SPLINE GALERKIN METHODS FOR THE SHERMAN-LAURICELLA EQUATION ON CONTOURS WITH CORNERS

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Abstract. Spline Galerkin approximation methods for the Sherman-Lauricella integral equation on simple closed piecewise smooth contours are studied, and necessary and sufficient conditions for their stability are obtained. It is shown that the method under consideration is stable if and only if certain operators associated with the corner points of the contour are invertible. Numerical experiments demonstrate a good convergence of the spline Galerkin methods and validate theoretical results. Moreover, it is shown that if all corners of the contour have opening angles located in interval $(0.1\pi, 1.9\pi)$, then the corresponding Galerkin method based on splines of order 0, 1 and 2 is always stable. These results are in strong contrast with the behaviour of the Nyström method, which has a number of instability angles in the interval mentioned.

Key words. Sherman–Lauricella equation, spline Galerkin method, stability, critical angles

AMS subject classifications. 65R20, 45L05

1. Introduction.

Let D be a simply connected planar domain bounded by a piecewise smooth curve Γ. It is well known that the solution of various boundary value problems for the biharmonic equation

$$\Delta^2 u(x,y) \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0, \quad (x,y) \in D,$$

where Δ is the Laplace operator, can be constructed via solutions of boundary integral equations. Consider, for example, the biharmonic Dirichlet problem

$$\begin{aligned} &\Delta^2|_D = 0, \\ &u|_\Gamma = f_1, \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_\Gamma = f_2, \end{aligned} \tag{1.1}$$

where $\partial u/\partial \mathbf{n}$ denotes the normal derivatives and f_1, f_2 are sufficiently smooth functions defined on the boundary Γ. This problem arises in various applications, in particular in the theory of viscous flows with small Reynolds numbers, bacteria movement, deflection of plates, elastic equilibrium of solids, sintering [3, 14, 17, 20, 22, 23, 24].

Setting $z = x + iy, i^2 = -1$, one can identify D with a domain in the complex plane $\mathbb{C}$. Let us equip the curve Γ with the counterclockwise orientation and consider the Sherman–Lauricella equation

$$\omega(t) + \frac{1}{2\pi i}\int_\Gamma \omega(\zeta)\, d\ln\left(\frac{\zeta - t}{\bar\zeta - \bar t}\right) - \frac{1}{2\pi i}\int_\Gamma \overline{\omega(\zeta)}\, d\left(\frac{\zeta - t}{\bar\zeta - \bar t}\right) = f(t), \quad t = x + iy \in \Gamma, \tag{1.2}$$

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where the bar denotes the complex conjugation and ω is an unknown function. Equation (1.2) originated in works of G. Lauricella (see [19]). He was the first who used the method of integral equations in elasticity. Later D.I. Sherman rewrites Lauricella equation in a complex form and proposes a new simple way to derive it [27]. The equation (1.2) is uniquely solvable in appropriate functional spaces, provided f satisfies certain smoothness conditions and

$$\operatorname{Re}\int_\Gamma \overline{f(t)}\,dt = 0, \tag{1.3}$$

[13, 20, 24]. Moreover, let $\alpha = \alpha(x,y)$, $(x,y)\in\Gamma$ denote the angle between the real axis $\mathbb{R}$ and the outward normal $\mathbf{n}$ to Γ at the point (x,y) and let $\mathbf{l}$ be the unit vector such that the angle between $\mathbf{l}$ and the real axis is $\alpha-\pi/2$. If one defines the function $f=f(t)=f(x,y)$, $t=x+iy$ by

$$f(t) := e^{-i\alpha}\left(f_2(t)+i\frac{\partial f_1}{\partial \mathbf{l}}(t)\right), \quad t\in\Gamma, \tag{1.4}$$

then the solution of the Sherman-Lauricella equation (1.2) with such right-hand side f can be used to determine a solution of the boundary value problem (1.1). More precisely, if ω is a solution of the equation (1.2) with the right-hand side (1.4), then consider two holomorphic functions $\varphi=\varphi(z)$ and $\psi=\psi(z)$, $z\in D$ defined by

$$\varphi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)}{\zeta - z}\,d\zeta, \quad z\in D, \tag{1.5}$$

$$\psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\omega(\zeta)}}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\zeta)}{\zeta - z}\,d\overline{\zeta} - \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\zeta}\omega(\zeta)}{(\zeta - z)^2}\,d\zeta, \quad z\in D. \tag{1.6}$$

According to [20], the boundary values of the functions φ and ψ satisfy the condition

$$\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)} = e^{-i\alpha}\left(f_2(t)+i\frac{\partial f_1}{\partial \mathbf{l}}(t)\right), \quad t\in\Gamma.$$

Therefore, by [11, Lemma 5.1.4] the function

$$u(x,y) := \operatorname{Re}\left(\overline{z}\varphi(z)+\psi(z)\right), \quad z=x+iy\in D \tag{1.7}$$

is the solution of the biharmonic Dirichlet problem (1.1).

Thus if an exact or an approximate solution of the integral equation (1.2) is known, then a solution of the biharmonic problem (1.1) can be obtained by using formulas (1.5), (1.6) and (1.7). Therefore, the main effort should be directed to the determination of solutions of the Sherman-Lauricella equation (1.2). Note that the Nyström method for the Sherman-Lauricella equation on smooth contours has been used in [14, 18] to find an approximate solution of biharmonic problems arising in fluid dynamics. However, the authors of these works do not provide any stability analysis for the method used. In the case of piecewise smooth curves, the study of the stability requires even more efforts since the integral operators in (1.2) are not compact. For piecewise smooth contours, conditions of the stability of the Nyström method are established in [6, 7]. These results have been used in [8] in order to construct a very accurate numerical method to find solutions of the biharmonic problem (1.1) in piecewise smooth domains in the case of piecewise continuous boundary conditions.

In the present paper, we consider spline Galerkin methods for the equation (1.2) and study their stability. It is shown that the corresponding method is stable if and only if certain operators R^τ from an algebra of Toeplitz operators are invertible. These operators depend on the spline space used and on the opening angles of the corner points $\tau \in \Gamma$. Unfortunately, nowadays there is no analytic tool to verify whether the operators in question are invertible or not. Nevertheless, we propose a numerical approach which can handle this problem. Thus spline Galerkin methods are applied to the Sherman–Lauricella equation on simple model curves and the behaviour of the corresponding approximation operators provide an information about the invertibility of the operators R^τ, $\tau \in \Gamma$. Note that in comparison to the Nyström method, the implementation of spline Galerkin methods requires more preparatory work. On the other hand, numerical experiments suggest that these methods have no "critical" angles located in the interval $[0.1\pi, 1.9\pi]$, i.e. if the boundary Γ does not possess corners with opening angles from the interval mentioned, then these methods are stable. In a sense, this is similar to the behaviour of the corresponding approximation methods for Sherman–Lauricella and Muskhelishvili equations in the case of smooth curves which always converge [5, 7, 10]. Of course, one also has to study the opening angles in the intervals $(0, 0.1\pi)$ and to $(1.9\pi, 2\pi)$ but this is a time consuming operation and will be considered elsewhere.

2. Splines and Galerkin method. We start this section with the construction of spline spaces on the contour Γ. Let $\gamma = \gamma(s), s \in \mathbb{R}$ be a 1-periodic parametrization of Γ, and let $\mathcal{M}_\Gamma$ denote the set of all corner points $\tau_0, \tau_1, \dots, \tau_{q-1}$ of Γ. Without loss of generality we can assume that $\tau_j = \gamma(j/q)$ for all $j = 0, 1, \dots, q-1$. In addition, we also suppose that the function γ is two times continuously differentiable on each interval $(j/q, (j+1)/q)$ and

$$\left|\gamma'\left(\frac{j}{q}+0\right)\right| = \left|\gamma'\left(\frac{j}{q}-0\right)\right|, \quad j = 0, 1, \dots, q-1.$$

Note that the last condition is not very restrictive and can always be satisfied by changing the parametrization of Γ in an appropriate way.

Let f and g be functions defined on the real line $\mathbb{R}$, and let $f * g$ denote the convolution

$$(f * g)(s) := \int_{\mathbb{R}} f(s-x)g(x)dx$$

of f and g. If χ is the characteristic function of the interval $[0, 1)$,

$$\chi(s) := \begin{cases} 1 & \text{if } s \in [0,1), \\ 0 & \text{otherwise,} \end{cases}$$

then $\widehat{\phi} = \widehat{\phi}^{(d)}(s)$ refers to the function defined by

$$\widehat{\phi}^{(d)}(s) := \begin{cases} \chi(s) & \text{if } d = 0, \\ (\chi * \widehat{\phi}^{(d-1)})(s) & \text{if } d = 1, 2 \dots . \end{cases}$$

Recall that for any given non-negative integer d, the function $\widehat{\phi}$ generates spline spaces on $\mathbb{R}$. Thus if an $n \in \mathbb{N}$ is fixed, then closure in the L^2-norm of the set of all finite linear combinations of the functions $\widehat{\phi}_{nj}(s) := \widehat{\phi}(ns - j)$, $j \in \mathbb{Z}$ constitutes a spline space on $\mathbb{R}$.

