (1.3) defines a smooth surface S in $H \times \mathbb{R}$. We record here the following relations that will be needed below:

$$(\partial F/\partial x)(x, t, u) = u \qquad \text{on } S, \tag{1.5}$$

$$(\partial F/\partial t)(x, t, u) = -f(u) \quad \text{on } S. \tag{1.6}$$

Let us introduce some notations and definitions. First let

$$A_\phi(x) = a(\phi(x)), \quad A'_\phi(x) = (a(\phi(x)))', \quad A^{(n)}_\phi(x) = \frac{d^n}{dx^n} A_\phi(x). \tag{1.7}$$

Using the above notations, we further let

$$L_1 = \{x \in \mathbb{R} \mid A'_\phi(x) < 0, A''_\phi(x) = 0, A'''_\phi(x) = 0\},$$
$$L_2 = \{x \in \mathbb{R} \mid A'_\phi(x) < 0, A''_\phi(x) = 0, \ldots, A^{(k)}_\phi(x) = 0, k \geq 3\}$$

and

$$\begin{aligned} L_3 = \{x \in \mathbb{R} \mid\ & A'_\phi(x) < 0, A''_\phi(x) = 0, \ldots, A^{(k)}_\phi(x) = 0, (k \geq 3) \text{ and} \\ & \exists\, \xi \in (x, x+\delta) \text{ such that } A''_\phi(\xi) < 0 \text{ or} \\ & \exists\, \eta \in (x-\delta, x) \text{ such that } A''_\phi(\eta) > 0\ \forall\, \delta > 0\}. \end{aligned}$$

It can be verified that L_2 is a proper subset of L_1 when $k > 3$ and L_3 is a proper subset of L_2.

Definition 1.1. Let u_0 be a minimizing value for $F(x_0, t_0, \bullet)$. Then u_0 is called non-degenerate (respectively, degenerate) if $F_{uu}(x_0, t_0, u_0) \neq 0$ (respectively, $= 0$).

Definition 1.2. The solution $u(x,t)$ of (1.1) is said to be piecewise C^k smooth if every bounded subset of $(-\infty, \infty) \times [0, \infty)$ intersects at most a finite number of shocks, every shock is piecewise C^{k+1} smooth. Moreover, $u(x,t)$ is C^k smooth on the complement of the shock set.

To better summarize some previously published results, more definitions are introduced below. Let

$$U = \{(x,t) \mid \exists \text{ a unique minimizer for } F(x,t,\bullet), \text{ at which } F_{uu} \neq 0\}, \tag{1.8a}$$
$$\Gamma_1 = \{(x,t) \mid \exists \text{ two minimizers for } F(x,t,\bullet), \text{ at which } F_{uu} \neq 0\}, \tag{1.8b}$$
$$\Gamma_0^{(c)} = \{(x,t) \mid \exists \text{ three minimizers for } F(x,t,\bullet), \text{ at which } F_{uu} \neq 0\}, \tag{1.8c}$$
$$\Gamma_0^{(f)} = \{(x,t) \mid \exists \text{ a unique minimizer for } F(x,t,\bullet), \text{ at which } F_{uu} = 0, F_u^{(4)} \neq 0\}. \tag{1.8d}$$

Moreover, let

$$\mathscr{M}_{(x,t)} = \{u \mid \text{ all the minimizing values for } F(x,t,\bullet)\}, \tag{1.9a}$$
$$\overline{\Gamma}_1 = \{(x,t) \mid \exists \text{ two connected components of } \mathscr{M}_{(x,t)}\}, \tag{1.9b}$$
$$\begin{aligned} \overline{\Gamma}_0^{(f)} = \{(x,t) \mid\ & \exists \text{ a unique connected component } [\alpha, \beta] \text{ of } \mathscr{M}_{(x,t)}; \\ & F_{uu}(x,t,\alpha) = 0\}, \end{aligned} \tag{1.9c}$$
$$\overline{\Gamma}_0^{(c)} = \{(x,t) \mid \exists\, n \text{ connected components of } \mathscr{M}_{(x,t)}, \text{ where } n \geq 3\}. \tag{1.9d}$$

It is shown by Schaeffer [9] that U is an open subset of $H = \mathbb{R} \times (0, \infty)$, on which the solution $u(x,t)$ is smooth; Γ_1 is a union of smooth curves across which the minimizing function has a jump discontinuity. $\Gamma_0^{(f)}$ consist of isolated points at which the curves in Γ_1 begin. $\Gamma_0^{(c)}$ consist of isolated points at which the curves in Γ_1 collide. It is obvious that

$$\Gamma_0^{(f)} \subset\subset \overline{\Gamma}_0^{(f)}, \quad \Gamma_1 \subset\subset \overline{\Gamma}_1, \quad \Gamma_0^{(c)} \subset\subset \overline{\Gamma}_0^{(c)}. \tag{1.10}$$

It is also proved by Schaeffer [9] that the solutions of (1.1) are piecewise smooth and the total number of shocks is finite for smooth initial data in $\phi \in \mathscr{S}(\mathbb{R})\backslash\Omega$, where $\mathscr{S}(\mathbb{R})$ is the Schwartz space and Ω is a certain subset of $\mathscr{S}(\mathbb{R})$ of the first category. In other words, there is a set $\Omega \subset \mathscr{S}(\mathbb{R})$ of the first category such that for $\phi \in \mathscr{S}(\mathbb{R})\backslash\Omega$

$$H = U \cup \Gamma_1 \cup \Gamma_0^{(c)} \cup \Gamma_0^{(f)}. \tag{1.11}$$

[9, Lemma 1.4] is very important, which shows that for any point $(x_0, t_0) \in \Gamma_0^{(f)}$ there exists a neighborhood Θ such that $\Gamma_1 \cap \Theta$ consists of a half-curve emanating from (x_0, t_0). The minimizing function is smooth on $\Theta'\backslash\Gamma_1$, where $\Theta' = \Theta\backslash\{(x_0, t_0)\}$. The proof is an adaption of standard techniques from the theory of singularities of differentiable mappings, especially Thom's theory of catastrophes [15]. Schaeffer made a serious attempt to make this material accessible to analysts, including reproving the so called universal unfolding of the Riemann–Hugoniot catastrophe, one of the seven elementary catastrophes of Thom [15]. [9, Sec. 2] is devoted to prove the unfolding theorem for the Riemann–Hugoniot catastrophe.

It is natural to ask whether the conclusions of [9, Lemma 1.4] are true when

$$F_u(x_0, t_0, u_0) = 0, \ldots, F_u^{(2n-1)}(x_0, t_0, u_0) = 0, \quad F_u^{(2n)}(x_0, t_0, u_0) > 0, \tag{1.12}$$

where $n \geq 3$ is some integer; or even $F_u^{(m)}(x_0, t_0, u_0) = 0, (m = 1, 2, \ldots)$. Unfortunately, the unfolding theorem for the Riemann–Hugoniot catastrophe is not applicable in this case since $F(x, t, u)$ is unstable as an unfolding of codimension two of $F(x_0, t_0, u)$ when

$$F_u(x_0, t_0, u_0) = 0, \quad F_{uu}(x_0, t_0, u_0) = 0, \quad F_u^{(3)}(x_0, t_0, u_0), \quad F_u^{(4)}(x_0, t_0, u_0) = 0. \tag{1.13}$$

One of the main purposes of this work is to show that the conclusions of [9, Lemma 1.4] are true under the conditions $L_3 = \emptyset$, which is more general than the condition (1.13). The method used in our proof is different from the one used by Schaeffer, which is elementary but technical.