Using the above defined spline functions, one can introduce spline spaces on the contour Γ. More precisely, for a fixed non-negative integer d and an $n \in \mathbb{N}$, $n \geq d+1$, we denote by $S_n^d = S_n^d(\Gamma)$ the set of all linear combinations of the functions

$$\widehat{\phi}_{nj}(t) := \widehat{\phi}(ns - j), \quad t = \gamma(s) \in \Gamma, \quad j = 0, 1, \ldots, n - (d+1), \quad s \in \mathbb{R},$$

the support of which belongs entirely to one of the arcs $[\tau_k, \tau_{k+1})$, $k = 0, \ldots, q$ and $\tau_{q+1} := \tau_0$. This definition is correct since the support $\operatorname{supp} \widehat{\phi}$ of the function $\widehat{\phi}$ is contained in the interval $[0, d+1]$ [26] and γ is a 1-periodic function.

In what follows, we also consider operators acting on various subspaces of the Hilbert space $\widetilde{l^2} = l^2(\mathbb{Z})$ of all sequences (ξ_k) of complex numbers $\xi_k, k \in \mathbb{Z}$ satisfying the condition

$$||(\xi_k)|| := \left(\sum_{k \in \mathbb{Z}} |\xi_k|^2 \right)^{1/2} < \infty.$$

The space $\widetilde{l^2}$ is closely connected to spline spaces on the real line $\mathbb{R}$. Indeed, the following result is true.

Lemma 2.1 ([4]). *Let $n \in \mathbb{N}$. Then there are constants c_1 and c_2 such that for any sequence $(\xi_k) \in \widetilde{l^2}$ the relations*

$$||(\xi_k)|| \leq c_1 \sqrt{n} \Big\| \sum_{k \in \mathbb{Z}} \xi_k \widehat{\phi}_{nk} \Big\|_{L^2(\mathbb{R})} \leq \frac{c_2}{\sqrt{n}} ||(\xi_k)||$$

hold.

Further, let $L^2(\Gamma)$ denote the set of all Lebesgue measurable functions f such that

$$||f||_{L^2} := \left(\int_\Gamma |f(t)|^2 \, |dt| \right)^{1/2} < \infty,$$

and let $A_\Gamma : L^2(\Gamma) \to L^2(\Gamma)$ be the operator corresponding to the Sherman-Lauricella equation (1.2). It is well known that the operator A_Γ is not invertible on the space $L^2(\Gamma)$ [20]. On the other hand, the invertibility of the corresponding operator is a necessary condition for the applicability of any Galerkin method to any operator equation. Therefore, for the approximate solution of the equation (1.2) we use the equation with an operator B_Γ instead of A_Γ and choose the right-hand sides f of the initial equation (1.2) from a suitable subspace of $L^2(\Gamma)$. More precisely, let $W_2^1(\Gamma)$ denote the closure of the set of all functions f with bounded derivatives in the norm

$$||f||_{W_2^1} := \left(\int_\Gamma |f(t)|^2 \, ds + \int_\Gamma |f'(t)|^2 \, ds \right)^{1/2},$$

and let $T_\Gamma : L^2(\Gamma) \to L^2(\Gamma)$ refer to the operator defined by

$$T_\Gamma \omega(t) := \frac{1}{(\overline{t} - \overline{a})} \frac{1}{2\pi i} \int_\Gamma \left(\frac{\omega(\zeta)}{(\zeta - a)^2} d\zeta + \frac{\overline{\omega(\zeta)}}{(\overline{\zeta} - \overline{a})^2} d\overline{\zeta} \right), \tag{2.1}$$

where a is a point in D.

Theorem 2.2. *If Γ is a simple closed piecewise smooth contour, then the equation*

$$B_\Gamma = (A_\Gamma + T_\Gamma)\omega = f \tag{2.2}$$

is uniquely solvable for any right-hand side $f \in L_2(\Gamma)$*. Moreover, if* $f \in W_2^1(\Gamma)$ *and satisfies condition* (1.3)*, then the solution of equation* (2.2) *is simultaneously a solution of the original Sherman-Lauricella equation* (1.2)*.*

Note that the proof of Theorem 2.2 involves information about the behaviour of the operators A_Γ and B_Γ in both $L^2(\Gamma)$ and $W_2^1(\Gamma)$ spaces. All the results concerning the operators A_Γ and B_Γ acting on $W_2^1(\Gamma)$ and similar operators connected with the Muskhelishvili equation follow from [13] where they proved even in a more general context of the spaces $W_p^1(\Gamma,\rho)$, $1 < p < \infty$ with Khvedelidze weights ρ. The same operators but acting in $L_p(\Gamma,\rho)$ have been considered in [10]. In the special case of L_2 space, Fredholm properties of the Muskhelishvili and Sherman-Lauricella operators are also presented in [7, 9].

Thus if the right hand sides $f \in W_2^1(\Gamma)$, an exact or an approximate solution of the equation (1.2) can be derived from the corrected Sherman-Lauricella equation (2.2). In the present paper, we employ spline based Galerkin methods to the equation (2.2) and study their stability and convergence. Let us describe these methods in more detail. First of all, we normalize all the basis spline functions used. If n is fixed, then for any $j \in \mathbb{Z}$ the norm $||\widehat{\phi}_{nj}||$ of any basis element $\widehat{\phi}_{nj}$ is

$$||\widehat{\phi}_{nj}||^2 = \frac{1}{n}\int_0^{d+1} \widehat{\phi}^2(s)\,ds.$$

Therefore, if ν_d refers to the number

$$\nu_d := \left(\int_0^{d+1} \widehat{\phi}^2(s)\,ds.\right)^{-1/2}, \tag{2.3}$$

then

$$\phi_{nj} := \nu_d \sqrt{n}\,\widehat{\phi}_{nj}, \quad j \in \mathbb{Z} \tag{2.4}$$

are unit norm vectors. An approximate solution of the equation (2.2) is sought in the form

$$\omega_n(t) = \sum_{\phi_{nk} \in S_n^d(\Gamma)} a_k \phi_{nk}(t), \tag{2.5}$$

the coefficients a_k of which are obtained from the following system of algebraic equations

$$(B_\Gamma \omega_n, \phi_{nj}) = (f, \phi_{nj}), \quad \phi_{nj} \in S_n^d(\Gamma), \tag{2.6}$$

where

$$(u,v) := \int_\Gamma u(t)\overline{v(t)}\,|dt|, \quad u,v \in L_2(\Gamma).$$

An important problem now is to study the solvability of the equations (2.6) and convergence of the approximate solutions to an exact solution of the original Sherman–Lauricella equation (1.2). In Section 3, this problem is discussed in a more detail but, at the moment, we would like to illustrate the method by a few numerical examples. Thus we present Galerkin solutions of the equation (1.2) with the right-hand side $f = f_1$,

$$f_1(z) = f(x,y) = 4x^3 - 12xy^2 + i(4y^3 - 12x^2y); \quad z = x + iy \in \Gamma, \tag{2.7}$$

on the unit square and rhombuses, and trace the evolution of the solution when the initial contour is transformed from the unit square into rhombuses with various opening angle α. Some of these contours have been used in [7] in order to illustrate the behaviour of the Nyström method. Note that in the corresponding examples from [7], approximate solutions of the equation (1.2) with the right-hand side

$$f_2(z) = |z|$$

have been determined. We apply the spline Galerkin method to the equations with such right-hand side, too. The results obtained have a very good correlation with [7] and the error evaluation for both cases are reported in Table 2.1, where $E^{f_i}_{n,\alpha}$ denotes the relative error $\|\omega_{2n} - \omega_n\|_2/\|\omega_{2n}\|_2$ computed for the righthand side f_i and equation (1.2) is considered on the rhombus with the opening angle α. In addition, Figures

Table 2.1
Relative error of the spline Galerkin methods

n	$E^{f_1}_{n,\pi/2}$	$E^{f_1}_{n,\pi/3}$	$E^{f_1}_{n,\pi/4}$	$E^{f_1}_{n,\pi/5}$	$E^{f_2}_{n,\pi/2}$	$E^{f_2}_{n,\pi/3}$	$E^{f_2}_{n,\pi/6}$
128	0.0373	0.6194	1.3577	2.1716	0.0121	0.0217	0.0205
256	0.0198	0.0268	0.2046	0.6169	0.0067	0.0112	0.0245
512	0.0096	0.0059	0.0616	0.1888	0.0045	0.0102	0.0193

2.1–2.4 show the convergence of the approximate solutions of the equation (1.2) with the right-hand side (2.7) obtained by the Galerkin method based on the splines of degree $d = 0$ and the transformation of these approximate solutions when n increases.

Let us mention a few technical details related to the examples below. Thus the rhombus with an opening angle α is parameterized as follows,

$$\gamma(s) = \begin{cases} 4s - \cos\left(\frac{\alpha}{2}\right) e^{i\alpha/2} & \text{if } 0 \le s < 1/4, \\ (4s-1)e^{i\alpha} - i\sin\left(\frac{\alpha}{2}\right) e^{i\alpha/2} & \text{if } 1/4 \le s < 1/2, \\ -(4s-2) + \cos\left(\frac{\alpha}{2}\right) e^{i\alpha/2} & \text{if } 1/2 \le s < 3/4, \\ -(4s-3)e^{i\alpha} + i\sin\left(\frac{\alpha}{2}\right) e^{i\alpha/2} & \text{if } 3/4 \le s \le 1. \end{cases} \tag{2.8}$$

We also have to compute the scalar products $(B_\Gamma \omega_n, \phi_{nj})$. Recall that $\operatorname{supp} \phi_{nj} \subset [j/n,\ (j+d+1)/n]$ and use the Gauss-Legendre quadrature rule with quadrature points which coincide with the zeros of the Legendre polynomial $P_{24}(x)$ on the canonical interval $[-1, 1]$, scaled and shifted to the interval $[j/n,\ (j+d+1)/n]$. More specifically, the corresponding formula is

$$(B_\Gamma \omega_n, \phi_{nj}) = \int_{j/n}^{(j+d+1)/n} B_\Gamma \omega_n(\gamma(s))\, \overline{\phi_{nj}(\gamma(s))} ds \approx \sum_{k=1}^{24} w_k B_\Gamma \omega_n(\gamma(s_k))\, \phi_{nj}(\gamma(s_k)), \tag{2.9}$$

where w_k, s_k are the Gauss-Legendre weights and the Gauss-Legendre points on the interval $[j/n,\ (j + d + 1)/n]$. In order to find the values of the corresponding line integrals at the Gauss-Legendre points, the composite Gauss-Legendre quadrature is used [7, Section 3], namely,

$$\begin{aligned}\int_\Gamma k(t,\tau)x(\tau)d\tau &= \int_0^1 k(\gamma(\sigma),\gamma(s))x(\gamma(s))\gamma'(s)ds \\ &\approx \sum_{l=0}^{m-1}\sum_{p=0}^{r-1} w_p k(\gamma(\sigma),\gamma(s_{lp}))x(\gamma(s_{lp}))\tau'_{lp}/m,\end{aligned} \tag{2.10}$$

where $\tau'_{lp} = \gamma'(s_{lp})$ with $m = 40$ and $r = 24$.

Table 2.1 and Figures 2.1-2.2 show a good convergence of approximate solutions if the corner point of the contour has an opening angle close or equal to $\pi/2$. On the other hand, the presence of opening angles of a small magnitude can cause problems and leads to a convergence slowdown (see Figures 2.3-2.4). Note that although the focus of this work is on the stability, the error estimates presented in Table 2.1 are comparable with estimates of the recent work [16] for fast Fourier–Galerkin method for an integral equation used to solve boundary value problem (1.1) in smooth domains. Moreover, further improvement of the convergence rate is possible if for the approximations of singular integrals and inner products arising in the Galerkin method one employs graded meshes of various kind [2, 15].

3. Galerkin method. Local operators and stability. Our next task is to find conditions of applicability of the spline Galerkin methods to the equation (2.2). It is worth mentioning that for smooth contours Γ, the methods considered here are always applicable and provide satisfactory results. For details the reader can consult [5], where similar methods for the Muskhelishvili equation on smooth

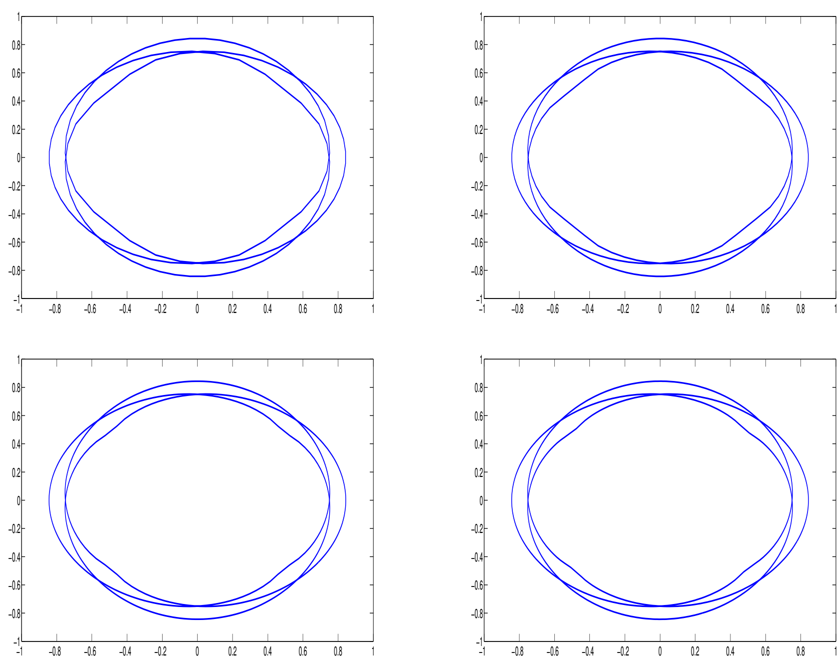

FIG. 2.1. Approximate solution $\omega_n(t)$ of the Sherman–Lauricella equation (1.2) on the unit square Γ with $f := f_1$ defined by (2.7) and $d = 0$. From the left to the right: $n = 128, 256, 512, 1024$

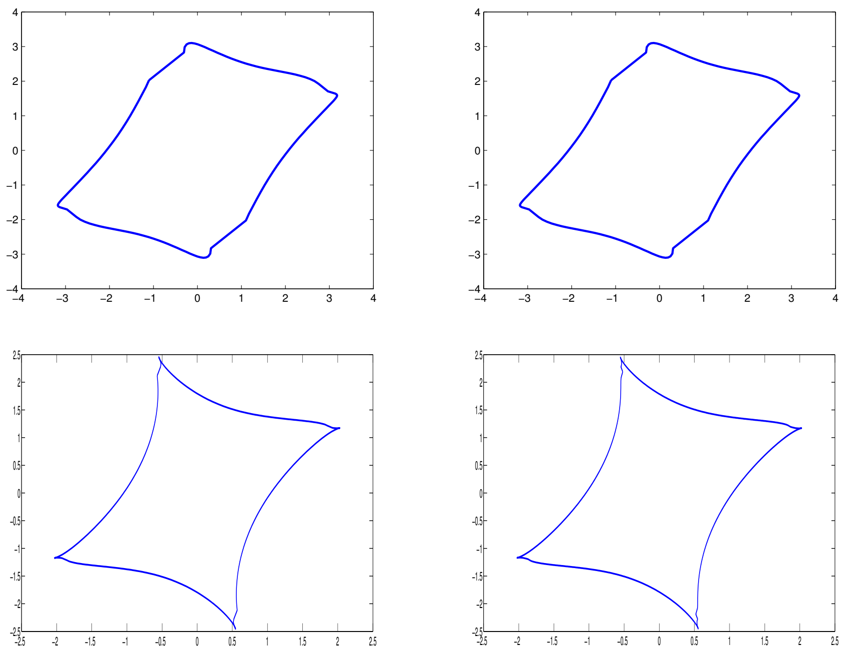

FIG. 2.2. Approximate solution $\omega_n(t)$ of the Sherman–Lauricella equation (1.2) on the rhombus Γ, $\alpha = \pi/3$ with $f := f_1$ defined by (2.7) and $d = 0$. From the left to the right: $n = 128, 256, 512, 1024$