It is shown by Li and Wang [6] that there is a set $\Omega_1 \subset\subset \Omega \subset \mathscr{S}(\mathbb{R})$ of first category such that for $\phi \in \mathscr{S}(\mathbb{R})\backslash\Omega_1$ the solutions are piecewise smooth. The authors also give an explicit conditions on Ω_1:

$$\Omega_1 = \{\phi \in \mathbb{R} \mid A'_\phi(x) < 0, A''_\phi(x) = 0, A'''_\phi(x) = 0, x \in \mathbb{R}\}. \tag{1.14}$$

Dafermos [2] introduced the concept of generalized characteristic to study the solution structures of hyperbolic conservation laws. It is proved that generically the solutions generated by initial data in C^k are piecewise smooth and do not contain centered compression waves. In other words, there is a set $\Omega_2 \subset C^k$ of the first category such that for $\phi \in C^k \backslash \Omega_2$ the solutions are piecewise smooth, where

$$\Omega_2 = \{\phi \in C^k \mid A'_\phi(x) < 0, A''_\phi(x) = 0, \ldots, A^{(k)}_\phi(x) = 0, (k \geq 3), x \in \mathbb{R}\}. \quad (1.15)$$

In this paper, we first generalize [9, Lemma 1.4], i.e. the case that $(x_0, t_0) \in \Gamma_0^{(f)}$, to the case that $(x_0, t_0) \in \overline{\Gamma}_0^{(f)}$. As a result, we will prove that there is some neighborhood of (x_0, t_0) such that a unique C^{k+1} smooth shock emanating from (x_0, t_0). We will show that $\overline{\Gamma}_1$ is a union of C^{k+1} curves, except the points at which at least one connected component of $\mathscr{M}$ is not an isolated point, across which the minimizing function has a jump discontinuity; $\overline{\Gamma}_0^{(f)}$ ($\overline{\Gamma}_0^{(c)}$) consists of isolated points at which the curves in Γ_1 begin (collide). We also show that for any point $(x_0, t_0) \in \overline{\Gamma}_0^{(c)}$, there exist finitely many, say n, connected components of $\mathscr{M}_{(x_0,t_0)}$ and a neighborhood Θ of (x_0, t_0) such that $\Gamma_1 \cap \Theta$ is the union of n half shocks, $n-1$ terminating at and one emanating from (x_0, t_0). The minimizing function is C^k smooth on each n components of $\Theta' \backslash \Gamma_1$. The shocks are piecewise C^{k+1} smooth and are not differentiable only at shock intersection points and at centers of centered compression waves. We prove that there is a set $\Omega_3 \subset C^k(\mathbb{R})$ of the first category such that for any $\phi \in C^k(\mathbb{R}) \backslash \Omega_3$, we have $H = U \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$ and the minimizing process leads to the piecewise C^k smooth solutions of (1.1) pointwise in H, where

$$\begin{aligned} \Omega_3 = \{\phi \in C^k \mid\ & A'_\phi(x) < 0, A''_\phi(x) = 0, \ldots, A^{(k)}_\phi(x) = 0, (k \geq 3) \text{ and} \\ & \exists\, \xi \in (x, x+\delta) \text{ such that } A''_\phi(\xi) < 0 \text{ or} \\ & \exists\, \eta \in (x-\delta, x) \text{ such that } A''_\phi(\eta) > 0\ \forall\, \delta > 0, x \in \mathbb{R}\}. \end{aligned} \quad (1.16)$$

It is obvious that Ω_3 is a proper subset of Ω_2. To our knowledge, Ω_3 is the smallest in the sense of inclusion relation of sets. We also prove that there is a set $\Omega_4 \subset \mathscr{S}(\mathbb{R})$ of the first category such that for any $\phi \in \mathscr{S}(\mathbb{R}) \backslash \Omega_4$, there are only finite number of shocks. To our knowledge, Ω_4 is also the smallest in the sense of inclusion relation of sets. Since the piecewise smooth solutions may contain centered compression waves, our results indicate that the class of piecewise smooth solutions obtained in this work is bigger than the solution class obtained by Dafermos [2].

This paper is organized as follows. We study the local structure of the solutions of (1.1) in Sec. 2. The piecewise smoothness of the solutions will be established in Sec. 3. Some concluding remarks will be made in the final section.

2. Local Solution Structure

In this section, we will study the local structure of the solutions of (1.1). Some main results in this section are listed below.

- In Theorem 2.11, we study the structure of the solutions in the neighborhoods of the point (x_0, t_0) at which a shock generates. We show that there is a neighborhood such that a unique C^{k+1} smooth shock exists.

- In Theorem 2.13, we study the points at which finitely many shocks collide to form a new shock. All of the shocks are C^{k+1} smooth except at the shock interaction points and points belonging to $\overline{\Gamma}_1 \backslash \Gamma_1$. Moreover, it is shown that the shocks are not differentiable at the shock interaction points and the centers of centered compression waves. In other words, all the shocks are piecewise C^{k+1} curves.

Let $y(x,t,u) = x - ta(u)$. For each $(x,t,u) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, it is easy to see that $y(x,t,u)$ is just the intersection point of a straight line passing through (x,t) with slope $a(u)$ with the line $t = 0$. Note that

$$F_u(x,t,u) = ta'(u)\{u - \phi(x - a(u)t)\}. \tag{2.1}$$

If $F_u(x,t,u) = 0$, then

$$F_{uu}(x,t,u) = ta'(u)[1 + A'_\phi(\bar{y})t]. \tag{2.2}$$

Here and below

$$\bar{y} = y(x,t,u). \tag{2.3}$$

If $F_u(x,t,u) = 0$ and $F_{uu}(x,t,u) = 0$, then

$$\begin{aligned} F_{uuu}(x,t,u) &= ta'(u)[A'_\phi(\bar{y})t]_u = t^2 a'(u)[A'_\phi(\bar{y})]_u \\ &= t^2 a'(u) A''_\phi(\bar{y})(y(x,t,u))_u \\ &= (-t)^3 [a'(u)]^2 A''_\phi(\bar{y}). \end{aligned} \tag{2.4}$$

By induction, if $F_u^{(m)}(x,t,u) = 0, (m = 1, \ldots, n-1)$, then we have

$$F_u^{(n)}(x,t,u) = (-t)^n [a'(u)]^{n-1} A_\phi^{(n-1)}(\bar{y}), \quad (n \geq 3). \tag{2.5}$$

In view of (2.1)–(2.5), it can be verified that if (x,t,u) satisfies

$$F_u^{(m)}(x,t,u) = 0 \ (m = 1, \ldots, 2n-1), \quad F_u^{(2n)}(x,t,u) > 0, \quad (n \geq 2), \tag{2.6}$$

then we have, for $\bar{y} = y(x,t,u) = x - ta(u)$,

$$\begin{array}{ll} A'_\phi(\bar{y}) < 0, \ A_\phi^{(m)}(\bar{y}) = 0, & (m = 2, 3, \ldots, 2n-2) \\ A_\phi^{(2n-1)}(\bar{y}) > 0, & (n \geq 2). \end{array} \tag{2.7}$$

On the other hand, if x_0 satisfies (2.7), then for $u = \phi(x_0)$,

$$t = -\frac{1}{A'_\phi(x_0)}, \quad x = x_0 - \frac{A_\phi(x_0)}{A'_\phi(x_0)}.$$

Consequently, (2.6) is satisfied.

2.1. *Some useful lemmas*

Lemma 2.1 [9]. *Let U and Γ_1 be defined in* (1.8a) *and* (1.8b)*. If $u(x,t)$ is the minimizing function of $F(x,t,\bullet)$, then we have*

- *U is an open subset of H, and $u(x,t)$ is smooth on U.*
- *Any point $(x_0,t_0) \in \Gamma_1$ has a neighborhood Θ such that $\Gamma_1 \cap \Theta$ is a smooth curve $x = \gamma(t)$ passing through (x_0,t_0). The minimizing function $u(x,t)$ is smooth on both components of $\Theta\backslash\Gamma_1$.*

The above lemma is from [9, Lemmas 1.1 and 1.2].

Suppose $(x_0,t_0) \in \Gamma_1$. Then according to Lemma 2.1, there exists some neighborhood Θ of (x_0,t_0) such that for $(x,t) \in \Theta$, the minimum of $F(x,t,\bullet)$ is assumed at either $u_1(x,t)$ or $u_2(x,t)$, or both. Hence every point of Θ belongs to U or Γ_1. Every point in the two components Θ_1 and Θ_2 of $\Theta\backslash\Gamma_1$ belongs to U. Let $u_i(x,t)$ be the unique minimizing value for $F(x,t,\bullet)$ for $(x,t) \in \Theta_i$ $(i = 1,2)$. Any curve in Γ_1 which separates two components of U is defined by an equation in the following form

$$F(x,t,u_2(x,t)) - F(x,t,u_1(x,t)) = 0 \tag{2.8}$$

and it follows from (1.5)–(1.6) that the jump relation

$$\dot{\gamma}(t) = [f(u)]/[u] \tag{2.9}$$

is satisfied along the curve, where $[u] = u_1 - u_2, [f(u)] = f(u_1) - f(u_2)$. Thus γ is C^{k+1}.

Now we turn to discuss the connection between the critical point and the characteristic. Suppose u_0 is a critical point of $F(x_0,t_0,\bullet)$, i.e. $F_u(x_0,t_0,u_0) = 0$. Then $u_0 = \phi(x_0 - t_0 a(u_0)) = \phi(y(x_0,t_0,u_0))$ due to (2.1). Consequently, there exists a characteristic

$$C_0 : x = y(x_0,t_0,u_0) + tA_\phi(y(x_0,t_0,u_0)) = x_0 + (t - t_0)a(u_0) \ (t > 0), \tag{2.10}$$

passing through (x_0,t_0) and $F_u(x,t,u_0) = 0$ for each $(x,t) \in C_0$. On the other hand, consider a characteristic $C_1 : x = \overline{x} + tA_\phi(\overline{x})$, $t > 0$. Then $F_u(x,t,\phi(\overline{x})) = 0$, for $(x,t) \in C_1$ due to $\phi(\overline{x}) = \phi(x - tA_\phi(\overline{x}))$ and (2.1). This implies that $\phi(\overline{x})$ is a critical point of $F(x,t,\bullet)$.

Naturally, it is asked if $\phi(\overline{x})$ is a minimizing value of $F(x,t,\bullet)$ for $(x,t) \in C_1$. The following lemma gives an answer.

Lemma 2.2 [6]. *Assume $\phi(x)$ is bounded and C^k smooth and let*

$$C = \{(x,t) \mid x = x_0 + tA_\phi(x_0),\ t > 0\}.$$

Then precisely one of the following statements must hold:

- *$C \subset U$ and $\phi(x_0)$ is the unique minimizing value for $F(x,t,\bullet)$; or*

- *there exists a point $(x_1, t_1) \in C$ such that $\phi(x_0)$ is either the unique minimizing value for $F(x_1, t_1, \bullet)$ which is degenerate or one of at least two minimizing values for $F(x_1, t_1, \bullet)$. Thus, $\phi(x_0)$ is the unique non-degenerate minimizing value for $F(x, t, \bullet)$ for $(x, t) \in C^- := C \cap \{(x, t) \mid t_1 > t > 0\} \subset U$ while $\phi(x_0)$ is no longer the minimizing value for $F(x, t, \bullet)$ for $(x, t) \in C^+ := C \cap \{(x, t) \mid t > t_1\}$.*

Definition 2.3. A characteristic segment C_{t_1}: $x = x_0 + A_\phi(x_0)t$ emanating from point $(x_0, 0)$, $0 < t < t_1 (\leq \infty)$ is called regular characteristic if $\phi(x_0)$ is a unique non-degenerate minimizing value for $F(x, t, \bullet)$ for $(x, t) \in C_{t_1}$ and $\phi(x_0)$ is no longer a minimizing value of $F(x, t, \bullet)$ for $(x, t) \in C^{t_1} = \{(x, t) \mid x = x_0 + A_\phi(x_0)t,\ t > t_1\}$.

Suppose u_0 is a minimizing value for $F(x_0, t_0, \bullet)$ (not necessarily unique). Consider a characteristic $C : x = y(x_0, t_0, u_0) + tA_\phi(y(x_0, t_0, u_0)) = x_0 + (t - t_0)a(u_0)(t > 0)$, which passes through (x_0, t_0) and $(y(x_0, t_0, u_0), 0)$. Consider the segment $C_{t_0} : x = x_0 + (t - t_0)a(u_0)(0 < t < t_0)$. It is a part of regular characteristic according to Lemma 2.2. Therefore, a minimizing value for $F(x_0, t_0, \bullet)$ defines a part of regular characteristic, on the other hand a regular characteristic provides a unique non-degenerate minimizing value for $F(x, t, \bullet)$ for each given point (x, t) belonging to a part of the regular characteristic. For each point $(x, t) \in H$, there exists at least one characteristic passing through it. Lemma 2.2 will be used to judge if $F(x, t, \bullet)$ has a unique non-degenerate (or degenerate) minimizing value, or several minimizing values.

Lemma 2.4. *If $L_3 = \emptyset$, then for each point $(x, t) \in H$, there are finitely many connected components of $\mathscr{M}_{(x,t)}$, where $\mathscr{M}_{(x,t)}$ is defined by* (1.9a).

Proof. Each connected component of $\mathscr{M}$ is either an isolated point or a closed interval. Let $\mathscr{M}_1 = [u_i^-, u_i^+]$, $\mathscr{M}_2 = [u_{i+1}^-, u_{i+1}^+]$ $(u_i^+ < u_{i+1}^-)$ be two neighboring connected components of $\mathscr{M}$. Thus there exists a point $u_i \in (u_{i+1}^-, u_i^+)$ such that

$$F_u(x, t, u_i) = 0, \quad F_{uu}(x, t, u_i) \leq 0, \tag{2.11}$$

where u_i is a local maximizing value for $F(x, t, \bullet)$. Set $y_i^\pm = y(x, t, u_i^\pm)$, $y_{i+1}^\pm = y(x, t, u_{i+1}^\pm)$ and $y_i = y(x, t, u_i)$. Now we claim

$$\exists\ x_i' \in [y_{i+1}^-, y_i] \text{ such that } A_\phi''(x_i') < 0. \tag{2.12}$$

If (2.12) is not true, then

$$A_\phi''(y) \geq 0,\ \forall\ y \in [y_{i+1}^-, y_i]. \tag{2.13}$$

Two cases need to be considered.

Case 1 ($F_{uu}(x, t, u_{i+1}^-) \neq 0$). In this case, we have $A_\phi'(y_{i+1}^-) > -1/t \geq A_\phi'(y_i)$ in view of (2.11) and $F_{uu}(x, t, u_{i+1}^-) \neq 0$. On the other hand, $A_\phi'(y_{i+1}^-) \leq A_\phi'(y_i)$ due to (2.13). This gives a contradiction, hence (2.12) holds for Case 1.

Case 2 ($F_{uu}(x,t,u_{i+1}^-) = 0$). In this case, we have

$$\exists\ \delta_0 > 0 \text{ such that } A_\phi''(y) \geq 0 \text{ for each } y \in [y_{i+1}^-, y_{i+1}^- + \delta_0] \tag{2.14}$$

according to $L_3 = \emptyset$ or (2.5). Next we claim that

$$\exists\ \xi \in (y_{i+1}^-, y_{i+1}^- + \delta) \text{ such that } A_\phi''(\xi) > 0 \text{ for each } \delta > 0. \tag{2.15}$$

If (2.15) is not true, thus according to (2.14), there exists a constant $\delta_1 > 0$ such that $A_\phi''(\xi) \equiv 0$ for each $\xi \in [y_{i+1}^-, y_{i+1}^- + \delta_1]$. Consequently, $\forall \xi \in [y_{i+1}^-, y_{i+1}^- + \delta_0]$, we have ${A_\phi}'(y) = {A_\phi}'(y_{i+1}^-) = -1/t$, which implies that $A_\phi(\xi) = -\xi/t + c$, where the constant c can be determined by letting $\xi = y_{i+1}^-$:

$$c = \frac{1}{t}[y_{i+1}^- + tA_\phi(y_{i+1}^-)] = \frac{x}{t}.$$

Therefore, we have

$$A_\phi(\xi) = -\frac{1}{t}\xi + \frac{x}{t}, \quad \xi \in [y_{i+1}^-, y_{i+1}^- + \delta_1]. \tag{2.16}$$