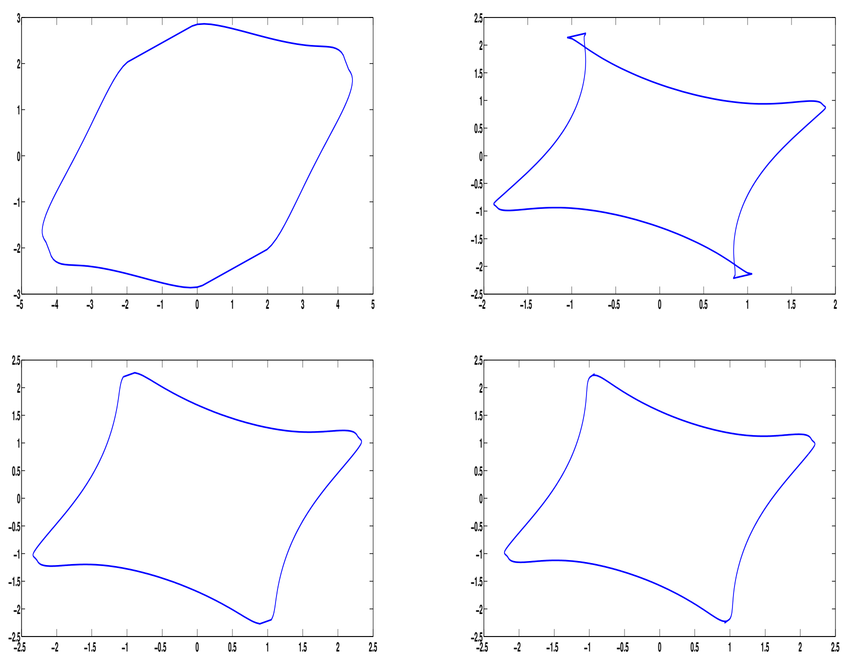

FIG. 2.3. Approximate solution $\omega_n(t)$ of the Sherman–Lauricella equation (1.2) on the rhombus Γ, $\alpha = \pi/4$ with $f := f_1$ defined by (2.7) and $d = 0$. From the left to the right: $n = 128, 256, 512, 1024$

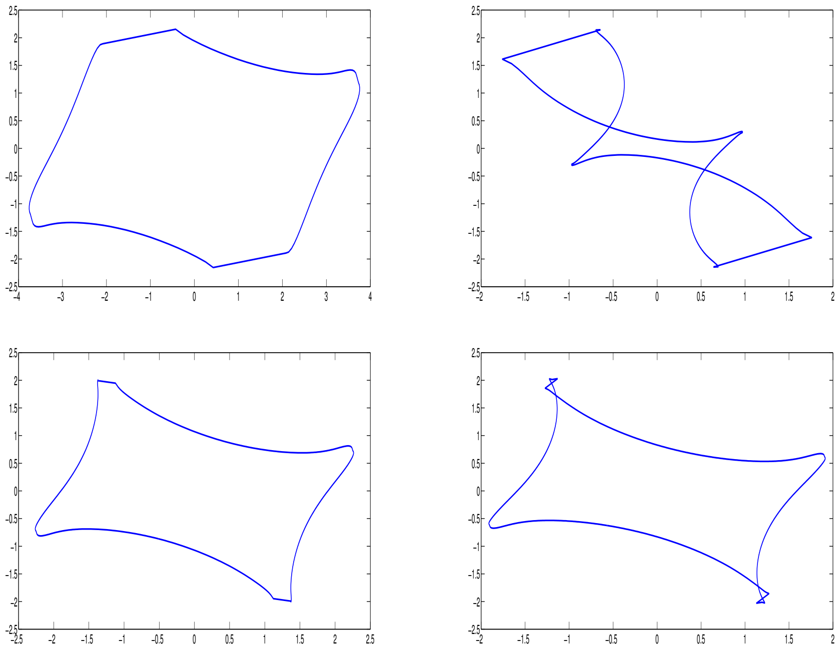

FIG. 2.4. Approximate solution $\omega_n(t)$ of the Sherman–Lauricella equation (1.2) on the rhombus Γ, $\alpha = \pi/5$ with $f := f_1$ defined by (2.7) and $d = 0$. From the left to the right: $n = 128, 256, 512, 1024$

contours are considered. On the other hand, the presence of corners changes the situation drastically, and the applicability of the approximation method is not always guaranteed.

Let P_n be the orthogonal projection from $L^2(\Gamma)$ on the subspace $S_n^d(\Gamma)$. Then the systems (2.6) are equivalent to the following operator equations

$$P_n B_\Gamma P_n \omega_n = P_n f, \quad n \in \mathbb{N}. \tag{3.1}$$

DEFINITION 3.1. *We say that the sequence* $(P_n B_\Gamma P_n)$ *is stable if there is an* $m \in \mathbb{R}$ *and an* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *the operators* $P_n B_\Gamma P_n : S_n^d(\Gamma) \to S_n^d(\Gamma)$ *are invertible and*

$$||(P_n B_\Gamma P_n)^{-1} P_n|| \leq m$$

for all $n \geq n_0$.

Recall that if the stability of the corresponding sequence $(P_n B_\Gamma P_n)$ is established, then the convergence of the Galerkin method and error estimates can be obtained from well known results, cf. [11, Section 1.6, inequality (1.30)]. Therefore, in this work we mainly deal with the stability and our approach is based on C^*-algebra methods often used in operator theory. Let $\mathcal{L}_{add}(L^2(\Gamma))$ refer to the real C^*-algebra of all additive continuous operators acting on the space $L^2(\Gamma)$. One can show [11] that every operator $A \in \mathcal{L}_{add}(L^2(\Gamma))$ admits the unique representation $A = A_1 + A_2 M$, where A_1, A_2 are linear operators and M is the operator of complex conjugation. This representation allows one to introduce the operation of involution on $\mathcal{L}_{add}(L^2(\Gamma))$ as follows

$$A^* := A_1^* + M A_2^*, \tag{3.2}$$

with A_1^*, A_2^* being the usual adjoint operators to the linear operators A_1, A_2, cf. [11, Theorem 1.3.8 and Example 1.3.9]. By $\mathcal{A}^\Gamma$ we denote the set of all bounded sequences (A_n) of bounded additive operators $A_n : \operatorname{im} P_n \to \operatorname{im} P_n$ such that there is an operator $A \in \mathcal{L}_{add}(L^2(\Gamma))$ with the property

$$s-\lim A_n P_n = A, \quad s-\lim (A_n P_n)^* P_n = A^*,$$

where $s-\lim A_n$ denotes the strong limit of the operator sequence (A_n).

Endowed by the natural operations of addition, multiplication, multiplication by scalars $\lambda \in \mathbb{C}$, by an involution introduced according to (3.2), and by the norm

$$||(A_n)|| := \sup_{n\in\mathbb{N}} ||A_n||,$$

the set $\mathcal{A}^\Gamma$ becomes a real C^*-algebra. Consider also the subset $\mathcal{J}^\Gamma \subset \mathcal{A}^\Gamma$ consisting of all sequences (J_n) of operators $J_n : \operatorname{im} P_n \to \operatorname{im} P_n$ which can be represented in the form

$$J_n = P_n T P_n + C_n, \quad n \in \mathbb{N},$$

where the operator T belongs to the ideal $\mathcal{K}_{add}(L^2(\Gamma)) \subset \mathcal{L}_{add}(L^2(\Gamma))$ of all compact operators and the sequence (C_n) tends to zero uniformly, i.e.

$$\lim_{n\to\infty} ||C_n|| = 0.$$

The stability of sequences from the algebra $\mathcal{A}^\Gamma$ can be characterized as follows.

Theorem 3.2 (cf. [11, Proposition 1.6.3]). *A sequence $(A_n) \in \mathcal{A}^\Gamma$ such that $A := s-\lim A_n P_n$ is stable if and only if the operator A is invertible in $\mathcal{L}_{add}(L^2(\Gamma))$ and the coset $(A_n) + \mathcal{J}^\Gamma$ is invertible in the quotient algebra $\mathcal{A}^\Gamma/\mathcal{J}^\Gamma$.*

Consider now the sequence $(P_n B_\Gamma P_n)$ of the Galerkin operators defined by the projection operators P_n. Recall that on the space $L^2(\Gamma)$ the sequence of the orthogonal projections (P_n) strongly converges to the identity operator I and $P_n^* = P_n, n \in \mathbb{N}$. This implies that for any operator $A \in \mathcal{L}_{add}(L^2(\Gamma))$ the following relations

$$s-\lim P_n A P_n = A, \quad s-\lim(P_n A P_n)^* P_n = A^*$$

hold [25]. Therefore, combining Theorem 2.2 and Theorem 3.2 one obtains the following result.

Corollary 3.3. *Let Γ be a simple closed piecewise smooth curve. The spline Galerkin method* (3.1) *is stable if and only if the coset $(P_n B_\Gamma P_n) + \mathcal{J}^\Gamma$ is invertible in the quotient algebra $\mathcal{A}^\Gamma/\mathcal{J}^\Gamma$.*

Thus in order to establish the stability of the Galerkin method, one has to study the invertibility of the coset $(P_n B_\Gamma P_n) + \mathcal{J}^\Gamma$ in the algebra $\mathcal{A}^\Gamma/\mathcal{J}^\Gamma$. This problem can be tackled more efficiently, if we restrict ourselves to a smaller algebra containing the coset $(P_n B_\Gamma P_n) + \mathcal{J}$. More precisely, let S_Γ be the Cauchy singular integral operator,

$$S_\Gamma \phi(t) := \frac{1}{\pi i} \int_\Gamma \frac{\phi(\zeta)}{\zeta - t}\, d\zeta.$$

Consider the smallest closed real C^*-subalgebra $\mathcal{B}^\Gamma$ of the algebra $\mathcal{A}^\Gamma$ which contains all operator sequences of the form $(P_n M P_n), (P_n S_\Gamma P_n)$ and also the sequences $(P_n f P_n), f \in \mathbb{C}_\mathbb{R}(\Gamma)$ and (G_n), where $\lim_{n\to\infty} ||G_n|| = 0$ and $\mathbb{C}_\mathbb{R}(\Gamma)$ is the set of all continuous real-valued functions on the contour Γ.

REMARK 3.4. *It follows from [10, 12, 21, 25] that $\mathcal{J}^\Gamma \subset \mathcal{B}^\Gamma$ and that the sequence $(P_n B_\Gamma P_n)$ belongs to $\mathcal{B}^\Gamma$. Therefore, $\mathcal{B}^\Gamma/\mathcal{J}^\Gamma$ is a real C^*-subalgebra of $\mathcal{A}^\Gamma/\mathcal{J}^\Gamma$, and by [11, Corollary 1.4.10] the coset $(P_n B_\Gamma P_n)+\mathcal{J}^\Gamma$ is invertible in $\mathcal{A}^\Gamma/\mathcal{J}^\Gamma$ if and only if it is invertible in $\mathcal{B}^\Gamma/\mathcal{J}^\Gamma$. Therefore, one can now study the invertibility of the coset $(P_n B_\Gamma P_n)+\mathcal{J}^\Gamma$ in the smaller algebra $\mathcal{B}^\Gamma/\mathcal{J}^\Gamma$. To this end we will employ a localizing principle.*

Thus with each point $\tau \in \Gamma$ we associate a model contour Γ_τ as follows. Let θ_τ be the angle between the right and the left semi-tangents to Γ at the point τ, and let β_τ refer to the angle between the right semi-tangent to Γ and the real line $\mathbb{R}$. Consider now the curve

$$\Gamma_\tau := e^{i(\beta_\tau+\theta_\tau)}\mathbb{R}^+_- \cup e^{i\beta_\tau}\mathbb{R}^+_+$$

where $\mathbb{R}^+_-$ and $\mathbb{R}^+_+$ denote the positive semi-axis $\mathbb{R}^+$ correspondingly directed to and out of the origin. Further, on each such contour Γ_τ, $\tau \in \Gamma$ we consider the corresponding Sherman-Lauricella operator

$$A_\tau = I + L_\tau - K_\tau M, \tag{3.3}$$

where

$$L_\tau\omega(t) := \frac{1}{2\pi i}\int_{\Gamma_\tau} \omega(\zeta)\, d\ln\left(\frac{\zeta-t}{\bar{\zeta}-\bar{t}}\right), \quad K_\tau\omega(t) := \frac{1}{2\pi i}\int_{\Gamma_\tau} \omega(\zeta)\, d\left(\frac{\bar{\zeta}-\bar{t}}{\zeta-t}\right).$$

Analogously to the algebra $\mathcal{B}^\Gamma$ and to the ideal $\mathcal{J}^\Gamma$ one can introduce algebras $\mathcal{B}^{\Gamma_\tau}$ and ideals $\mathcal{J}^{\Gamma_\tau} \subset \mathcal{B}^{\Gamma_\tau}$, $\tau \in \Gamma$, which allow to establish conditions of the applicability of the corresponding Galerkin method for the operator (3.3). For this we also need appropriate spline spaces on both the contour Γ_τ and the positive semi-axis $\mathbb{R}^+ := \mathbb{R}^+_+$. These spline spaces can be constructed by using the functions (2.4) again. More precisely, consider the functions

$$\widetilde{\phi}_{nj}(t) := \begin{cases} \begin{cases} \phi_{nj}(s) & \text{if} \quad t = e^{i\beta_\tau}s \\ 0 & \text{otherwise} \end{cases} & j \geq 0 \\ \begin{cases} \phi_{n,j-d}(s) & \text{if} \quad t = e^{i(\beta_\tau+\theta_\tau)}s \\ 0 & \text{otherwise} \end{cases} & j < 0 \end{cases}. \tag{3.4}$$

Let $S_n^d(\Gamma_\tau)$ and $S_n^d(\mathbb{R}^+)$ be, respectively, the smallest closed subspaces of $L_2(\Gamma_\tau)$ and $L^2(\mathbb{R}^+)$ which contains all functions (3.4) and all functions ϕ_{nj}, $j \geq 0$ of (3.4) for $\beta_\tau = 0$. Moreover, let P_n^τ, $n \in \mathbb{N}$ and P_n^+ denote the orthogonal projections onto the subspaces $S_n^d(\Gamma_\tau)$ and $S_n^d(\mathbb{R}^+)$, respectively. In order to study the stability of the sequence $(P_n^\tau A_\tau P_n^\tau)$, one can apply Theorem 3.2 and Remark 3.4 to obtain the following result.

COROLLARY 3.5. *The sequence $(P_n^\tau A_\tau P_n^\tau) \in \mathcal{B}^{\Gamma_\tau}$ is stable if and only if the operator A_τ is invertible in $\mathcal{B}^\tau$ and the coset $(P_n^\tau A_\tau P_n^\tau) + \mathcal{J}^{\Gamma_\tau}$ is invertible in the quotient algebra $\mathcal{B}^{\Gamma_\tau}/\mathcal{J}^{\Gamma_\tau}$.*

Further, let $L_2^2(\mathbb{R}^+)$ be the space of all pairs $(\varphi_1,\varphi_2)^T$, $\varphi_1,\varphi_2 \in L^2(\mathbb{R}^+)$ endowed by the norm

$$||(\varphi_1,\varphi_2)^T|| := (||\varphi_1||^2 + ||\varphi_2||^2)^{1/2},$$

and let $\eta : L^2(\Gamma_\tau) \to L_2^2(\mathbb{R}^+)$ be the mapping defined by

$$\eta(\varphi) = (\varphi(se^{i(\beta_\tau+\omega_\tau)}), \varphi(se^{i\beta}))^T, \quad s \in \mathbb{R}^+,$$

where a^T denotes the transposition of the vector a. It is clear that η is a linear isometry from $L^2(\Gamma_\tau)$ onto $L_2^2(\mathbb{R}^+)$. Moreover, the mapping $\Psi : \mathcal{L}_{add}(L^2(\Gamma_\tau)) \to \mathcal{L}_{add}(L_2^2(\mathbb{R}^+))$ defined by

$$\Psi(A) = \eta A \eta^{-1}, \tag{3.5}$$

is an isometric algebra isomorphism. In particular, straightforward calculations show that

$$\Psi(P_n^\tau) = \operatorname{diag}(P_n^+, P_n^+), \tag{3.6}$$

$$\Psi(M) = \operatorname{diag}(\widetilde{M}, \widetilde{M}), \tag{3.7}$$

$$\Psi(L_\tau) = \begin{pmatrix} 0 & \mathcal{N}_{\theta_\tau} \\ \mathcal{N}_{\theta_\tau} & 0 \end{pmatrix}, \tag{3.8}$$

$$\Psi(K_\tau) = \begin{pmatrix} 0 & e^{i2\beta_\tau}\mathcal{M}_{2\pi-\theta_\tau} \\ -e^{i2(\beta_\tau+\theta_\tau)}\mathcal{M}_{\theta_j} & 0 \end{pmatrix}, \tag{3.9}$$

where

$$\mathcal{N}_{\theta_\tau}\varphi(\sigma) = \frac{1}{2}\frac{1}{2\pi i}\int_0^\infty \left(\frac{1}{1-(\sigma/s)e^{i\theta_\tau}} - \frac{1}{1-(\sigma/s)e^{i(2\pi-\theta_\tau)}}\right)\varphi(s)\frac{ds}{s},$$

$$\mathcal{M}_{\theta_\tau}\varphi(\sigma) := \frac{1}{\pi}\int_0^\infty \left(\frac{\sigma}{s}\right)\frac{\sin\theta_\tau}{(1-(\sigma/s)e^{i\theta_\tau})^2}\varphi(s)\frac{ds}{s},$$

and the symbol $\widetilde{M}$ in the right-hand side of (3.7) refers to the operator of the complex conjugation on the space $L^2(\mathbb{R}^+)$. Moreover, one can observe that the operators $\mathcal{N}_{\theta_\tau}$ and $\mathcal{M}_{\theta_\tau}$ have a special form – viz.