Set $u = \phi(\xi)$, for $\xi \in [y_{i+1}^-, y_{i+1}^- + \delta_1]$. According to (2.16),

$$\begin{aligned} F_u(x,t,u)|_{u=\phi(\xi)} &= ta'(u)[u - \phi(x - ta(u))]|_{u=\phi(\xi)} \\ &= ta'(\phi(\xi))[\phi(\xi) - \phi(x - tA_\phi(\xi))] \\ &= ta'(u)(\phi(\xi) - \phi(\xi)) = 0. \end{aligned} \tag{2.17}$$

Thus $F(x,t,u)$ is a constant for $u \in [\phi(y_{i+1}^- + \delta_1), \phi(y_{i+1}^-)] = [\alpha', u_{i+1}^-]$, where $\alpha' = \phi(y_{i+1}^- + \delta_1)$. This leads to a contradiction since $[u_{i+1}^-, u_{i+1}^+]$ is a connected component, hence (2.15) holds. Similarly, it can be shown that

$$\exists\ \eta \in (y_{i+1}^+ - \delta, y_{i+1}^+) \text{ such that } A_\phi''(\eta) < 0 \text{ for each } \delta > 0. \tag{2.18}$$

We have $A_\phi'(y_i) > A_\phi'(y_{i+1}^-)$ according to (2.13) and (2.15). This is a contradiction since $A_\phi'(y_i) \leq A_\phi'(y_{i+1}^-) = -1/t$ according to $F_{uu}(x,t,u_{i+1}^-) = 0$ and $F_{uu}(x,t,u_i) \leq 0$. Hence (2.12) holds for Case 2. Thus (2.12) is true.

Similar to the proof of (2.12), we can prove that it can be shown that there exists $x_i'' \in [y_i, y_i^+]$ such that $A_\phi''(x_i'') > 0$. This result, together with (2.12), yield

$$\exists\ x_i' \in [y_{i+1}^-, y_i] \text{ and } x_i'' \in [y_i, y_i^+] \text{ such that } A_\phi''(x_i') < 0 \text{ and } A_\phi''(x_i'') > 0. \tag{2.19}$$

Next we claim there are finitely many connected components of $\mathscr{M}$. If not, then without loss of generality, there exists a monotone increasing sequence $\{\mathscr{M}_i\} = \{[u_i^-, u_i^+]\}$. The sequence $\{[y_i^+, y_i^-]\}$ is monotone decreasing and bounded, then it must be convergent to a point x_0. It is easy to know that

$$\lim_{i\to\infty} x_i' = \lim_{i\to\infty} x_i'' = \lim_{i\to\infty} y_i = x_0.$$

It follows from $A_\phi'(y_i) \leq -1/t$ that $A_\phi'(x_0) \leq -1/t < 0$. On the other hand, $A_\phi''(x_0) = \cdots = A_\phi^{(k)}(x_0) = 0$ in light of (2.19). For each $\delta > 0$, $\exists\ i_0 > 0$ such that $[y_i^+, y_i^-] \subset (x_0, x_0 + \delta)$ for each $i > i_0$ due to the fact $x_0 < x_i' < x_i''$ for each

$i > i_0$ since the sequence $\{[y_i^+, y_i^-]\}$ is monotone decreasing. Observing $A''_\phi(x'_i) < 0$, $x'_i \in (x_0, x_0 + \delta)$, which implies $x_0 \in L_3$. This contradicts $L_3 = \emptyset$. The proof is complete. □

The same result by Li and Wang [7] as to Lemma 2.4 was obtained under the hypothesis that ϕ is locally finite to f.

Definition 2.5. Suppose (x_0, t_0) lies on a characteristic $C : x = y_0 + tA_\phi(y_0)$, (x_0, t_0) is called a degenerate point of the characteristic C, if $F_{uu}(x_0, t_0, \phi(y_0)) = 0$.

Let $x = \xi + tA_\phi(\xi),\ t > 0$, where $A_\phi(\xi) = a(\phi(\xi))$. Set $F_u(x, t, \phi(\xi)) = 0$ and $F_{uu}(x, t, \phi(\xi)) = 0$. By a direct computation, we have

$$x(\xi) = \xi - \frac{A_\phi(\xi)}{A'_\phi(\xi)}, \quad t(\xi) = -\frac{1}{A'_\phi(\xi)},$$

which is just the degenerate point on the characteristic emanating from ξ according to Definition 2.5. This motivation leads to the following result.

Lemma 2.6. *Suppose that there exists a unique connected component $[\alpha, \beta]$ of $\mathscr{M}_{(x_0,t_0)}$, and $F_{uu}(x_0, t_0, \alpha) = 0$. If $L_3 = \emptyset$, then the locus of the degenerate points on all the characteristics emanating from some neighborhood of $[y(x_0, t_0, \beta), y(x_0, t_0, \alpha)]$ form two half curves in the neighborhood Θ of (x_0, t_0) for $t \geq t_0$ with a unique common point (x_0, t_0). Each of them is continuously differentiable.*

Proof. By (2.1), the critical set of $F(x, t, \bullet)$ is contained in the compact interval $J = \{u : |u| \leq M\}$, where $M = \sup_y |\phi(y)|$. First we claim that

$$\begin{aligned}&\text{for each } (\alpha - \varepsilon, \beta + \varepsilon),\ \exists \text{ an open neighborhood } \Theta \text{ of } (x_0, t_0) \text{ such that}\\ &\text{all the minimizing values of } F(x, t, \bullet) \text{ belong to } (\alpha - \varepsilon, \beta + \varepsilon) \text{ for } (x, t) \in \Theta.\end{aligned} \tag{2.20}$$

If the above claim is not true, then there exists a sequence (x_n, t_n) converging to (x_0, t_0) and a sequence $u_n \in J \backslash (\alpha - \varepsilon, \beta + \varepsilon)$ such that

$$F(x_n, t_n, u_n) = \min_{u \in \mathbb{R}} F(x_n, t_n, u), \quad (n = 1, 2, \ldots).$$

Since the set $J \backslash (\alpha - \varepsilon, \beta + \varepsilon)$ is compact, we can choose a subsequence of u_n, written again as u_n for convenience, convergent to $u_1 \in J \backslash (\alpha - \varepsilon, \beta + \varepsilon)$. Then

$$F(x_0, t_0, \alpha) = \lim_{n \to \infty} F(x_n, t_n, u_n) = F(x_0, t_0, u_1).$$

Since $F(x, t, u)$ is a continuous function of (x, t) for fixed u, the function $m(x, t) = \min_{u \in \mathbb{R}} F(x, t, u)$ is continuous. This implies that there are at least two connected components of $\mathscr{M}_{(x_0,t_0)}$, which contradicts to the assumption that there is a unique connected component of $\mathscr{M}_{(x_0,t_0)}$. Thus the assertion (2.20) is true. According to (2.20), there exists a sufficiently small constant $\varepsilon_0 > 0$ and a neighborhood Θ of

(x_0, t_0) such that for each $(x,t) \in \Theta$ and $u \in (\alpha - \varepsilon_0, \beta + \varepsilon_0)$, we have $x - ta(u) \in (y(x_0, t_0, \beta) - \delta, y(x_0, t_0, \alpha) + \delta)$, for each given $\delta > 0$.

Let $I(\alpha, \delta_0) := [y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0]$ and $J(\beta, \delta_0) := [y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta)]$. We can choose δ_0 sufficiently small such that $A''_\phi(\xi) \geq 0$ for each $\xi \in I(\alpha, \delta_0)$ and $A''_\phi(\xi) \leq 0$ for each $\xi \in J(\beta, \delta_0)$ according to $L_3 = \emptyset$ or (2.5).

The locus of the degenerate points on all the characteristics that emanate from the interval $I(\alpha, \delta_0)$ can be written in the form

$$\left.\begin{aligned} x(\xi) &= \xi - \frac{A_\phi(\xi)}{A'_\phi(\xi)}, \\ t(\xi) &= -\frac{1}{A'_\phi(\xi)}, \end{aligned}\right. \quad \xi \in I(\alpha, \delta_0). \tag{2.21}$$

Similarly, the locus of the degenerate points on all the characteristics that emanate from the interval $J(\beta, \delta_0)$ can be written in the form

$$\left.\begin{aligned} x(\xi) &= \xi - \frac{A_\phi(\xi)}{A'_\phi(\xi)}, \\ t(\xi) &= -\frac{1}{A'_\phi(\xi)}, \end{aligned}\right. \quad \xi \in J(\beta, \delta_0). \tag{2.22}$$

They have a common point (x_0, t_0). Moreover, the tangents to the curves (2.21) and (2.22) at (x_0, t_0) are $a(\alpha)$ and $a(\beta)$, respectively.