$$K\varphi(\sigma) := \int_0^\infty \mathsf{k}_{\theta_\tau}\left(\frac{\sigma}{s}\right)\varphi(s)\frac{ds}{s} \tag{3.10}$$

and

$$\mathsf{k}_{\theta_\tau} = \mathsf{k}_{\theta_\tau}(u) := \mathsf{n}_{\theta_\tau}(u) = \frac{1}{2\pi}\frac{u\sin\theta_\tau}{|1-ue^{i\theta_\tau}|^2}, \quad \text{if} \quad K = \mathcal{N}_{\theta_\tau}, \tag{3.11}$$

$$\mathsf{k}_{\theta_\tau} = \mathsf{k}_{\theta_\tau}(u) := \mathsf{m}_{\theta_\tau}(u) = \frac{1}{\pi}\frac{u\sin\theta_\tau}{(1-ue^{i\theta_\tau})^2}, \quad \text{if} \quad K = \mathcal{M}_{\theta_\tau}. \tag{3.12}$$

On the space l^2 of the sequences (ξ_k) of complex numbers $\xi_k, k = 0, 1, \ldots,$

$$l^2 := \{(\xi_k)_{k=0}^\infty : \sum_{k=0}^\infty |\xi_k|^2 < \infty\},$$

the function k_{θ_τ} defines a bounded linear operator $A(\mathsf{k}_{\theta_\tau})$ with the matrix representation

$$A(\mathsf{k}_{\theta_\tau}) = \left(\nu_d^2 \int_0^{d+1} \widehat{\phi}(t) \int_0^{d+1} \mathsf{k}_{\theta_\tau}\left(\frac{u+l}{t+q}\right)\widehat{\phi}(u)\frac{du}{u+q}\,dt\right)_{q,l=0}^\infty, \tag{3.13}$$

where ν_d is the constant (2.3).

THEOREM 3.6. *Let* n_{θ_τ} *and* m_{θ_τ} *be the functions defined by* (3.11) *and* (3.12), *respectively. The spline Galerkin method* (3.1) *is stable if and only if the operators* $R^\tau : l^2 \times l^2 \to l^2 \times l^2$,

$$R^\tau :=$$

$$\begin{pmatrix} I & A(\mathsf{n}_{\theta_\tau}) \\ A(\mathsf{n}_{\theta_\tau}) & I \end{pmatrix} + \begin{pmatrix} 0 & e^{i\beta_\tau} A(\mathsf{m}_{2\pi-\theta_\tau}) \\ -e^{-i(\beta_\tau+\theta_\tau)} A(\mathsf{m}_{\theta_\tau}) & 0 \end{pmatrix} \begin{pmatrix} \overline{M} & 0 \\ 0 & \overline{M} \end{pmatrix} \tag{3.14}$$

are invertible for all $\tau \in \mathcal{M}_\Gamma$.

Proof. By Corollary 3.3 the sequence $(P_n B_\Gamma P_n)$ is stable if and only if the coset $(P_n B_\Gamma P_n) + \mathcal{J}^\Gamma$ is invertible. Moreover, since T_Γ of (2.1) is a compact operator, the sequences $(P_n A_\Gamma P_n)$ and $(P_n B_\Gamma P_n)$ belong to the same coset $(P_n A_\Gamma P_n) + \mathcal{J}^\Gamma$ of the quotient algebra $\mathcal{B}^\Gamma/\mathcal{J}^\Gamma$. However, by a version of the Allan's Local Principle [1] for real C^*-algebras [11, Theorem 1.9.5], the coset $(P_n A_\Gamma P_n) + \mathcal{J}^\Gamma$ is invertible if and only if for every $\tau \in \Gamma$ the coset $(P_n^\tau A_{\Gamma_\tau} P_n^\tau) + \mathcal{J}^{\Gamma_\tau}$ is invertible in the corresponding algebra $\mathcal{B}^{\Gamma_\tau}/\mathcal{J}^{\Gamma_\tau}$. Therefore, the stability of our operator sequence will be established if we manage to show the invertibility of all cosets $(P_n^\tau A_{\Gamma_\tau} P_n^\tau) + \mathcal{J}^{\Gamma_\tau}$, $\tau \in \Gamma$. Let us start with the case where τ is not a corner point of Γ. If $\tau \notin \mathcal{M}_\Gamma$, then $\theta_\tau = \pi$, and straightforward calculations show that L_τ and K_τ are the zero operators. Hence, A_τ is just the identity operator I in the corresponding space, so that $P_n^\tau A_\tau P_n^\tau = P_n^\tau$. The sequence (P_n^τ) is obviously stable so that the corresponding coset $(P_n^\tau) + \mathcal{J}^\tau$ is invertible.

Consider next the case where $\tau \in \mathcal{M}_\Gamma$. The operator A_τ is invertible on the space $L^2(\Gamma_\tau)$, [7, Theorem 2.2]. Note that the invertibility of the operator A_τ in $L_2(\Gamma)$ also follows from [10, 9]. Therefore, by Corollary 3.5 the coset $(P_n^\tau A_\tau P_n^\tau) + \mathcal{J}^\tau$ is invertible in $\mathcal{B}^{\Gamma_\tau}/\mathcal{J}^{\Gamma_\tau}$ if and only if the sequence $(P_n^\tau A_\tau P_n^\tau)$ is stable. However, the stability of this sequence is equivalent to the stability of the sequence $(\Psi(P_n^\tau A_\tau P_n^\tau))$, where the mapping Ψ is defined by (3.5). Consider also the operators $\Lambda_n : S_n^d(\mathbb{R}^+) \to l^2$ defined by

$$\Lambda_n \left(\sum_{j=0}^{\infty} \xi_j \phi_{nj} \right) = (\xi_0, \xi_1, \ldots,).$$

By Lemma 2.1 these operators are bounded and continuously invertible. Set $\Lambda_{-n} := \Lambda_n^{-1}$ and note that the sequence $(\Psi(P_n^\tau A_\tau P_n^\tau))$ is stable if and only if so is the sequence (R_n^τ), where

$$R_n^\tau = \operatorname{diag}(\Lambda_n, \Lambda_n) \cdot \Psi(P_n^\tau A_\tau P_n^\tau) \cdot \operatorname{diag}(\Lambda_{-n}, \Lambda_{-n}) : l^2 \times l^2 \to l^2 \times l^2.$$

From the definition of the mappings Ψ and $\Lambda_{\pm n}$ one obtains that the operators R_n^τ have the form

$$R_n^\tau = (A_{lp}^{(n,\tau)})_{l,p=1}^2 + (D_{lp}^{(n,\tau)})_{l,p=1}^2 \operatorname{diag}(\overline{M}, \overline{M}),$$

with the operators $A_{lp}^{(n,\tau)}, D_{lp}^{(n,\tau)} : l^2 \to l^2$ defined according to the relations (3.6)-(3.9), (3.13)-(3.14). However, these operators do not depend on the parameter n at all. Really, consider the matrix representations of the operators $A_{12}^{(n,\tau)}, A_{21}^{(n,\tau)}, D_{12}^{(n,\tau)}$,

$D_{21}^{(n,\tau)}$. It follows from (3.10) that the entries a_{lq} of the corresponding matrices $(a_{lq})_{l,q=0}^{\infty}$ are

$$
\begin{aligned}
a_{pq} &= \int_{\mathbb{R}^+} K\phi_{qn}(\sigma)\phi_{ln}(\sigma)\,d\sigma = \int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \mathsf{k}_{\theta_\tau}\left(\frac{\sigma}{s}\right)\phi(ns-q)\frac{ds}{s}\phi(n\sigma-l)\,d\sigma \\
&= \frac{1}{n}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \mathsf{k}_{\theta_\tau}\left(\frac{u+l}{t+q}\right)\phi(u)\frac{du}{u+q}\phi(t)\,dt \\
&= \frac{1}{n}\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} \mathsf{k}_{\theta_\tau}\left(\frac{u+l}{t+q}\right)(\nu_d\sqrt{n}\widehat{\phi}(u))\frac{du}{u+q}(\nu_d\sqrt{n}\widehat{\phi}(t))\,dt \\
&= \nu_d^2\int_0^{d+1}\widehat{\phi}(t)\int_0^{d+1} \mathsf{k}_{\theta_\tau}\left(\frac{u+l}{t+q}\right)\widehat{\phi}(u)\frac{du}{u+q}\,dt,
\end{aligned}
$$

hence these operators are independent of n. Moreover, $D_{11}^{(n,\tau)}, D_{22}^{(n,\tau)} = 0$ and $A_{11}^{(n,\tau)} = A_{22}^{(n,\tau)} = I$. Combining all the above representations, one obtains that the operators R_n^τ do not depend on the parameter n. Therefore, (R_n^τ) is a constant sequence and it is stable if and only if any of its members, say R_1^τ, is invertible. It remains to observe that $R^\tau = R_1^\tau$, which completes the proof. □