Next we claim that

$$\text{(2.21) and (2.22) are one to one continuous mappings.} \tag{2.23}$$

In fact, if there exist ξ_1, ξ_2 $(\xi_1 < \xi_2)$ such that $t(\xi_1) = t(\xi_2)$ and $[\xi_1, \xi_2] \subset I(\alpha, \delta_0)$, then there exists $\xi_0 \in I(\alpha, \delta_0)$, $\xi_0 < \xi_1$ such that $A''_\phi(\xi_0) < 0$ according to $L_3 = \emptyset$, which is contradictory to the fact that $A''_\phi(\xi) \geq 0$ $\forall \xi \in I(\alpha, \delta_0)$. A similar argument can be applied to the case (2.22). Thus (2.23) is true.

Therefore, (2.21) and (2.22) define two continuous curves $x = x_l(t)$ and $x = x_r(t)$, respectively. By a direct computation, we have

$$x'_l(t) = A_\phi(\xi). \tag{2.24}$$

Since $A_\phi(\xi)$ is strictly decreasing, $x = x_l(t)$ is strictly concave. Similarly, $x = x_r(t)$ is strictly convex. Let $(x_l(t_1), t_1), (x_r(t_1), t_1) \in \Theta$ $(t_1 > t_0)$. Then $x_r(t_1) - x_l(t_1) > 0$. Thus these two curves have a unique common point (x_0, t_0). □

Remark 2.7. In fact $x_l(t)$ is the envelope of all the characteristics emanating from $I(\alpha, \delta_0)$; $x_r(t)$ is the envelope of all the characteristics emanating from $J(\beta, \delta_0)$.

Lemma 2.8. *Assume $[u_1^-, u_1^+]$ and $[u_2^-, u_2^+]$ are two neighboring connected components of $\mathscr{M}_{(x_1,t_1)}$, where $u_1^- \leq u_1^+ < u_2^- \leq u_2^+$. If $L_3 = \emptyset$, then there exists an open neighborhood $\widetilde{\Theta}$ of (x_1, t_1) such that $\widetilde{\Theta} \cap G \cap \Gamma_1$ contains a half curve $x = \gamma^-(t)$ terminating at (x_1, t_1), where G is the triangle domain formed by $t = 0$, the characteristics $x = y(x_1, t_1, u_2^-) + ta(u_2^-)$ and $x = y(x_1, t_1, u_1^+) + ta(u_1^+)$.*

Proof. By an argument similar to the proof of assertion (2.20), there exists a neighborhood $\widetilde{\Theta}$ of (x_1, t_1) and two intervals

$$J_1 := (y(x_1, t_1, u_1^+) - \delta, y(x_1, t_1, u_1^+))$$

and

$$J_2 := (y(x_1, t_1, u_2^-), y(x_1, t_1, u_2^-) + \delta)$$

such that for each given $(x, t) \in \widetilde{\Theta}' \cap G$ and any minimizing value, say u, for $F(x, t, \bullet)$, $y(x, t, u)$ belongs to J_1 and/or J_2. Now we claim that:

$$\begin{gathered}\exists \ \text{ at most one minimizing value } u(x,t) \text{ for } F(x,t,\bullet) \\ \text{such that } y(x,t,u(x,t)) \in J_1, \forall (x,t) \in \widetilde{\Theta}' \cap G, \\ \text{and } u(x,t) \text{ is non-degenerate if it exists.}\end{gathered} \tag{2.25}$$

In fact, there are only two cases to be considered

Case 1 ($F_{uu}(x_1, t_1, u_1^+) = 0$). In this case, assume there has a point $(\tilde{x}, \tilde{t}) \in \widetilde{\Theta}' \cap G$ such that there exist at least two minimizing values u^* and u^{**} for $F(\tilde{x}, \tilde{t}, \bullet)$ and $y(\tilde{x}, \tilde{t}, u^*), y(\tilde{x}, \tilde{t}, u^{**}) \in J_1$. Thus there exist at least two characteristics passing through $(\tilde{x}, \tilde{t})$:

$$x = y(\tilde{x}, \tilde{t}, u^*) + ta(u^*); \quad x = y(\tilde{x}, \tilde{t}, u^{**}) + ta(u^{**}). \tag{2.26}$$

On the other hand, using the fact that $A'_\phi(x)$ is strictly monotone decreasing on the interval J_1 (by the assumption $L_3 = \emptyset$ and (2.18), or (2.5)), we have

$$\tilde{t} = -\frac{y(\tilde{x}, \tilde{t}, u^*) - y(\tilde{x}, \tilde{t}, u^{**})}{a(u^*) - a(u^{**})} = -\frac{1}{A'_\phi(\tilde{\eta})} > t_1,$$

where $\tilde{\eta}$ is between $y(\tilde{x}, \tilde{t}, u^*)$ and $y(\tilde{x}, \tilde{t}, u^{**})$. This is contradictory to the fact that $\tilde{t} < t_1$. Furthermore, suppose $\tilde{u}$ is the unique minimizing value for $F(x, t, \bullet)$ such that $y(x, t, \tilde{u}) \in J_1$ and $F_{uu}(x, t, \tilde{u}) = 0$. Thus $t = -1/A'_\phi(y(x, t, \tilde{u})) > t_1$. Since $A'_\phi(x)$ is monotone, $\tilde{u}$ must be non-degenerate. Hence the assertion (2.1) is true for Case 1.

Case 2 ($F_{uu}(x_1, t_1, u_1^+) \neq 0$). In this case, since $F_{uu}(x_1, t_1, u_1^+) \neq 0$, it follows from the implicit function theorem that there is an open neighborhood $U(u_1^+)$ of u_1 such that for (x, t) sufficiently close to (x_1, t_1) the equation $(\partial F/\partial u)(x, t, u) = 0$ has a unique solution $u(x, t) \in U(u_1^+)$. Consequently, there is a unique minimizing value $u(x, t)$ such that $y(x, t, u(x, t)) \in J_1$ and $u(x, t)$ must be non-degenerate. Thus the assertion (2.25) is true for Case 2. Consequently, (2.25) is true.

Similarly, we can deduce that

$$\begin{gathered}\exists \ \text{ at most one minimizing value } u(x,t) \text{ for } F(x,t,\bullet) \\ \text{such that } y(x,t,u(x,t)) \in J_2, \forall (x,t) \in \widetilde{\Theta}' \cap G, \\ \text{and } u(x,t) \text{ is non-degenerate if it exists.}\end{gathered} \tag{2.27}$$

Let

$$\begin{aligned}A_{t_2} = \{x \mid\ & (x,t_2) \in \widetilde{\Theta}' \cap G, \\ & \exists\, \text{a regular characteristic emanating from } (y(x_1,t_1,u_1^+),0) \\ & \text{or a point on the left of } (y(x_1,t_1,u_1^+),0) \text{ and passes through } (x,t_2)\},\end{aligned} \tag{2.28}$$

$$\begin{aligned}B_{t_2} = \{x \mid\ & (x,t_2) \in \widetilde{\Theta}' \cap G, \\ & \exists\, \text{a regular characteristic emanating from } (y(x_1,t_1,u_2^-),0) \\ & \text{or a point on the right of } (y(x_1,t_1,u_2^-),0) \text{ and passes through } (x,t_2)\}.\end{aligned} \tag{2.29}$$

Let (x_{l2},t_2) be the point on the characteristic $x = y(x_1,t_1,u_2^-) + ta(u_2^-)$, (x_{r2},t_2) be the point on the characteristic $x = y(x_1,t_1,u_1^+) + ta(u_1^+)$. Then $(x_{l2},t_2) \in B_{t_2}$, $(x_{r2},t_2) \in A_{t_2}$. Thus $A_{t_2} \neq \emptyset$, $B_{t_2} \neq \emptyset$. Let $\overline{x}_1$ be the infimum of the set A_{t_2} and $\overline{x}_2$ be the supremum of the set B_{t_2}. Next we will show

$$\exists \ \text{ two minimizing values for } F(\overline{x}_1,t_2,\bullet). \tag{2.30}$$