4. Numerical approach to the invertibility of local operators. .

As was already mentioned, there is no efficient analytic method to verify the invertibility of the local operators R^τ. On the other hand, numerical approaches turn out to be surprisingly fruitful. Recall that the operators R^τ, $\tau \in \mathcal{M}_\Gamma$ do not depend on the shape of the contour Γ but only on the relevant angles θ_τ and β_τ. Therefore, for contours having only one corner point, Theorem 3.6 can be reformulated as follows.

COROLLARY 4.1. *If τ is the only corner point of the contour Γ, then the operator R^τ is invertible if and only if the Galerkin method $(P_n B_\Gamma P_n)$ is stable.*

Thus in order to determine the critical angles, i.e. the opening angles θ for which the operators R^τ are not invertible, one can consider the behaviour of the spline Galerkin methods on special contours. A family of such contours Γ_1^θ, $\theta \in (0, 2\pi)$,

$$
\Gamma_1^\theta := \{t \in \mathbb{C} : t = \gamma_1(s) = \sin(\pi s)\exp(i\theta(s-0.5)),\ s \in [0,1]\}
$$

has been used in [6, 9] to study the local operators of the Nyström method for Sherman–Lauricella and Muskhelishvili equations. Changing the parameter θ in the interval $(0, 2\pi)$, one obtains contours located at the origin and having only one corner of various magnitude. In the present paper, we use the same contours to detect the critical angles of the spline Galerkin methods. It is worth mentioning that the operator R^τ depends not only on θ_τ but also on the angle β_τ between the right semi-tangent to the contour Γ_1^θ at the point τ and the real line $\mathbb{R}$. However, numerical experiments conducted for both the Nyström and spline Galerkin methods show that, in fact, the angle β_τ does not influence the invertibility of the operator R^τ (see Figure 4.4 below and Remark 4.3). This opens a way for verifying the results obtained for contour Γ_1^θ by conducting similar tests for equations on contours with two or more corners, all of the same magnitude. To this end, we will use another contour Γ_2^θ, which is the union of two circular arcs with the parametrization

$$
\begin{aligned}
\gamma_1(s) &= -0.5\cot(0.5\theta) + 0.5/\sin(0.5\theta)\,\exp(i\theta(s-0.5)), \quad 0 \le s \le 1, \\
\gamma_2(s) &= 0.5\cot(0.5\theta) - 0.5/\sin(0.5\theta)\,\exp(i\theta(s-0.5)), \quad 0 \le s \le 1.
\end{aligned}
$$

To find the angles of instability, the interval $[0.1\pi,\ 1.9\pi]$ has been divided by the points $\theta_k := \pi(0.1+0.01k)$ and for each opening angle θ_k we constructed the matrices of the corresponding approximation operators for the Galerkin methods based on the splines of degree $d = 0, d = 1$ and $d = 2$. Note that we consider Galerkin

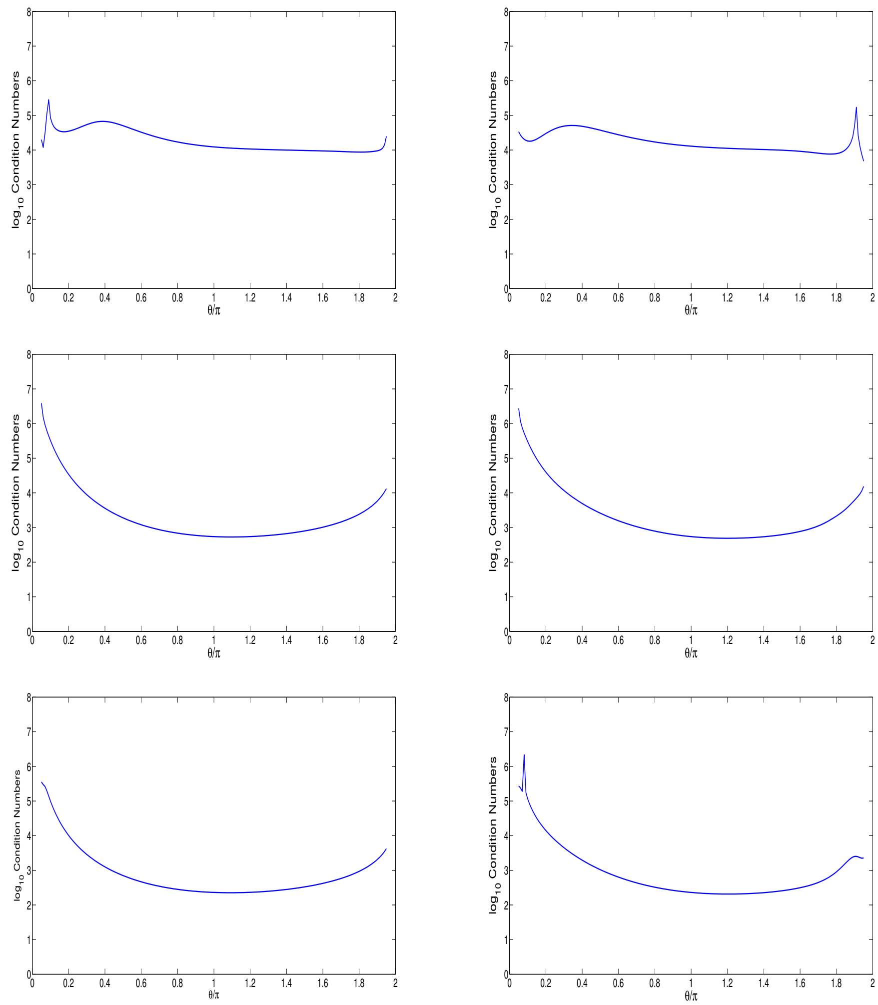

FIG. 4.1. Condition numbers vs. opening angles in case $n = 128$. From row 1 to row 3: splines of degree 0, 1 and 2, respectively. Left column: one-corner geometry, right column: two-corner geometry.

methods for two choices of n, namely for $n = 128$ and $n = 256$, and the integrals arising in the equation (2.2) and in the method (2.6) have been approximated by quadrature formulas (2.9), (2.10). Further, to verify the stability of the method, for each angle θ_k we compute the condition numbers of the corresponding matrices and the results of these computations are presented in Figures 4.1-4.3, where possible presence of peaks might indicate critical angles. Thus it seems that inside of the interval $(0.1\pi, 1.9\pi)$ neither of the Galerkin methods based on splines of degree $0, 1$ or 2 has critical angles. This differs from the Nyström method, where critical angles have been discovered for both Sherman–Lauricella and Muskhelishvili equations [6, 9].

Contrariwise, information about the critical angles at the interval ends is not so conclusive. Thus in the case $n = 256$, the computation of the condition numbers for both one and two corner geometry shows that for the Galerkin method based on the splines of degree zero there can be a critical angle at the right end of the interval mentioned.