First we claim that

$$\exists \ \text{ at least two minimizing values for } F(\overline{x}_1,t_2,\bullet). \tag{2.31}$$

If this is not true, then there is a unique minimizing value u_1 for $F(\overline{x}_1,t_2,\bullet)$. In this case, there exists a unique characteristic passing through $(\overline{x}_1,t_2)$, which satisfies one of the following possibilities:

Case (i): It emanates from a point on the left of $(y(x_1,t_1,u_1^+),0)$ and u_1 is non-degenerate. In this case, according to Lemma 2.1, we can find a neighborhood $U(\overline{x}_1)$ of $\overline{x}_1$ such that for each $x \in U(\overline{x}_1)$, there exists only one characteristic that emanates from a point on the left of $(y(x_1,t_1,u_1^+),0)$ and passes through (x,t_2). Thus there exists a point $\tilde{x} \in U(\overline{x}_1)$, $\tilde{x} < \overline{x}_1$ such that there exists a regular characteristic passing through $(\tilde{x},t_2)$ that emanates from a point on the left of $(y(x_1,t_1,u_1^+),0)$, which implies that $\tilde{x} \in A_{t_2}$. This is contradictory to the fact that $\overline{x}_1$ is the infimum of the set A_{t_2}.

Case (ii): It emanates from a point on the right of $(y(x_1,t_1,u_2^-),0)$ and u_2 is non-degenerate. Similar to Case (i) above, a contradiction can be also obtained.

Next, we claim that

$$\exists \text{ at most two minimizing values for } F(\overline{x}_1, t_2, \bullet). \tag{2.32}$$

If this is not true, then there exist two minimizing values u_1 and u_2 for $F(\overline{x}_1, t_2, \bullet)$ such that both $y(\overline{x}_1, t_2, u_1)$ and $y(\overline{x}_1, t_2, u_2)$ belong to J_1 or J_2. Without loss of generality, suppose both $y(\overline{x}_1, t_2, u_1)$, $y(\overline{x}_1, t_2, u_2) \in J_1$. However, this is impossible according to (2.25). Therefore, (2.32) is true. Combining (2.31) and (2.32), we obtain (2.30).

Similarly, we can prove that

$$\exists \text{ only two minimizing values for } F(\overline{x}_2, t_2, \bullet) \text{ and they are non-degenerate.} \tag{2.33}$$

Now we claim

$$\overline{x}_1 = \overline{x}_2. \tag{2.34}$$

If not, i.e. $\overline{x}_1 \neq \overline{x}_2$, then it follows from (2.30) and (2.33) that there is a characteristic passing through $(\overline{x}_1, t_2)$ and a characteristic passing through $(\overline{x}_2, t_2)$, both are regular before $t = t_2$, such that they intersect with each other at a time earlier than t_2. This is a contradiction since they will not be regular after they intersect according to Lemma 2.2. Therefore $\overline{x}_1 = \overline{x}_2$.

In summary, a unique curve $x = \gamma^-(t)$ terminating at (x_1, t_1) is defined in $\widetilde{\Theta}' \cap G$ such that there are only two minimizing values for $F(\gamma^-(t), t, \bullet)$. This completes the proof of Lemma 2.8. $\square$

We point out that (x_1, t_1) is the center of the centered compression wave when $u_1^- < u_1^+$ or/and $u_2^- < u_2^+$. Therefore, there does not exist one to one correspondence between the shock generation point and the center of a centered compression waves in general.

Lemma 2.9. *Assume the assumptions in Lemma* 2.6 *hold. If* $L_3 = \emptyset$, *then all the characteristics that emanate from the interval* $I(\alpha, \delta_0)$ (*respectively*, $J(\beta, \delta_0)$) *can only intersect with each other after* $t = t_0$. *Moreover, for any two characteristics emanating from the points on the right* (*left*) *of* $(y(x_0, t_0, \alpha), 0)$ (*respectively*, $(y(x_0, t_0, \beta), 0)$), *the time when the one closer to* $(y(x_0, t_0, \alpha), 0)$ (*respectively*, $(y(x_0, t_0, \beta), 0)$) *touches the curve* (2.21) (*respectively*, (2.22)) *is earlier than the time when they intersect with each other, where* $\delta_0 > 0$ *is the same constant given in Lemma* 2.6.

Proof. Consider the following two characteristics:

$$x = \overline{x}_1 + tA_\phi(\overline{x}_1), \tag{2.35}$$

$$x = \overline{x}_2 + tA_\phi(\overline{x}_2), \tag{2.36}$$

where $\overline{x}_1, \overline{x}_2 \in (y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0)$. Suppose (x_{12}, t_{12}) is their intersection point; i.e.

$$t_{12} = -\frac{\overline{x}_1 - \overline{x}_2}{A_\phi(\overline{x}_1) - A_\phi(\overline{x}_2)} = -\frac{1}{A'_\phi(\overline{x}_{12})},$$

where $\overline{x}_1 < \overline{x}_{12} < \overline{x}_2$. Let $t_1 = -1/A'_\phi(\overline{x}_1)$ and $t_2 = -1/A'_\phi(\overline{x}_2)$. Thus $F_u(x_i, t_i, \phi(\overline{x}_i)) = 0$ and $F_{uu}(x_i, t_i, \phi(\overline{x}_i)) = 0$, where $x_i = \overline{x}_i + t_i A_\phi(\overline{x}_i)$ $(i = 1, 2)$. It can be verified that

$$t_0 < t_1 < t_{12} < t_2 \tag{2.37}$$

according to (2.14) and (2.23). A similar conclusion can be also obtained if $\overline{x}_1, \overline{x}_2 \in (y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta))$. Thus the proof of this lemma is complete. □

Lemma 2.10. *Assume the assumptions in Lemma* 2.6 *hold. If $L_3 = \emptyset$, then all the minimizing values for $F(x, t, \bullet)$ are non-degenerate for $(x, t) \in \Theta' = \Theta \backslash \{(x_0, t_0)\}$. Moreover, there exists a unique minimizing value for $F(x, t, \bullet)$ for $(x, t) \in \Theta'$ lying under and on the curves* (2.21) *and* (2.22), *i.e.* $(x, t) \in \Theta' \cap \{x \leq x_l(t)$ *or* $x \geq x_r(t)\}$.

Proof. Recall $I(\alpha, \delta_0) = [y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0]$ and $J(\beta, \delta_0) = [y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta)]$. We consider two possible cases.

Case 1 (under the curves). The points of characteristics emanating from the interval $I(\alpha, \delta_0)$ and $J(\beta, \delta_0)$ are non-degenerate in $\theta_{l,r}$, where $\theta_{l,r} = \Theta' \cap \{(x, t) | x < x_l(t)$ or $x > x_r(t)\}$. Furthermore, they will not intersect with each other in $\theta_{l,r}$ by Lemma 2.9. Thus there is a unique non-degenerate minimizing value for $F(x, t, \bullet)$ for $(x_1, t_1) \in \theta_{l,r}$.