For splines of the degree $d = 0$ and $d = 1$, the one and two corner geometries give contradictory results (see Figure 4.2). To clarify the situation one has to refine

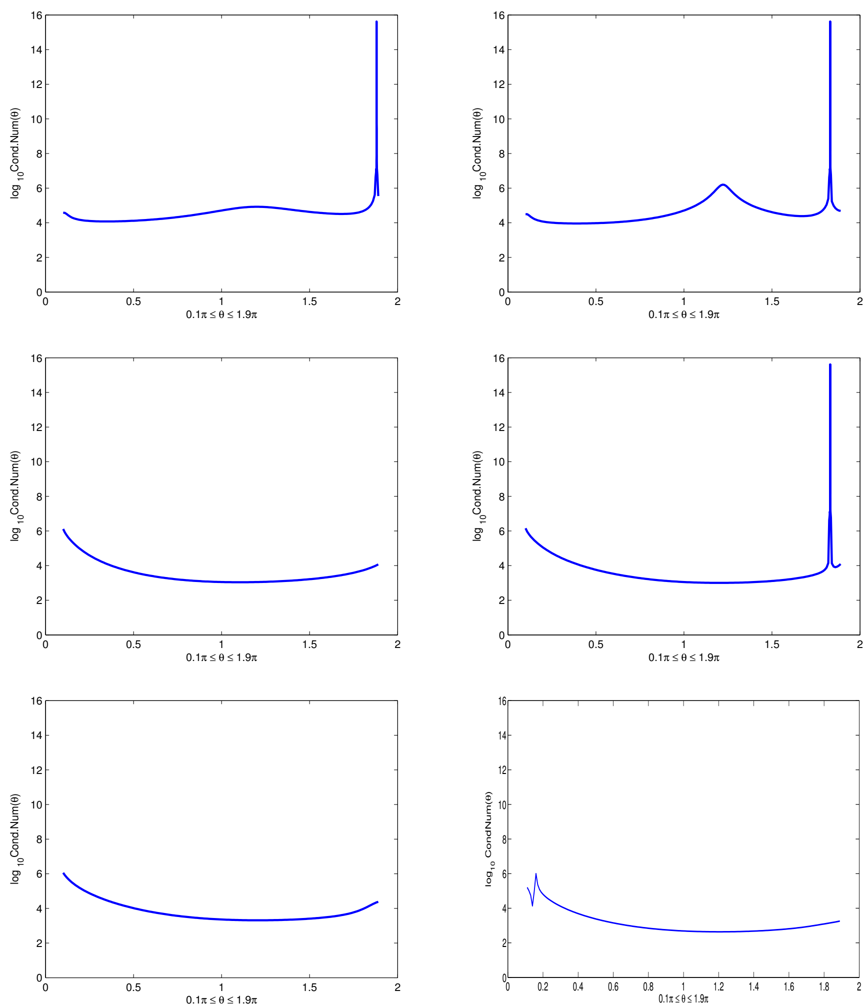

FIG. 4.2. Condition numbers vs. opening angles in case $n = 256$. From row 1 to row 3: splines of degree 0, 1 and 2, respectively. Left column: one-corner geometry, right column: two-corner geometry.

the mesh $\{\theta_k\}$ and essentially increase the dimension of the matrices used. Note that while discovering a suspicious critical angle for $n = 256$, we refined the mesh $\{\theta_k\}$ in a neighbourhood of that angle by reducing its step to 0.001π, and calculated the condition numbers for the corresponding Galerkin methods with n changed to 512. This allows us to show that, in fact, there are no critical angles in the interval mentioned. However, the computing time increases drastically.

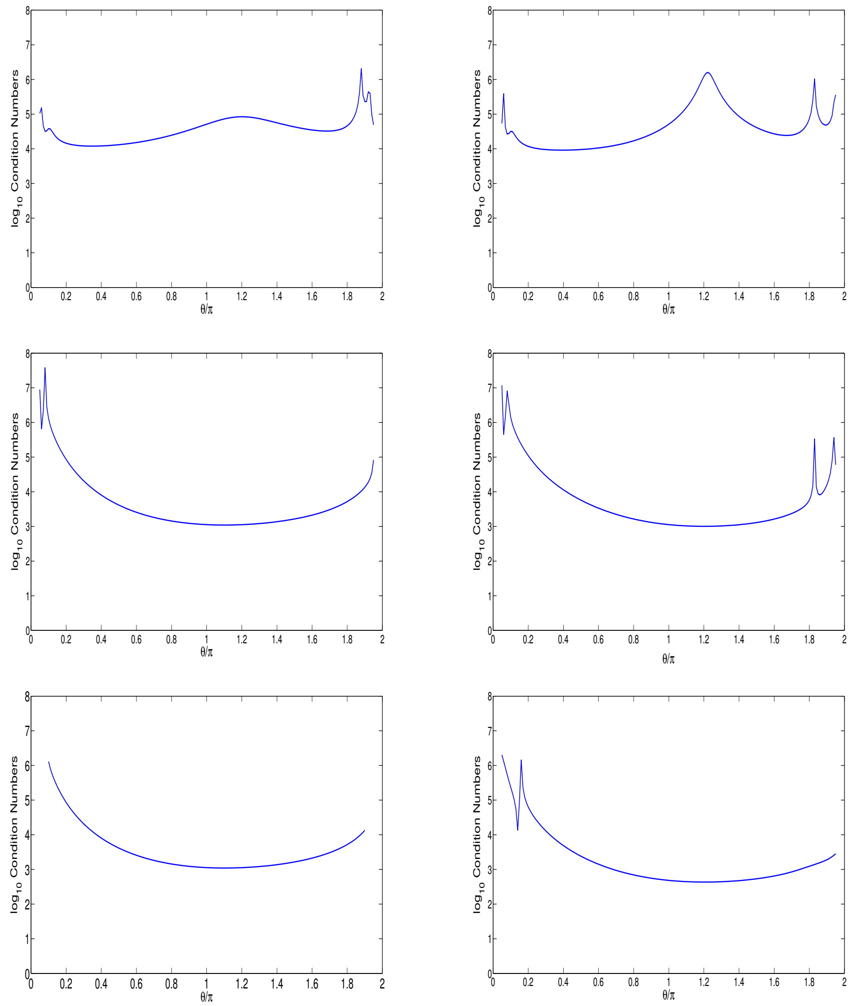

FIG. 4.3. Condition numbers vs. opening angles in case $n = 256$ and $n = 512$ after refining the mesh in neighbourhoods of suspicious points. From row 1 to row 3: splines of degree 0, 1 and 2, respectively. Left column: one-corner geometry, right column: two-corner geometry.

REMARK 4.2. *Note that that condition numbers of the Galerkin methods for the Sherman-Lauricella equation are quite large and one can raise a question, whether the methods are stable for at least one curve* Γ_1^θ, $0 < \theta < 2\pi$. *The answer to this question is affirmative: The method is stable for the curve* Γ_1^π, *since in this case the corresponding integral operators are compact. Taking into account the invertibility of the corrected Sherman-Lauricella operator and strong convergence of the projections to the identity operators, one obtains the claim.*

REMARK 4.3. *As was already mentioned, numerical experiments do not show that the stability depends on the angle* β. *The results obtained for both curves show the same behavior of condition numbers even if the parameter* β *is different for* Γ_1^θ *and* Γ_2^θ. *Moreover, we rotate the curve* Γ_1^θ *by various angles and compute the corresponding condition numbers. The results of the numerical experiments are presented in Figure*

4.4. Note that although for the rotated curves, the condition numbers corresponding to the same angle θ are different, the graphs in Figure 4.4 do not indicate the presence of any "infinite" peaks. Let us emphasize that if any suspicious point was discovered, then the initial mesh has been refined in a neighbourhoud of such a point and the experiment was repeated with a modified mesh.

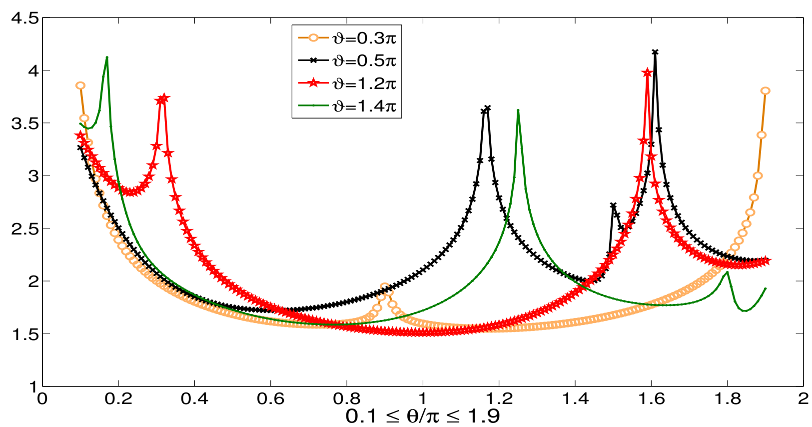

FIG. 4.4. Condition numbers vs. opening angles in case $n = 256$ and the curve Γ_1^θ rotated by $0.3\pi, 0.5\pi, 1.2\pi$ and 1.4π. The mesh has been refined in neighbourhoods of suspicious points and no "infinite" peaks are discovered.

The numerical experiments are performed in MATLAB environment (version 7.9.0) and executed on an Acer Veriton M680 workstation equipped with a Intel Core i7 vPro 870 Processor and 8GB of RAM, and it took from one to two weeks of computer work in order to obtain every single graph presented in Figure 4.1, 4.2 or 4.3.

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