Case 2 (on the curves). If $x_1 = x_l(t_1)$, we claim that

$$\exists \text{ a unique minimizing value for } F(x_1, t_1, \bullet) \text{ and it is non-degenerate.} \tag{2.38}$$

In fact, there exists a unique minimizing value u_1 for $F(x_1, t_1, \bullet)$ such that $y(x_1, t_1, u_1) \in J(\beta, \delta_0)$ according to Lemma 2.9. Assume there exists another minimizing value u_2 for $F(x_1, t_1, \bullet)$ such that $y(x_1, t_1, u_2) \in I(\alpha, \delta_0)$. Let $\xi_2 = y(x_1, t_1, u_2)$ and $\xi_1 = y(x_1, t_1, u_1)$. Then there exists a characteristic passing through (x_1, t_1):

$$x = \xi_2 + t A_\phi(\xi_2), \quad 0 < t \leq t_1, \tag{2.39}$$

which is the characteristic tangent to the curve (2.21) at the point $(x_1, t_1) = (x_l(t_1), t_1)$, i.e. $x'_l(t_1) = A_\phi(\xi_2)$. Let

$$x = \xi_1 + t A_\phi(\xi_1), \quad 0 < t \leq t_1. \tag{2.40}$$

Now u_1 and u_2 are the only two minimizing values for $F(x_1, t_1, \bullet)$. In light of Lemma 2.8, there exists an open neighborhood $\widetilde{\Theta}$ of (x_1, t_1) such that $\widetilde{\Theta} \cap \Gamma_1$ contains a half curve $x = \gamma^-(t)$ terminating at (x_1, t_1). Furthermore, we have

$$\frac{d\gamma^-(t)}{dt} = \frac{f(u_1(x, t)) - f(u_2(x, t))}{u_1(x, t) - u_2(x, t)}, \tag{2.41}$$

$y(x,t,u_1(x,t)) \in (\xi_1,\xi_1+\delta)\subset J(\beta,\delta_0)$, and $y(x,t,u_2(x,t)) \in (\xi_2-\delta,\xi_2)\subset I(\alpha,\delta_0)$, $(t<t_1)$. Letting $t\to t_1-$, we have

$$\lim_{t\to t_1-0}\frac{d\gamma^-(t)}{dt}=\frac{f(u_1)-f(u_2)}{u_1-u_2}\in (a(u_2),a(u_1)).$$

On the other hand, the slope of the line tangent to the curve (2.21) at (x,t) tending to (x_1,t_1) converges to $a(u_2)$. Consequently, the curve $x=\gamma^-(t)$ lies between the characteristic (2.39) and the characteristic (2.40) for t close to t_1. Thus the curve $x=\gamma^-(t)$ must lie on the left of the curves (2.21), which is impossible according to Case 1. Hence (2.38) is proved for $x_1 = x_l(t_1)$. Similarly, (2.38) is true for $x_1=x_r(t_1)$. □

2.2. *Local piecewise smoothness*

Theorem 2.11. *Assume that there is a unique connected component* $[\alpha,\beta]$ *of* $\mathscr{M}_{(x_0,t_0)}$. *If* $L_3=\emptyset$ *and* $F_{uu}(x_0,t_0,\alpha)=0$, *then* (x_0,t_0) *has a neighborhood* Θ *such that* $\Gamma_1\cap\Theta$ *consists of a* C^{k+1} *half-curve emanating at* (x_0,t_0). *The minimizing function is smooth on* $\Theta'\backslash\Gamma_1$.

Proof. We will prove that there exists a C^{k+1} smooth shock emanating at (x_0,t_0). For each given $t_1>t_0$, let

$$\begin{aligned} A_{t_1} = \{x \mid (x,t_1)\in\Theta,\ \exists\ &\text{a regular characteristic emanating from a point}\\ &\text{on the left of } (y(x_0,t_0,\beta),0) \text{ and passes through } (x,t_1)\}, \end{aligned}\tag{2.42}$$

$$\begin{aligned} B_{t_1} = \{x \mid (x,t_1)\in\Theta,\ \exists &\text{a regular characteristic emanating from a point}\\ &\text{on the right of } (y(x_0,t_0,\alpha),0) \text{ and passes through } (x,t_1)\}. \end{aligned}\tag{2.43}$$

According to (2.38), there exists a unique regular characteristic passing through $(x_l(t_1),t_1)$, then $(x_l(t_1),t_1)\in A_{t_1}$, which implies that $A_{t_1}\neq\emptyset$. Similarly, $B_{t_1}\neq\emptyset$. Let x_1 be the supremum of the set A_{t_1} and x_2 be the infimum of the set B_{t_1}.

Next we will show that

$$\exists \text{ two minimizing values for } F(x_1,t_1,\bullet).\tag{2.44}$$

First we claim that

$$\exists \text{ at least two minimizing values for } F(x_1,t_1,\bullet).\tag{2.45}$$

The proof is the same as that for (2.31). Next, we claim that

$$\exists \text{ at most two minimizing values for } F(x_1,t_1,\bullet).\tag{2.46}$$

If (2.46) is not true, thus there exist two minimizing values u_1 and u_2 for $F(x_1,t_1,\bullet)$ such that $y(x_1,t_1,u_1)=x_1-t_1a(u_1)$ and $y(x_1,t_1,u_2)=x_1-t_1a(u_2)\in J(\beta,\delta_0)$

or $I(\alpha, \delta_0)$. Without loss of generality, suppose $y(x_1, t_1, u_1), y(x_1, t_1, u_2) \in I(\alpha, \delta_0)$. Then there exist two characteristics passing through (x_1, t_1)

$$x = \overline{x}_1 + tA_\phi(\overline{x}_1), \tag{2.47}$$

$$x = \overline{x}_2 + tA_\phi(\overline{x}_2), \tag{2.48}$$

(x_1, t_1) is the intersection point of them, where $\bar{x}_1 < \bar{x}_2$. On the other hand, the points of the two characteristics are non-degenerate before and at the time t_1, thus characteristics (2.47) and (2.48) cannot intersect with each other at (x_1, t_1) according to Lemma 2.9. It is a contradiction. Consequently, the assertion (2.46) is true.

Combining (2.45) and (2.46), we conclude that *there are only two minimizing values for $F(x_1, t_1, \bullet)$ and they are non-degenerate*. By a similar argument to the assertion (2.44), it can be demonstrated that *there are two minimizing values for $F(x_2, t_1, \bullet)$*. Now we claim that $x_1 = x_2$. The proof is the same as the proof of assertion (2.34). In this way a unique curve $x = \gamma(t)$ is defined for $t > t_0$. Therefor there are only two minimizing values for $F(\gamma(t), t, \bullet)$.

Consider any point $(x_1, t_1) \in \Theta \cap \{t > t_0\}$. If $x_1 < \gamma(t_1)$, then there is a unique u which minimizes $F(x_1, t_1, \bullet)$ since (x_1, t_1) lies on the left of $(\gamma(t_1), t_1)$. Thus the minimizing function is smooth at (x, t) in view of Lemma 2.1. The same result holds for the case when $x_1 > \gamma(t_1)$. If $x_1 = \gamma(t_1)$, then there are only two values of u which minimizes $F(x_1, t_1, \bullet)$ and they are non-degenerate according to the above arguments. These results, together with Lemma 2.2, complete the proof of this theorem. □

Remark 2.12. [9, Lemma 1.4] is a special case of Theorem 2.11, even if we set $\alpha = \beta$ in Theorem 2.11. In particular, this theorem allows our solutions to contain centered compression waves, and if this is the case then the generation point of the shock is the center of centered compression wave if $\alpha \neq \beta$.

Theorem 2.13. *Assume that $[u_1^-, u_1^+], [u_2^-, u_2^+], \ldots, [u_n^-, u_n^+]$ are the n connected components of $\mathscr{M}_{(x_0,t_0)}$ (suppose $u_1^- \leq u_1^+ < u_2^- \leq u_2^+ \cdots < u_n^- \leq u_n^+$), where $n \geq 2$. If $L_3 = \emptyset$, then (x_0, t_0) has a neighborhood Θ such that $\Gamma_1 \cap \Theta$ consists of n half-curves, one emanating from (x_0, t_0) and the other $(n-1)$'s terminating at (x_0, t_0). Moreover, the minimizing function is smooth on $\Theta' \backslash \Gamma_1$, where $\Theta' = \Theta \backslash \{(x_0, t_0)\}$.*

Proof. Set

$$l_i^+ : x = y(x_0, t_0, u_i^+) + tA_\phi(y(x_0, t_0, u_i^+)), \tag{2.49}$$

$$l_i^- : x = y(x_0, t_0, u_i^-) + tA_\phi(y(x_0, t_0, u_i^-)), \tag{2.50}$$

where $i = 1, \ldots, n$, $0 < t < t_0$. In light of Lemma 2.8, there exists a unique half-curve terminating at (x_0, t_0), denoted by $x = \gamma_i^-(t)$, in a neighborhood Θ of (x_0, t_0). More precisely, $x = \gamma_i^-(t)$ is defined in the triangle domain G_{i+1}^i formed by the line

$t = 0$, the characteristic l_i^+ and l_{i+1}^-. Moreover, for each $x = \gamma_i^-(t)$, $1 \le i \le n-1$, there exist two minimizing values for $F(x, t, \bullet)$, which are non-degenerate.

Similar to the proof of Theorem 2.11, there exists a half-curve emanating from (x_0, t_0) denoted by $x = \gamma^+(t)$, $(t > t_0)$. Thus, the proof of this theorem is complete. □

Corollary 2.14. *Any point* $(x_0, t_0) \in \overline{\Gamma}_1$ *has a neighborhood* Θ *such that* $\overline{\Gamma}_1 \cap \Theta$ *is a curve* $x = \gamma(t)$ *passes through* (x_0, t_0) *is* C^{k+1} *smooth at each point except at* $t = t_0$. *The minimizing function* $u(x, t)$ *is smooth on both components of* $\Theta \backslash \overline{\Gamma}_1$.

The smoothness of the curve $x = \gamma(t)$ can be decided by the following cases:

Case 1 ($\mathscr{M}_{(x_0,t_0)} = \{u_1, u_2\}$ for $F(x_0, t_0, \bullet)$). In this case,

- If $F_{uu}(x_0, t_0, u_1) = 0$ or $F_{uu}(x_0, t_0, u_2) = 0$, then the curve $x = \gamma(t)$ is C^{k+1} at each point except $t = t_0$. In fact, $x = \gamma(t)$ is only C^1 on the line $t = t_0$ since $u_1(x, t)$ and $u_2(x, t)$ are continuous, but $u_{1x}(x, t_0) \to \infty$ or $u_{2x}(x, t_0) \to \infty$ as $x \to x_0 - 0$ or $x \to x_0 + 0$.
- If $F_{uu}(x_0, t_0, u_1) \neq 0$ and $F_{uu}(x_0, t_0, u_2) \neq 0$, then the curve $x = \gamma(t)$ is C^{k+1} smooth.

Case 2 ($\mathscr{M}_{(x_0,t_0)} = [u_1^-, u_1^+] \cup [u_2^-, u_2^+]$, where $u_1^- < u_1^+$ or $u_2^- < u_2^+$). In this case, the curve $x = \gamma(t)$ is continuous at the point $t = t_0$.

3. Piecewise Smoothness

In the last section, we demonstrated that the shocks are C^{k+1} smooth except at the shock interaction points and points belonging to $\overline{\Gamma}_1 \backslash \Gamma_1$; the shocks are not differentiable at the shock interaction points and at the center points of the centered compression waves. In other words, all the shocks are piecewise C^{k+1} smooth. In this section, more precise statements on the piecewise smoothness will be given based on the local structure analysis given in Sec. 2. In particular, we show that the total number of shocks is finite when the initial data belongs to $\mathscr{S}(\mathbb{R}) \backslash \Omega_4$, where $\mathscr{S}(\mathbb{R})$ is the Schwartz space and Ω_4 is a set of first category. To our knowledge, Ω_4 is the smallest set to be excluded in obtaining the piecewise smoothness and finiteness of shock numbers. In particular, the set $\mathscr{S}(\mathbb{R}) \backslash \Omega_4$ allows our solutions to contain centered compression waves, which is not possible in the previous works.

Let Ω_2 and Ω_3 be defined in (1.15) and (1.16), respectively. Dafermos in [2] has proved that the set Ω_2 is of first category in C^k. Thus Ω_3 as a proper subset of Ω_2 is also of first category in C^k. Now we will show all of the results mentioned above.

Theorem 3.1. *Let* Ω_2 *and* Ω_3 *be defined in* (1.15) *and* (1.16), *respectively. Consider the initial value problem* (1.1). *For any initial data* $\phi \in C^k(\mathbb{R}) \backslash \Omega_3$, *we have* $H = U \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$, *where these sets are defined in Sec.* 1. *In particular,* $\Omega_3 \subset C^k(\mathbb{R})$ *of the first category is a proper subset of* Ω_2. *Moreover, the solutions of* (1.1) *are piecewise* C^k *smooth.*

Below we outline the proof of the above theorem. First we can show that $(x,t) \in U \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$ for each $(x,t) \in H$, namely, $H = U \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$ provided that $L_3 = \emptyset$, since there are only finitely many connected components of $\mathscr{M}_{(x,t)}$ for each given $(x,t) \in H$, where $\mathscr{M}_{(x,t)}$ is defined by (1.9a). By Lemma 2.1, $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$ is a closed subset of H, and Γ is covered by open neighborhoods of the type as described in Theorems 2.11 and 2.13. For any compact set $K \subset H$, by choosing a finite subcover of $K \cap \Gamma$ such that $K \cap \Gamma$ consists of the union of a finite number of shocks where each shock is piecewise C^{k+1} smooth. Therefore, if $\phi \in C^k(\mathbb{R}) \backslash \Omega_3$, then the minimizing function $u(x,t)$ is piecewise C^k smooth since $C^k(\mathbb{R}) \backslash \Omega_3 = \{\phi \in C^k(\mathbb{R}) \mid L_3 = \emptyset\}$.

Theorem 3.2. *Let $\Omega_4 = (\Omega_3 \cap \mathscr{S}(\mathbb{R})) \cup \triangle^c \subset \mathscr{S}(\mathbb{R})$, where Ω_3 is defined by* (1.16), *$\mathscr{S}(\mathbb{R})$ is the Schwartz space, $\triangle^c = \mathscr{S}(\mathbb{R}) \backslash \triangle$, $\triangle$ is an open and dense set. For $\phi \in \mathscr{S}(\mathbb{R}) \backslash \Omega_4$, we have $H = U \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$. Moreover, the solutions of* (1.1) *are piecewise C^∞ smooth and the total number of possible shocks is finite.*

We outline the proof of Theorem 3.2. Li–Wang in [6, Theorems 4 and 5] proved that there is an open and dense set $\triangle \subset \mathscr{S}(\mathbb{R})$ such that for any $\phi \in \triangle$, the associated function Φ attains its minimum over $\mathbb{R}$ only at points a_i ($|a_i| < \infty$) and $\phi^{(l_i)}(a_i) \neq 0$, $\phi^{(l_i-1)}(a_i) = \cdots = \phi(a_i) = 0$, where l_i ($i = 1, \ldots, m$) is some integer. Then for sufficiently large t there are precisely $m+1$ smooth shocks. The proofs of these results are the refinement of [9, Lemma 4.1 and Proposition 4.2]. Consequently, we see there is an open and dense set $\triangle \subset \mathscr{S}(\mathbb{R})$ such that for any $\phi \in \triangle$, there are finitely many shock curves for sufficiently large t, say $t > T$. It is easy to show that there is a constant X such that no shock can be formed in the region $\{(x,t) : |x| \geq X, 0 \leq t \leq T\}$. Consequently, only finitely many shocks can be formed in the compact region $\{(x,t) : |x| \leq X, 0 \leq t \leq T\}$ by Theorem 3.1. This completes the proof of the above theorem.

4. Concluding Remarks

In this work, we proved that if the initial data do not belong to a very small subset of C^k then the solutions of scalar conservation laws are piecewise C^k smooth. It is important to understand the conditions under which the solution of the conservation law (1.1) is *piecewise smooth* since most practical cases deal with the piecewise smooth solutions. For this reason, there have many studies on approximation methods for conservation laws whose solutions are piecewise smooth. For example, for systems of conservation laws, Goodman and Xin [3] proved that the viscosity methods approximating piecewise smooth solutions with finitely many *noninteracting* shocks have a local first-order rate of convergence away from the shocks; on the other hand, for scalar conservation laws, the global rate of convergence for the viscosity methods can be obtained [13, 14], and the point-wise rate of convergence for the viscosity methods has been obtained [11, 12].

In this work, we have introduced a new approach for studying the solution structures for the conservation laws, which is particularly suitable for handling the larger class of initial data considered in this works. We point out that Thom's theory of catastrophes [15], which plays a key role in Schaeffer [9], cannot be used to analyze the larger class of initial data. The main motivation of this study is to develop a new analysis approach which can be extended to study the solution structures for the Hamilton–Jacobi equations. The study along this direction is under investigation, and some relevant results will be reported elsewhere.

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